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ON THE ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF AN ANALYTIC OPERATOR IN THE SENSE OF KATO AND APPLICATIONS

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Abstract. In the present paper, we derive a precise description to the behavior of the spectrum of a self-adjoint operator T_0 on a separable Hilbert space $\mathcal{H}$ after a perturbation by an analytic operator

$$B(\varepsilon) := \varepsilon T_1 + \varepsilon^2 T_2 + \cdots + \varepsilon^k T_k + \cdots ,$$

where $\varepsilon \in \mathbb{C}$ and $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying a specific growing inequality; while the spectrum of T_0 is discrete and its eigenvalues are not condense. Moreover, we apply the obtained results to a Gribov operator in Bargmann space and to a problem of radiation of a vibrating structure in a light fluid.

1. Introduction.

The problem of determining the asymptotic behavior of the spectrum of unbounded normal operator has been of great importance and considerable difficulty. Since it presents an important tool to study the stability of many systems and to prove the existence of bases for a class of non-normal

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perturbations of normal operator, it has long been studied by several authors such as A. Jeribi [12] and A. S. Markus and V. I. Matsaev [17]. Mainly in [17], the authors often considered an unbounded operator G on a separable Hilbert space $\mathcal{H}$ writing as a perturbation of a leading normal component T by its p-subordinate B, i.e.,

$$\|B\varphi\| \leq b\|T\varphi\|^p \|\varphi\|^{1-p} \quad \text{for every } \varphi \in \mathcal{D}(T), \tag{1.1}$$

where $p \in [0, 1]$ and b is strictly positive constant.

It should be noted here that the sharpest results on the comparison of the spectrum which are due to A. S. Markus and V. I. Matsaev [17] dealt with the subordinate condition (1.1). In fact, under Eq. (1.1), the authors in [16] and [17] developed several approaches on the distribution of the eigenvalues of the perturbed operator G. More precisely, they established asymptotic relations between the eigenvalue-counting functions of G and T and provided sufficient conditions to prove the existence of bases.

Now, if we try to weaken Eq. (1.1). Then, could we obtain an estimate about the distribution of the eigenvalues of G in terms of the one of T?

A positive answer to this question was given by A. A. Shkalikov [20]. Indeed, he replaced Eq. (1.1) by

$$\|B\psi_n\| \leq b, \tag{1.2}$$

where $(\psi_n)_{n\in\mathbb{N}^*}$ is an orthonormal system of eigenvectors of T associated to the eigenvalues $(\mu_n)_{n\in\mathbb{N}^*}$ and assumed that T is positive and self-adjoint and the spectrum of T is discrete and its eigenvalues $(\mu_n)_{n\in\mathbb{N}^*}$ are not condense, i.e.,

$$\mu_{n+q} - \mu_n \geq 1, \text{ for some } q \in \mathbb{N}^*. \tag{1.3}$$

Using Eqs (1.2) and (1.3), he gave a precise description to the localization of the spectrum of the perturbed operator G. More precisely, he constructed a half-strip containing all the eigenvalues of T and G. As a consequence, he claimed that the difference between the eigenvalue-counting functions of G and T is bounded by a constant, i.e.,

$$n(r, G) - n(r, T) = O(1), \tag{1.4}$$

where $n(r, T)$ (respectively, $n(r, G)$) denotes the sum of multiplicities of all eigenvalues of T (respectively, G) contained in the disk $\{\lambda \in \mathbb{C} \text{ such that } |\lambda| \leq r\}$ (see [20, Theorem 1]).

It is interesting to note that this outcome is of importance on itself. In fact, it is quite hard to control the jump of the eigenvalue-counting function of G. For instance, Theorems 8.2 and 8.4 stated in [16] guarantee the equivalence between the eigenvalue-counting functions of G and T if the operator B is

T-compact and the growth of $n(r,T)$ is regular. Further, [16, Theorem 9.2] characterize the behavior of $n(r,G)$ in terms of $n(r,T)$ if Eq. (1.1) holds. However, [20, Theorem 1] improves these theorems in the sense that Eqs (1.2) and (1.3), under which Eq. (1.4) holds, are much weaker.

Although, in certain applications, it is quite hard to verify Eq. (1.2). For example, let us consider the following Gribov operator originated from Reggeon field theory (see [1], [2], [5], [8], [11] and [13]):

$$(A^*A)^3 + \varepsilon A^*(A+A^*)A + \varepsilon^2(A^*A)^{3u_2} + \cdots + \varepsilon^k(A^*A)^{3u_k} + \cdots, \tag{1.5}$$

where $\varepsilon \in \mathbb{C}$ and $(u_k)_{k\in\mathbb{N}}$ is a strictly decreasing sequence with strictly positive terms such that $u_0 = 1$ and $u_1 = \dfrac{1}{2}$; while A (respectively, A^*) is the annihilation (respectively, creation) operator acting in Bargmann space

$$E = \left\{\varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2}|\varphi(z)|^2 dxdy < \infty \text{ and } \varphi(0) = 0\right\},$$

where $dxdy$ is the Lebesgue measure on $\mathbb{R}^2$ identified with $\mathbb{C}$ and the expressions of A and A^* are given by

$$\begin{cases} A : \mathcal{D}(A) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in E \text{ such that } A\varphi \in E\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in E \text{ such that } A^*\varphi \in E\}. \end{cases}$$

This space is equipped with the following scalar product:

$$\begin{cases} \langle\cdot,\cdot\rangle : E \times E \longrightarrow \mathbb{C} \\ \qquad (\varphi,\psi) \longrightarrow \langle\varphi,\psi\rangle = \dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}} e^{-|z|^2}\varphi(z)\overline{\psi(z)}dxdy \end{cases}$$

and its associated norm is denoted by $\|.\|$.

Since $\{\varphi_n := \dfrac{z^n}{\sqrt{n!}}\}_{n\geq 1}$ is an orthonormal basis of eigenvectors of $(A^*A)^3$ associated to the eigenvalues $\{n^3\}_{n\geq 1}$, so for $|\varepsilon| < 1$ we get

$$\left\| \left(\varepsilon A^*(A+A^*)A + \sum_{k=2}^{\infty} \varepsilon^k (A^*A)^{3u_k} \right) \varphi_n \right\| \leq \frac{|\varepsilon|}{1-|\varepsilon|}(1+2\sqrt{2})(1+n^3). \tag{1.6}$$

As we observe, Eq. (1.6) could not obey to the criteria (Eq. (1.2)) of A. A. Shkalikov [20]. Consequently, we risk to loose [20, Theorem 1].

Furthermore, let us consider the following integro-differential operator suggested by P. J. T. Filippi et al. [10] (see also [6], [7], [9] and [13]) to describe the radiation of a vibrating structure in a light fluid:

$$(I+\varepsilon K)^{-1}\frac{d^4\varphi}{dx^4}+\varepsilon\,(I+\varepsilon K)^{-1}K\left(\frac{d^4}{dx^4}-\left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}}\right)\varphi=\lambda(\varepsilon)\varphi.$$

Here K is the integral operator with kernel the Hankel function of the first kind and order 0 and ε is a complex number such that $|\varepsilon|<\dfrac{1}{\|K\|}$.

If we denote by $(\varphi_n)_{n\geq1}$ the system of eigenvectors of the operator

$$\begin{cases}\dfrac{d^4}{dx^4}:\mathcal{D}\left(\dfrac{d^4}{dx^4}\right)\subset L^2(]-L,L[)\longrightarrow L^2(]-L,L[)\\ \qquad\qquad\qquad\qquad\qquad\qquad\varphi\longrightarrow\dfrac{d^4\varphi}{dx^4}\\ \mathcal{D}\left(\dfrac{d^4}{dx^4}\right)=H_0^2(]-L,L[)\cap H^4(]-L,L[)\end{cases}$$

associated to the eigenvalues $(\lambda_n)_{n\geq1}$, where $\left(\dfrac{(2n+1)\pi}{4L}\right)^4\leq\lambda_n\leq\left(\dfrac{(2n+3)\pi}{4L}\right)^4$, we can easily see that $(\varphi_n)_{n\geq1}$ forms an orthonormal basis of $L^2(]-L,L[)$ and for $|\varepsilon|<\dfrac{1}{\|K\|}$, we have

$$\left\|\sum_{k=1}^{\infty}(-1)^k\varepsilon^kK^k\left(\frac{d^4\varphi_n}{dx^4}\right)^{\frac{1}{2}}\right\|\leq\frac{|\varepsilon|\|K\|}{1-|\varepsilon|\|K\|}\;\left(\frac{(2n+3)\pi}{4L}\right)^4.$$

Hence, it is quite difficult to verify Eq. (1.2) and consequently [20, Theorem 1] could not be applied.

As a tentative approach to grapple with such difficulty, we had the idea to extend the outcomes obtained by A. A. Shkalikov [20] to an abstract analytic operator introduced by B. Sz. Nagy in [19] and considered later in [3], [4] and [6]–[9]. More precisely, we consider the following operator

$$T(\varepsilon):=T_0+\varepsilon T_1+\varepsilon^2T_2+\cdots+\varepsilon^kT_k+\cdots, \tag{1.7}$$

satisfying the following hypotheses:

$(H1)$ T_0 is self-adjoint, positive and with domain $\mathcal{D}(T_0)$ on a separable Hilbert space $\mathcal{H}$.

$(H2)$ The resolvent of T_0 is compact and its eigenvalues $(\lambda_n)_{n\geq1}$ do not condense, i.e.,

$$\lambda_{n+p} - \lambda_n \geq 1, \text{ for some } p \in \mathbb{N}^*.$$

Setting $T_1, T_2, T_3, \ldots$ linear operators on $\mathcal{H}$ having the same domain $\mathcal{D}$ and verifying:

$(H3)$ $\mathcal{D} \supset \mathcal{D}(T_0)$ and there exist $a, b, q > 0$ and $\beta \in]0, \frac{1}{2}[$ such that for all $k \geq 1$

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}), \text{ for all } \varphi \in \mathcal{D}(T_0).$$

Following some ideas of [20], we derive a precise description to the localization of the eigenvalues of the perturbed operator (1.7) and we give an asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$. More precisely, we prove that:

Theorem 1.1. *Assume that hypotheses* $(H1)$–$(H3)$ *are satisfied. Then, for* $|\varepsilon|$ *enough small, the spectrum of the operator* $T(\varepsilon)$ *is constituted by isolated eigenvalues satisfying*

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1) \text{ i.e., } |n(r, T(\varepsilon)) - n(r, T_0)| < C \text{ as } r \to \infty,$$

where $n(r, T_0)$ (*respectively,* $n(r, T(\varepsilon))$) *denotes the sum of multiplicities of all eigenvalues of* T_0 (*respectively,* $T(\varepsilon)$) *contained in the disk* $\{\lambda \in \mathbb{C}$ *such that* $|\lambda| < r\}$ *and* C *is a constant.*

We shall emphasize on the fact that Theorem 1.1 improves [8, Theorem 3.1] since the spectral analysis method developed in [8] is based on the fact that the eigenvalues $(\lambda_n)_{n\geq 1}$ of T_0 are with multiplicity one, whereas in Theorem 1.1, we control the jump of $n(r, T(\varepsilon))$ in terms of $n(r, T_0)$ by assuming that the eigenvalues $(\lambda_n)_{n\geq 1}$ of T_0 are with finite multiplicity and do not condense.
Moreover, Theorem 1.1 extends the result of [20, Theorem 1] to an analytic operator instead of the sum of two operators and this generalization enables us to study the asymptotic behavior of the eigenvalues of some analytic operators where the criteria of A. A. Shkalikov [20] can not be applied (see Section 4).

Now, let us outline briefly the contents of this paper: Section 2 is devoted to present some basic definitions and auxiliary results connected to the main body of the paper. In the third section, we prove the objective of this paper. In the last section, we apply the obtained results to a Gribov operator in Bargmann space and to a problem of radiation of a vibrating structure in a light fluid.

2. Preliminaries.

In this section, we introduce some definitions and preliminary results that we will need in the sequel. To this end, let us consider a Hilbert space $\mathcal{H}$.

Definition 2.1 ([16, p. 55]). *Let U be a domain in $\mathbb{C}$, an operator-valued function $A(z)$ $(z \in U)$ with values in $L(\mathcal{H})$ is said to be uniformly holomorphic if for any $z_0 \in U$ the uniform limit*

$$\lim_{z \to z_0} \frac{A(z) - A(z_0)}{z - z_0} \ (= A'(z_0))$$

exists.

Definition 2.2. *Let U be a domain in $\mathbb{C}$, an operator-valued function $A(z)$ $(z \in U)$ with values in $L(\mathcal{H})$ is said to be meromorphic if it is holomorphic in U except in certain points which are poles for it.*

Lemma 2.1 ([20]). *Let $f(\lambda)$ be a bounded holomorphic function in the rectangle*

$$\Pi = \{\lambda \in \mathbb{C} \text{ such that } |\operatorname{Re}\lambda - r| < a_1 \text{ and } |\operatorname{Im}\lambda| < c_1\}.$$

For $\delta \in (0,1)$, we set $a' = a_1(1-\delta)$ and $c' = c_1(1-\delta)$, and we denote by Π' the rectangle defined by

$$\Pi' = \{\lambda \in \mathbb{C} \text{ such that } |\operatorname{Re}\lambda - r| < a' \text{ and } |\operatorname{Im}\lambda| < c'\}.$$

Let $M := \sup_{\lambda \in \Pi} |f(\lambda)|$ and $M' := \sup_{\lambda \in \Pi'} |f(\lambda)|$. Then:
(i) The number of zeros of the function $f(\lambda)$ in the rectangle Π', $n_f(\Pi')$, satisfies the estimate

$$n_f(\Pi') \leq C(\ln M - \ln M'),$$

(ii) if γ is a straight line segment contained in Π' that does not pass through the zeros of f in Π', then the variation of the argument of the function f along γ is estimated as

$$|[\arg f(\lambda)]_\gamma| \leq C'(\ln M - \ln M'),$$

where the constants C and C' depend only on δ and the ratio $\frac{a_1}{c_1}$ and do not depend on r and f.

An important role in the comparison of spectra of operators is played by the concept of determinant of operators having the form $I + K$, where K is a finite rank operator on $\mathcal{H}$.

Definition 2.3 ([17, p. 144]). *The determinant of the operator $I + K$ is the product of all its eigenvalues with multiplicity taken into account. The determinant of $I + K$ is denoted by $\det(I + K)$.*

Proposition 2.1 ([16, p. 14]). *The following assertions hold:*
(i) $|\det(I+K)| \leq (1+\|K\|)^{rk(K)}$, *where* $rk(K)$ *designates the rank of* K.
(ii) If $A(z)$ *is a holomorphic operator-valued function with values in* $L(\mathcal{H})$, *then the function* $\det(I+KA(z))$ *is holomorphic.*

Remark 2.1. We have

$$|\det(I+K)| \geq |1-\|K\||^{rk(K)}.$$

Lemma 2.2 ([17, p. 149]). *Let* γ *be a Jordan curve that is not closed (possibly unbounded) and* $K(z)$ $(z \in \gamma)$ *be a continuous operator-valued function with* $\|K(z)\| < 1$ *and* $rk(K(z)) \leq n$. *Then, we have*

$$|[\arg\det(I+K(z))]_\gamma| \leq n\pi.$$

Lemma 2.3 ([17, p. 149]). *Let* A *and* $A+K$ *be two operators on* $\mathcal{H}$ *with compact resolvent and* Δ *a bounded domain with rectifiable boundary* Γ.

If $\sigma(A) \cap \Gamma = \emptyset$ *or* $\sigma(A+K) \cap \Gamma = \emptyset$, *then we have*

$$n(\Delta, A+K) - n(\Delta, A) = \frac{1}{2\pi}[\arg\det(I-KR_\lambda(A))]_\Gamma,$$

where $n(\Delta, A)$ *(respectively,* $n(\Delta, A+K)$*) denotes the sum of multiplicities of all eigenvalues of* A *(respectively,* $A+K$*) in* Δ.

In the remaining part of this section, we will state some basic results on the behavior of the spectrum under a relatively compact perturbation (see [16]).

Let A and B be two linear operators on $\mathcal{H}$ such that the resolvent set of A, $\rho(A)$, is not empty.

Definition 2.4 ([16, p. 16]). *The operator* B *is said to be* A*-compact if its domain* $\mathcal{D}(B)$ *contains* $\mathcal{D}(A)$ *and the operator* $BR_\lambda(A)$ *is compact, where* $\lambda \in \rho(A)$.

Lemma 2.4 ([16, Lemma 3.6, p. 17]). *Suppose that* A *is normal with compact resolvent,* B *is* A*-compact and* $\sigma(A) \cap \Omega(\theta) \subset \mathbb{R}$, *where* $\Omega(\theta)$ *is the angle* $\{\lambda : |\arg\lambda| < \theta\}$ *and* $0 < \theta \leq \frac{\pi}{2}$. *Then the resolvent of the operator* $G = A+B$ *is also compact and for* $\delta > 0$ *only finitely many eigenvalues of* G *lie in the angles* $\Phi_\pm(\delta) = \{\lambda : \delta \leq \pm\arg\lambda \leq \theta - \lambda\}$.

Proposition 2.2 ([16, Lemma 8.1, p. 41]). *Assume that* A *is with compact resolvent and* B *is* A*-compact. If* Δ *is a bounded domain with rectifiable boundary* Γ *and* $\sigma(A+tB) \cap \Gamma = \emptyset$, *for all* $t \in [0,1]$, *then the operator* $A+B$ *also has a compact resolvent and* $n(\Delta, A+B) = n(\Delta, A)$, *where* $n(\Delta, A)$ *(respectively,* $n(\Delta, A+B)$*) designates the sum of multiplicities of all eigenvalues of* A *(respectively,* $A+B$*) in* Δ.

3. Main results. In this section, we follow some ideas of [20] to prove that for $|\varepsilon|$ enough small, the spectrum of the perturbed operator $T(\varepsilon)$ (see [19] and [6]–[9]) is constituted by isolated eigenvalues such that

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1).$$

3.1. The behavior of the spectrum of the perturbed operator $T(\varepsilon)$ under a finite rank perturbation. In order to prove Theorem 1.1, we will study the comportment of the spectrum of the perturbed operator $T(\varepsilon)$ under a specific perturbation by a finite rank operator that we denote by K_r. More precisely, we prove that

$$n(r, T(\varepsilon) - K_r) = n(r, T_0).$$

To this end, we shall introduce some notations and state some preliminary results.

We have T_0 is self-adjoint with compact resolvent, so let

$$T_0 f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n$$

be its spectral decomposition, where λ_n is the n^{th} eigenvalue of T_0 associated to the eigenvector φ_n. Denoting by

$$K_r := 2l \sum_{r-2l<\lambda_n<r} \langle \cdot, \varphi_n \rangle \varphi_n - 2l \sum_{r \le \lambda_n < r+2l} \langle \cdot, \varphi_n \rangle \varphi_n,$$

with $r > 4l > 0$ and $l \in \mathbb{N}^*$, we have K_r is a finite rank operator and $rk(K_r) \le 4lp$.

Let us consider $T_r := T_0 - K_r$.

Lemma 3.1. *T_r is positive and self-adjoint.*

Moreover, T_r preserves the system of eigenvectors $(\varphi_n)_n$ and the eigenvalues $(\lambda_n)_n$ lying outside the interval $]r-2l, r+2l[$ and changes only the $(\lambda_n)_n$ lying inside the interval (see Fig. 1).

Fig. 1

More precisely, if we denote by $(\nu_n)_n$ the eigenvalues of T_r numbered in the increasing order, we get

$$\nu_n = \begin{cases} \lambda_n - 2l & \text{if } \lambda_n \in]r-2l, r[\\ \lambda_n + 2l & \text{if } \lambda_n \in [r, r+2l[\\ \lambda_n & \text{if } \lambda_n \notin]r-2l, r+2l[. \end{cases}$$

Thus, $]r-2l, r+2l[$ doesn't contain any eigenvalues ν_n of T_r.

Lemma 3.2. *The eigenvalues* $(\nu_n)_n$ *of* T_r *verify:*

$$\nu_{n+2p} - \nu_n \geq 1, \ \ n = 1, 2, \dots .$$

Remark 3.1. As the spectrum of T_0 is discrete then the spectrum of T_r. Further, we have

$$n(r, T_r) = n(r, T_0).$$

Now, we shall recall the following theorem developed in [8].

Theorem 3.1 ([8, Theorem 2.1])**.** *Suppose that hypotheses* $(H1)$ *and* $(H3)$ *hold. Then for* $|\varepsilon| < q^{-1}$*, the series* $\sum_{i\geq 0} \varepsilon^i T_i \varphi$ *converges for all* $\varphi \in \mathcal{D}(T_0)$*. If* $T(\varepsilon)\varphi$ *denotes its limit, then* $T(\varepsilon)$ *is a linear operator with domain* $\mathcal{D}(T_0)$ *and for* $|\varepsilon| < (q+\beta b)^{-1}$*, the operator* $T(\varepsilon)$ *is closed.*

In the sequel, we denote by $B(\varepsilon) := \sum_{i\geq 1} \varepsilon^i T_i$. The first result of this section asserts:

Proposition 3.1. *Assume that hypotheses* $(H1)$*–*$(H3)$ *hold. Then, for* $|\varepsilon| < \dfrac{1}{q+\beta b}$ *we have:*

(i) The operator $B(\varepsilon)$ *is* T_0*-compact. Further, the operator* $T(\varepsilon)$ *is with compact resolvent.*

(ii) The operator $B(\varepsilon)$ *is* T_r*-compact. Moreover, the operator* $T(\varepsilon) - K_r$ *is with compact resolvent.*

P r o o f. (i) Let $n \in \mathbb{N}^*$ and λ_n the n^{th} eigenvalue of T_0. Using hypothesis $(H3)$, we get

$$\begin{aligned} \|B(\varepsilon)\varphi_n\| &\leq \sum_{i=1}^{\infty} \|\varepsilon^i T_i \varphi_n\| \\ &\leq \sum_{i=1}^{\infty} |\varepsilon|^i q^{i-1} (a\|\varphi_n\| + b\|T_0\varphi_n\|^{\beta} \|\varphi_n\|^{1-\beta}) \\ &\leq \sum_{i=1}^{\infty} |\varepsilon|^i q^{i-1} (a + b\lambda_n^{\beta}). \end{aligned} \tag{3.1}$$

Then, for $|\varepsilon| < \frac{1}{q}$, Eq. (3.1) implies that

$$\|B(\varepsilon)\varphi_n\| \leq \frac{|\varepsilon|}{1-|\varepsilon|q}\,(a+b\lambda_n^{\beta}). \tag{3.2}$$

Hence,

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{\lambda_n^2} \leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\frac{(a+b\lambda_n^{\beta})^2}{\lambda_n^2}.$$

Taking into account hypothesis $(H2)$, we obtain

$$\begin{aligned}\lambda_{n+1}-\lambda_1 = \underbrace{\lambda_{n+1}-\lambda_{(n+1)-p}}_{\geq 1} + \underbrace{\lambda_{(n+1)-p}-\lambda_{(n+1)-2p}}_{\geq 1} + \cdots \\ + \underbrace{\lambda_{1+p}-\lambda_{(n+1)-\frac{n}{p}p(=1)}}_{\geq 1} \geq \frac{n}{p}.\end{aligned} \tag{3.3}$$

Thus, Eq. (3.3) yields $\lambda_n \geq \frac{n-1}{p} + \lambda_1$. So, writing $\frac{(a+b\lambda_n^{\beta})^2}{\lambda_n^2}$ as

$$\frac{(a+b\lambda_n^{\beta})^2}{\lambda_n^2} = \frac{a^2}{\lambda_n^2} + \frac{2ab\lambda_n^{\beta}}{\lambda_n^2} + \frac{b^2\lambda_n^{2\beta}}{\lambda_n^2},$$

we conclude that the series $\sum_n \frac{2ab\lambda_n^{\beta}}{\lambda_n^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{\lambda_n^2}$ are convergent since $\beta \in \left]0, \frac{1}{2}\right[$. So, $\sum_n \frac{\|B(\varepsilon)\varphi_n\|^2}{\lambda_n^2}$ is also convergent. Consequently,

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{\lambda_n^2} < \infty.$$

Then, by [20, Lemma 1] the operator $B(\varepsilon)$ is T_0-compact and the spectrum of the operator $T(\varepsilon)$ is discrete for $|\varepsilon| < \frac{1}{q+\beta b}$. Moreover, since T_0 is self-adjoint with compact resolvent and its spectrum is positif, then in view of Lemma 2.4, the operator $T(\varepsilon)$ is with compact resolvent.

(ii) Making the same reasoning as the one in *(i)*, we obtain $B(\varepsilon)$ is T_r-compact and $T(\varepsilon) - K_r$ is with compact resolvent. □

Corollary 3.1. *Suppose that hypotheses $(H1)$ and $(H3)$ are verified. Moreover, assume that*

$$\lambda_{n+p}^{1-\alpha} - \lambda_n^{1-\alpha} \geq 1, \ \textit{where } 0 \leq \alpha < 1. \tag{3.4}$$

Hence, for $\beta \in]0, 1 + \frac{\alpha - 1}{2}[$ *and* $|\varepsilon| < \frac{1}{q + \beta b}$ *the spectrum of the operator* $T(\varepsilon)$ *is discrete.*

Proof. In view of Eq. (3.4), we have $\lambda_n^{1-\alpha} \geq \frac{n-1}{p} + \lambda_1^{1-\alpha}$. Hence, the series $\sum_n \frac{2ab\lambda_n^\beta}{\lambda_n^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{\lambda_n^2}$ are convergent since $\beta \in]0, 1 + \frac{\alpha - 1}{2}[$. So, making the same reasoning as the one in the proof of Proposition 3.1, we conclude that the spectrum of the operator $T(\varepsilon)$ is discrete for $|\varepsilon| < \frac{1}{q + \beta b}$. □

Lemma 3.3. *For* $|\varepsilon| < \frac{1}{q}$ *and* $r - l \leq \operatorname{Re} \lambda \leq r + l$*, there exists a positive number* $M(\varepsilon, a, 2p, q, l)$ *such that*

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} < M(\varepsilon, a, 2p, q, l),$$

where ν_n *is the eigenvalue number* n *of* T_r*.*

Proof. Let $n \in \mathbb{N}^*$ and $\sigma = \operatorname{Re} \lambda$. Then, there exists $k \in \mathbb{N}^*$ such that $\nu_{k-1} \leq \sigma$ and $\nu_k > \sigma$. Thus, we obtain

$$\nu_k - \sigma > \nu_k^\beta \left(\nu_k^{1-\beta} - \sigma^{1-\beta}\right) \geq C_1 \nu_k^\beta \ (C_1 > 0) \tag{3.5}$$

and

$$|\lambda - \nu_{k-1}| \geq ||\lambda| - \nu_{k-1}| > |\lambda|^\beta \left(|\lambda|^{1-\beta} - \nu_{k-1}^{1-\beta}\right) \geq C_2 \nu_{k-1}^\beta \ (C_2 > 0). \tag{3.6}$$

Further, we have

$$\sigma - \nu_n > \nu_{k-1} - \nu_n \ (n < k-1), \ \nu_n - \sigma > \nu_n - \nu_k \ (n > k).$$

Hence, we get

$$\nu_{k-1} - \nu_n \geq \nu_n^\beta(\nu_{k-1}^{1-\beta} - \nu_n^{1-\beta}) \ (n < k-1), \ \nu_n - \nu_k \geq \nu_n^\beta(\nu_n^{1-\beta} - \nu_k^{1-\beta}) \ (n > k). \tag{3.7}$$

Consequently, using Eqs (3.5), (3.6) and (3.7) we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \nu_n|^2} \leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\sigma - \nu_n|^2}$$
$$< \frac{c}{C_1^2} + \frac{c}{C_2^2} + \sum_{n<k-1} \frac{\lambda_n^{2\beta}}{\nu_n^{2\beta}\left(\nu_{k-1}^{1-\beta} - \nu_n^{1-\beta}\right)^2}$$

$$
\begin{aligned}
&+\sum_{n>k}\frac{\lambda_n^{2\beta}}{\nu_n^{2\beta}\left(\nu_n^{1-\beta}-\nu_k^{1-\beta}\right)^2}\\
&\leq \frac{2c}{C_3^2}+\sum_{n<k-1}\frac{c}{\left(\nu_{k-1}^{1-\beta}-\nu_n^{1-\beta}\right)^2}\\
&+\sum_{n>k}\frac{c}{\left(\nu_n^{1-\beta}-\nu_k^{1-\beta}\right)^2}, \qquad (3.8)
\end{aligned}
$$

where $0<c\leq 1+\dfrac{2l}{\nu_1}$ and $C_3:=\min\{C_1,C_2\}$. As $\beta\in]0,\dfrac{1}{2}[$, we get in view of [16, p. 33]

(3.9) $\nu_{k-1}^{1-\beta}-\nu_n^{1-\beta}\geq(1-\beta)(\nu_{k-1}-\nu_n)\nu_{k-1}^{-\beta}\geq c_1(1-\beta)(\nu_{k-1}-\nu_n)^{1-\beta},\ 0<c_1<1$

and

(3.10) $\nu_n^{1-\beta}-\nu_k^{1-\beta}\geq(1-\beta)(\nu_n-\nu_k)\nu_n^{-\beta}\geq c_2(1-\beta)(\nu_n-\nu_k)^{1-\beta},\ 0<c_2<1.$

Then, it follows from Eqs (3.8), (3.9) and (3.10) that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda-\nu_n|^2} &< \frac{2c}{C_3^2}+\sum_{n<k-1}\frac{c}{c_1^2(1-\beta)^2\left(\nu_{k-1}-\nu_n\right)^{2(1-\beta)}}+\\
&\sum_{n>k}\frac{c}{c_2^2(1-\beta)^2\left(\nu_n-\nu_k\right)^{2(1-\beta)}}\\
&\leq \frac{2c}{C_3^2}+\sum_{n<k-1}\frac{c}{c'^2(1-\beta)^2\left(\nu_{k-1}-\nu_n\right)^{2(1-\beta)}}+\\
&\sum_{n>k}\frac{c}{c'^2(1-\beta)^2\left(\nu_n-\nu_k\right)^{2(1-\beta)}}, \qquad (3.11)
\end{aligned}
$$

where $c':=\min\{c_1,c_2\}$. Hence, using the fact that $\nu_n\geq\dfrac{n-1}{2p}+\nu_1$, Eq. (3.11) yields

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda-\nu_n|^2} &< \frac{2c}{C_3^2}+\sum_{n<k-1}\frac{c(2p)^{2(1-\beta)}}{c'^2(1-\beta)^2\left(k-1-n\right)^{2(1-\beta)}}\\
&+\sum_{n>k}\frac{c(2p)^{2(1-\beta)}}{c'^2(1-\beta)^2\left(n-k\right)^{2(1-\beta)}}\\
&\leq \frac{2c}{C_3^2}+\frac{c(2p)^{2(1-\beta)}}{c'^2(1-\beta)^2}\left(\sum_{n<k-1}\frac{1}{\left(k-1-n\right)^{2(1-\beta)}}+\sum_{m=1}^{\infty}\frac{1}{m^{2(1-\beta)}}\right)
\end{aligned}
$$

$$< \frac{2c}{C_3^2} + \frac{2c(2p)^{2(1-\beta)}}{c'^2(1-\beta)^2} \sum_{m=1}^{\infty} \frac{1}{m^{2(1-\beta)}} < \infty.$$

Further, we have

$$\sum_{n=1}^{\infty} \frac{\lambda_n^\beta}{|\lambda - \nu_n|^2} < \frac{2c}{C_3^2} + \frac{8c(2p)^{2-\beta}}{c'^2(2-\beta)^2} \sum_{m=1}^{\infty} \frac{1}{m^{2-\beta}} < \infty.$$

Hence, the series $\sum_n \frac{2ab\lambda_n^\beta}{|\lambda - \nu_n|^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{|\lambda - \nu_n|^2}$ are convergent. So, let ξ be a positive constant such that

$$\xi := \sum_{n=1}^{\infty} \frac{2ab\lambda_n^\beta}{|\lambda - \nu_n|^2} + \sum_{n=1}^{\infty} \frac{b^2\lambda_n^{2\beta}}{|\lambda - \nu_n|^2}. \tag{3.12}$$

On the other hand, since the interval $]r-2l, r+2l[$ does not contain any eigenvalue ν_n of T_r, then there exists $k \in \mathbb{N}^*$ such that $\nu_{k-1} \leq r - 2l$ and $\nu_k \geq r + 2l$. Moreover, by Lemma 3.2 and for any $\sigma \in [r - l, r + l]$ we obtain

$$\nu_{k+j+2sp} - \sigma \geq l + s \text{ and } \sigma - \nu_{k-j-2sp-1} \geq l + s, \quad \text{where } j = 0, 1 \ldots, 2p - 1 \text{ and } s = 0, 1 \ldots. \tag{3.13}$$

In view of hypothesis $(H3)$ and Eqs (3.2) and (3.12), we get for $|\varepsilon| < \frac{1}{q}$

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} < \frac{|\varepsilon|^2}{(1 - |\varepsilon| q)^2} \left(\xi + \sum_{n=1}^{\infty} \frac{a^2}{|\sigma - \nu_n|^2} \right). \tag{3.14}$$

Now, let $\sigma \in [r-l, r+l]$. Using Eqs (3.13) and (3.14), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\nu_n|^2} &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + a^2\left(\sum_{n=1}^{k-1}\frac{1}{|\sigma-\nu_n|^2} + \sum_{n=k}^{\infty}\frac{1}{|\nu_n-\sigma|^2}\right)\right)\\ &\le \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + a^2\left(\sum_{s=0}^{\frac{k}{2p}-1}\frac{2p}{(l+s)^2} + \sum_{s=0}^{\infty}\frac{2p}{(l+s)^2}\right)\right)\\ &\le \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + 2a^2\left(\sum_{s=1}^{\infty}\frac{2p}{(l+s)^2} + \frac{2p}{l^2}\right)\right)\\ &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + 4pa^2\left(\int_0^{\infty}\frac{dx}{(l+x)^2} + \frac{1}{l^2}\right)\right)\\ &\le \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + 4pa^2\frac{l+1}{l^2}\right) =: M(\varepsilon, a, 2p, q, l). \qquad \square\end{aligned}$$

Lemma 3.4. *Let τ be an arbitrary positive number. If $|\operatorname{Im}\lambda| \ge \tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists a positive constant $N(\varepsilon, a, 2p, q, \tau)$ such that*

$$\sum_{n=1}^{\infty}\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\nu_n|^2} < N(\varepsilon, a, 2p, q, \tau). \tag{3.15}$$

If $\operatorname{Re}\lambda \le -\tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists also a positive constant $N_1(\varepsilon, a, 2p, q, \tau)$ such that

$$\sum_{n=1}^{\infty}\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\nu_n|^2} < N_1(\varepsilon, a, 2p, q, \tau). \tag{3.16}$$

Proof. Let $n \in \mathbb{N}^*$, $\sigma = \operatorname{Re}\lambda$ and $\lambda = \sigma \pm i\tau$, where $\tau > 0$. Then, there exists $k \in \mathbb{N}^*$ such that $\nu_{k-1} \le \sigma$ and $\nu_k > \sigma$. In view of hypothesis $(H3)$ and Eq. (3.2), we get for $|\varepsilon| < \frac{1}{q}$

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\nu_n|^2} \le \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\frac{a^2}{|\lambda-\nu_n|^2} + \frac{2ab\lambda_n^{\beta}}{|\lambda-\nu_n|^2} + \frac{b^2\lambda_n^{2\beta}}{|\lambda-\nu_n|^2}\right). \tag{3.17}$$

• Let us consider $|\operatorname{Im}\lambda| \geq \tau$. In view of Lemma 3.2, we obtain

$$\nu_{k+j+2sp} - \sigma \geq s \text{ and } \sigma - \nu_{k-j-2sp-1} \geq s, \qquad \text{where } j = 0, 1, \ldots, 2p-1 \text{ and } s = 0, 1, \ldots. \tag{3.18}$$

Hence, for $|\varepsilon| < \frac{1}{q}$, Eqs (3.12), (3.17) and (3.18) yield

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + a^2 \sum_{n=1}^{\infty} \frac{1}{(\nu_n - \sigma)^2 + \tau^2}\right) \\ &\leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2a^2 p \left(\sum_{s=0}^{\frac{k}{2p}-1} \frac{1}{s^2+\tau^2} + \sum_{s=0}^{\infty} \frac{1}{s^2+\tau^2}\right)\right) \\ &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 4pa^2 \left(\frac{1}{\tau^2} + \int_0^{\infty} \frac{dx}{x^2+\tau^2}\right)\right).\end{aligned}$$

Consequently, for $|\varepsilon| < \frac{1}{q}$ we get

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} < \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2a^2 \frac{p}{\tau}\left(\pi + \frac{2}{\tau}\right)\right) =: N(\varepsilon, a, 2p, q, \tau).$$

• Now, if $\operatorname{Re}\lambda \leq -\tau$. Then, for $|\varepsilon| < \frac{1}{q}$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} &\leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\sum_{s=0}^{\infty} \frac{1}{(\tau+s)^2}\right)\right) \\ &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\int_0^{\infty} \frac{dx}{(\tau+x)^2} + \frac{1}{\tau^2}\right)\right) \\ &\leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2a^2 \frac{p}{\tau}\left(1 + \frac{1}{\tau}\right)\right) =: N_1(\varepsilon, a, 2p, q, \tau). \qquad \square\end{aligned}$$

Now, let us estimate the relation between $\|B(\varepsilon)(\lambda - T_r)^{-1}\|$ and $\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2}$.

Proposition 3.2. *We have*

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\|^2 \leq \sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2}.$$

We denote by $\Omega_{r,h}$ the intersection of S_h with the half-plane $\operatorname{Re}\lambda < r$, where

$$S_h := \{\lambda \in \mathbb{C} \text{ such that } |\operatorname{Im}\lambda| < h \text{ and } \operatorname{Re}\lambda > -h\} \text{ and } h > 3l$$

(see Fig. 2)

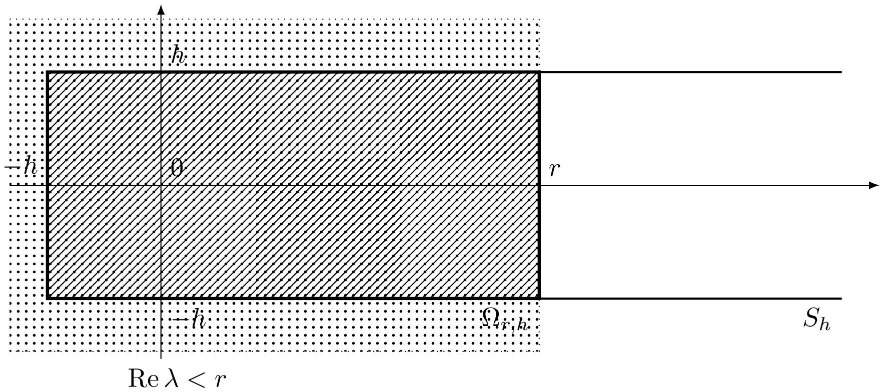

Fig. 2

Now, we are ready to prove the following result.

Proposition 3.3. *Assume that hypotheses* $(H1)$-$(H3)$ *are verified. Then, for* $|\varepsilon|$ *enough small, we have*

$$n(r, T(\varepsilon) - K_r) = n(r, T_0).$$

P r o o f. Since the interval $]r - 2l, r + 2l[$ does not contain any eigenvalue ν_n of T_r and T_r is positive, we have

$$\sigma(T_r) \cap \partial\Omega_{r,h} = \emptyset.$$

Now, let $\lambda \in \partial\Omega_{r,h}$ and $t \in [0,1]$. We have

$$\lambda - T_r - tB(\varepsilon) = [I - tB(\varepsilon)(\lambda - T_r)^{-1}](\lambda - T_r). \tag{3.19}$$

Due to Lemma 3.3 and Proposition 3.2, for $\operatorname{Re}\lambda \in [r - l, r + l]$ and $|\varepsilon| < \dfrac{1}{q}$ we have

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\|^2 < M(\varepsilon, a, 2p, q, l).$$

Then, it is easy to see that for $\operatorname{Re}\lambda \in [r - l, r + l]$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2\frac{l+1}{l^2}}}$ we obtain

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < 1. \tag{3.20}$$

Similarly, by Eq. (3.15) and Proposition 3.2, we get for $|\operatorname{Im}\lambda| \geq h$ and $|\varepsilon| < \dfrac{1}{q}$

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\|^2 < N(\varepsilon, a, 2p, q, h).$$

So, it is easy to check that for $|\operatorname{Im}\lambda| \geq h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 2a^2\frac{p}{h}(\pi + \frac{2}{h})}}$ we have

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < 1. \tag{3.21}$$

Furthermore, for $\operatorname{Re}\lambda \leq -h$ and $|\varepsilon| < \dfrac{1}{q}$, Eq. (3.16) and Proposition 3.2 imply that

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\|^2 < N_1(\varepsilon, a, 2p, q, h).$$

Hence, for $\operatorname{Re}\lambda \leq -h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 2a^2\frac{p}{h}(1 + \frac{1}{h})}}$ we obtain

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < 1. \tag{3.22}$$

Consequently, since $h > 3l$ Eqs (3.20), (3.21) and (3.22) yield for $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2\frac{l+1}{l^2}}}$

$$\|tB(\varepsilon)(\lambda - T_r)^{-1}\|_{\partial\Omega_{r,h}} \leq \|B(\varepsilon)(\lambda - T_r)^{-1}\|_{\partial\Omega_{r,h}} < 1.$$

Hence, $I - tB(\varepsilon)(\lambda - T_r)^{-1}$ is invertible with bounded inverse. So, Eq. (3.19) implies that $\lambda - T_r - tB(\varepsilon)$ is invertible with bounded inverse. Then,

$$\sigma(T_r + tB(\varepsilon)) \cap \partial\Omega_{r,h} = \emptyset.$$

On the other hand, T_r is self-adjoint with discrete spectrum, then due to [17, p. 146] T_r is with compact resolvent. Moreover, by Proposition 3.1 (*ii*) the operator $B(\varepsilon)$ is T_r-compact. Hence, Proposition 2.2 allows us to conclude that

$$n(r, T(\varepsilon) - K_r) = n(r, T_r).$$

In view of Remark 3.1, we obtain

$$n(r, T(\varepsilon) - K_r) = n(r, T_0). \qquad \square$$

3.2. The behavior of the spectrum of the perturbed operator $T(\varepsilon)$. In this part, we will study the comportment of the eigenvalues of $T(\varepsilon)$ in terms of the one of T_0. To this end, we shall emphasize on the fact that Lemma 2.3 and Proposition 3.3 imply that

$$n(r, T(\varepsilon)) = n(r, T_0) + \frac{1}{2\pi}[\arg D_\varepsilon(\lambda)]_{\partial\Omega_{r,h}},$$

where $D_\varepsilon(\lambda) := \det(I - K_\varepsilon(\lambda))$ and $K_\varepsilon(\lambda) := K_r(\lambda - T(\varepsilon) + K_r)^{-1}$.

So to prove Theorem 1.1, it suffices to estimate the increment of the argument of $D_\varepsilon(\lambda)$ along the boundary of $\Omega_{r,h}$.

To this interest, we shall need the following results enabling us to get such estimates.

Remark 3.2. $D_\varepsilon(\lambda)$ is well defined for $\lambda \notin \sigma(T(\varepsilon) - K_r)$. In fact, due to [15, p. 160], $K_\varepsilon(\lambda)$ is with finite rank for $\lambda \notin \sigma(T(\varepsilon) - K_r)$. Furthermore, $D_\varepsilon(\lambda)$ is a meromorphic function.

Proposition 3.4. *Suppose that hypotheses* $(H1)$–$(H3)$ *hold. Then:*
(i) *In the strip* $\operatorname{Re}\lambda \in [r-l, r+l]$ *and for* $|\varepsilon|$ *enough small, the function* $D_\varepsilon(\lambda)$ *is holomorphic and is estimated as*

$$|D_\varepsilon(\lambda)| \le \left(1 + 2\left(1 - \sqrt{M(\varepsilon, a, 2p, q, l)}\right)^{-1}\right)^{4lp}.$$

(ii) *In the same strip, for* $\operatorname{Im}\lambda = \pm h$ *and for* $|\varepsilon|$ *enough small, the following lower estimate holds*

$$|D_\varepsilon(\lambda)| \ge \left(1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}\right)^{4lp}.$$

Proof. (i) Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \in [r-l, r+l]$. Since $[r-l, r+l]$ does not contain any eigenvalue of T_r, then we have

$$\lambda - T_r - B(\varepsilon) = [I - B(\varepsilon)(\lambda - T_r)^{-1}](\lambda - T_r).$$

Furthermore, combining Lemma 3.3 and Proposition 3.2, we obtain for $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2\frac{l+1}{l^2}}}$

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < \sqrt{M(\varepsilon, a, 2p, q, l)} < 1.$$

Hence, $I - B(\varepsilon)(\lambda - T_r)^{-1}$ is invertible with bounded inverse. Consequently, for $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2\frac{l+1}{l^2}}}$ the operator $K_\varepsilon(\lambda)$ can be written as

$$K_\varepsilon(\lambda) := K_r(\lambda - T_r)^{-1}[I - B(\varepsilon)(\lambda - T_r)^{-1}]^{-1}. \tag{3.23}$$

Taking into account the fact that $D_\varepsilon(\lambda)$ is meromorphic and the strip $\operatorname{Re}\lambda \in [r-l, r+l]$ does not contain any eigenvalue of T_r, we get $D_\varepsilon(\lambda)$ is holomorphic in the strip $\operatorname{Re}\lambda \in [r-l, r+l]$.

Now, we are going to estimate $|D_\varepsilon(\lambda)|$ for $\operatorname{Re}\lambda \in [r-l, r+l]$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$.

Due to Proposition 2.1 (i), we have

$$|D_\varepsilon(\lambda)| \le (1 + \|K_\varepsilon(\lambda)\|)^{rk(K_\varepsilon)} \le (1 + \|K_\varepsilon(\lambda)\|)^{4lp}. \tag{3.24}$$

On the other hand, Eq. (3.23) implies that

$$\|K_\varepsilon(\lambda)\| \le \|K_r\| \; \|(\lambda - T_r)^{-1}\| \; \|[I - B(\varepsilon)(\lambda - T_r)^{-1}]^{-1}\|. \tag{3.25}$$

As

$$\|K_r\| \le 2l, \; \|(\lambda - T_r)^{-1}\| \le \frac{1}{d(\lambda, \sigma(T_r))} \le \frac{1}{l}$$

and

$$\left\| \left[I - B(\varepsilon)(\lambda - T_r)^{-1}\right]^{-1} \right\| \le \left(1 - \sqrt{M(\varepsilon, a, 2p, q, l)}\right)^{-1},$$

we obtain in view of Eqs (3.24) and (3.25)

$$|D_\varepsilon(\lambda)| \le \left(1 + 2\left(1 - \sqrt{M(\varepsilon, a, 2p, q, l)}\right)^{-1}\right)^{4lp}.$$

(ii) Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \in [r-l, r+l]$ and $\operatorname{Im}\lambda = \pm h$.

In view of Eq. (3.15) and Proposition 3.2, we have for $|\operatorname{Im}\lambda| \ge h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 2a^2 \frac{p}{h}(\pi + \frac{2}{h})}}$

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < \sqrt{N(\varepsilon, a, 2p, q, h)} < 1.$$

So, for $|\operatorname{Im}\lambda| \ge h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$ we get

$$\left\| \left[I - B(\varepsilon)(\lambda - T_r)^{-1}\right]^{-1} \right\| \le \left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}.$$

Moreover, we have

$$\|(\lambda - T_r)^{-1}\| \le \frac{1}{|\operatorname{Im}\lambda|} = \frac{1}{h} \text{ and } \|K_r\| \le 2l.$$

Hence, for $\operatorname{Im}\lambda = \pm h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$, Eq. (3.23) implies that

$$\|K_\varepsilon(\lambda)\| \le \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}. \tag{3.26}$$

Consequently, by Remark 2.1 we obtain for $\operatorname{Im}\lambda = \pm h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$

$$\left|D_\varepsilon(\lambda)\right| \ge \left|1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}\right|^{rk(K_\varepsilon(\lambda))}.$$

Since $h > 3l$, we have for $|\varepsilon| < W$

$$\frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1} < 1, \tag{3.27}$$

where $W := \min\left\{\dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}, \dfrac{1}{q + \frac{h}{h-2l}\sqrt{\xi + 2a^2 \frac{p}{h}(\pi + \frac{2}{h})}}\right\}$.

So, for $|\varepsilon| < W$ we obtain

$$\left|D_\varepsilon(\lambda)\right| \ge \left(1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}\right)^{4lp}. \qquad \square$$

A key tool in this section is the following proposition.

Proposition 3.5. *Assume that hypotheses* $(H1)$*–*$(H3)$ *are verified. Suppose that the line* $\operatorname{Re}\lambda = r$ *does not contain any eigenvalue of the operator* $T(\varepsilon)$*. Then, for* $|\varepsilon|$ *enough small the variation of the argument of the function* $D_\varepsilon(\lambda)$ *along the boundary of* $\Omega_{r,h}$ *satisfies the estimate*

$$\left|[\arg D_\varepsilon(\lambda)]_{\partial\Omega_{r,h}}\right| \le C_{1,\varepsilon},$$

where $C_{1,\varepsilon}$ *is a constant.*

Proof. • For $\operatorname{Re}\lambda \in [r - l, r + l]$, $\operatorname{Im}\lambda = \pm h$ and $|\varepsilon| < W$, Eqs (3.26) and (3.27) imply that

$$\|K_\varepsilon(\lambda)\| \le \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1} < 1. \tag{3.28}$$

• On the other hand, for $\operatorname{Re}\lambda = -h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$ it follows from

Eq. (3.16) and Proposition 3.2 that

$$\|B(\varepsilon)(\lambda - T_r)^{-1}\| < \sqrt{N_1(\varepsilon, a, 2p, q, h)} < 1 \text{ and } \|(\lambda - T_r)^{-1}\| < \frac{1}{h}.$$

So, for $\operatorname{Re}\lambda = -h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}$, Eq. (3.23) implies that

$$\|K_\varepsilon(\lambda)\| \leq \frac{2l}{h}\left(1 - \sqrt{N_1(\varepsilon, a, 2p, q, h)}\right)^{-1}.$$

As $h > 3l$, we have for $|\varepsilon| < W_1$

$$\frac{2l}{h}\left(1 - \sqrt{N_1(\varepsilon, a, 2p, q, h)}\right)^{-1} < 1, \tag{3.29}$$

where $W_1 := \min\left\{\dfrac{1}{q + \sqrt{\xi + 4pa^2 \frac{l+1}{l^2}}}, \dfrac{1}{q + \frac{h}{h-2l}\sqrt{\xi + 2a^2 \frac{p}{h}(1 + \frac{1}{h})}}\right\}.$

Hence, for $|\varepsilon| < \min\{W, W_1\} =: W$, Eqs (3.28) and (3.29) yield

$$\|K_\varepsilon(\lambda)\|_{\partial S_h} < 1.$$

Since $rk(K_\varepsilon(\lambda)) \leq 4lp$, then in virtue of Lemma 2.2 we obtain for $|\varepsilon| < W$

$$|[\arg\det(I - K_\varepsilon(\lambda))]_{\partial S_h}| \leq 4lp\pi. \tag{3.30}$$

To complete the proof, we apply Lemma 2.1. We consider the rectangle $\Gamma_{l,h}$ bounded by the straight lines $\operatorname{Re}\lambda = r \pm l$ and $\operatorname{Im}\lambda = \pm 2h$ and the second contracted rectangle $\Gamma'_{l,h}$ with the same center r and bounded by the lines $\operatorname{Re}\lambda = r \pm \dfrac{l}{2}$ and $\operatorname{Im}\lambda = \pm h$ (see Fig. 3).

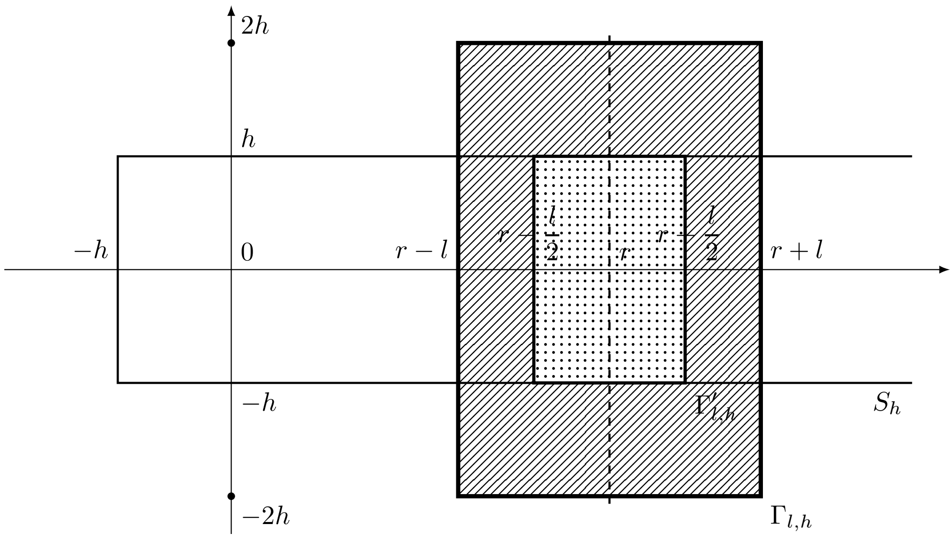

Fig. 3

In view of Proposition 3.4, we have $D_\varepsilon(\lambda)$ is a bounded and holomorphic function in the strip $\operatorname{Re}\lambda \in [r-l, r+l]$.

We denote by $m_1 = \sup\limits_{\lambda\in\Gamma_{l,h}} |D_\varepsilon(\lambda)|$ and $m_2 = \sup\limits_{\lambda\in\Gamma'_{l,h}} |D_\varepsilon(\lambda)|$.

Taking into account that the line $\operatorname{Re}\lambda = r$ does not contain any eigenvalue of $T(\varepsilon)$ (i.e., does not contain zeros of $D_\varepsilon(\lambda) := \det\big((\lambda - T(\varepsilon))(\lambda - T(\varepsilon) + K_r)^{-1}\big)$, we get in view of Lemma 2.1 (ii)

$$|[\arg D_\varepsilon(\lambda)]_{\operatorname{Re}\lambda=r}| \le C'(\ln m_1 - \ln m_2), \text{ where } C' > 0. \tag{3.31}$$

Hence, using Proposition 3.4, we obtain for $|\varepsilon| < W$

$$\begin{aligned}\ln m_1 - \ln m_2 \le \Bigg[&\ln\left(\left(1 + 2\left(1 - \sqrt{M(\varepsilon, a, 2p, q, l)}\right)^{-1}\right)^{4lp}\right) - \\ &\ln\left(\left(1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}\right)^{4lp}\right)\Bigg].\end{aligned} \tag{3.32}$$

So, Eqs (3.31) and (3.32) imply that for $|\varepsilon| < W$

$$\begin{aligned}|[\arg D_\varepsilon(\lambda)]_{\operatorname{Re}\lambda=r}| \le C'\Bigg[&\ln\left(\left(1 + 2\left(1 - \sqrt{M(\varepsilon, a, 2p, q, l)}\right)^{-1}\right)^{4lp}\right) - \\ &\ln\left(\left(1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon, a, 2p, q, h)}\right)^{-1}\right)^{4lp}\right)\Bigg].\end{aligned}$$

Then, for $|\varepsilon| < W$, we have

$$|[\arg D_\varepsilon(\lambda)]_{\mathrm{Re}\,\lambda=r}| \le C'_\varepsilon, \tag{3.33}$$

where

$$C'_\varepsilon := 4lp\ C' \ln\left(\frac{1+2\left(1-\sqrt{M(\varepsilon,a,2p,q,l)}\right)^{-1}}{1-\frac{2l}{h}\left(1-\sqrt{N(\varepsilon,a,2p,q,h)}\right)^{-1}}\right).$$

Consequently, for $|\varepsilon| < W$, Eqs (3.30) and (3.33) yield

$$|[\arg D_\varepsilon(\lambda)]_{\partial\Omega_{r,h}}| \le C_{1,\varepsilon},$$

where $C_{1,\varepsilon} := \max\{4lp\pi, C'_\varepsilon\}$. □

Using the results described above, we can now prove, by the similar way to [20, Theorem 1], the objective of this section.

Theorem 3.2. *Suppose that hypotheses* $(H1)$-$(H3)$ *are satisfied. Then, for* $|\varepsilon|$ *enough small, the spectrum of the operator* $T(\varepsilon)$ *is constituted by isolated eigenvalues satisfying*

$$n(r,T(\varepsilon)) = n(r,T_0) + O(1) \ \text{i.e.,}\ |n(r,T(\varepsilon)) - n(r,T_0)| < C \ \text{as}\ r \to \infty,$$

where C *is a constant.*

Remark 3.3. (i) Theorem 3.2 improves [8, Theorem 3.1] since we give an asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$ and we obtain results on the distribution of the eigenvalues of $T(\varepsilon)$ in terms of the one of T_0 where the eigenvalues $(\lambda_n)_n$ of T_0 are not necessarily with multiplicity one.

(ii) Theorem 3.2 generalizes the result of [20, Theorem 1] to an analytic operator instead of the sum of two operators.

P r o o f o f T h e o r e m 3.2. Due to Proposition 3.1, the operators $T(\varepsilon)$ and $T(\varepsilon) - K_r$ are with compact resolvent, for $|\varepsilon| < \dfrac{1}{q+\beta b}$. Further, making the same reasoning as the one in the proof of Proposition 3.3, we have for $|\varepsilon| < \dfrac{1}{q+\sqrt{\xi+4pa^2\frac{l+1}{l^2}}}$

$$\sigma(T(\varepsilon) - K_r) \cap \partial\Omega_{r,h} = \emptyset.$$

Consequently, for $|\varepsilon| < \dfrac{1}{q+\sqrt{\xi+4pa^2\frac{l+1}{l^2}}}$ Lemma 2.3 implies that

$$n(r,T(\varepsilon)) - n(r,T(\varepsilon)-K_r) = \frac{1}{2\pi}[\arg D_\varepsilon(\lambda)]_{\partial\Omega_{r,h}}.$$

Taking into account Proposition 3.3, we obtain for $|\varepsilon| < \dfrac{1}{q+\sqrt{\xi+4pa^2\frac{l+1}{l^2}}}$

$$n(r,T(\varepsilon)) - n(r,T_0) = \frac{1}{2\pi}[\arg D_\varepsilon(\lambda)]_{\partial\Omega_{r,h}}.$$

On the other hand, due to Proposition 3.5 we have $[\arg D_\varepsilon(\lambda)]$ is bounded for any r by $C_{1,\varepsilon}$ when λ varies along the boundary of $\Omega_{r,h}$, for $|\varepsilon| < W$. As a consequence, we get

$$|n(r,T(\varepsilon)) - n(r,T_0)| \leq \frac{1}{2\pi}C_{1,\varepsilon} = O(1) \text{ as } r\to\infty.$$

But, we obtain this result when we exclude the values of r for which the straight line $\operatorname{Re}\lambda = r$ passes through the zeros of $D_\varepsilon(\lambda)$. However, all such zeros lie in the rectangle $\Gamma'_{l,h}$ (see Fig. 4).

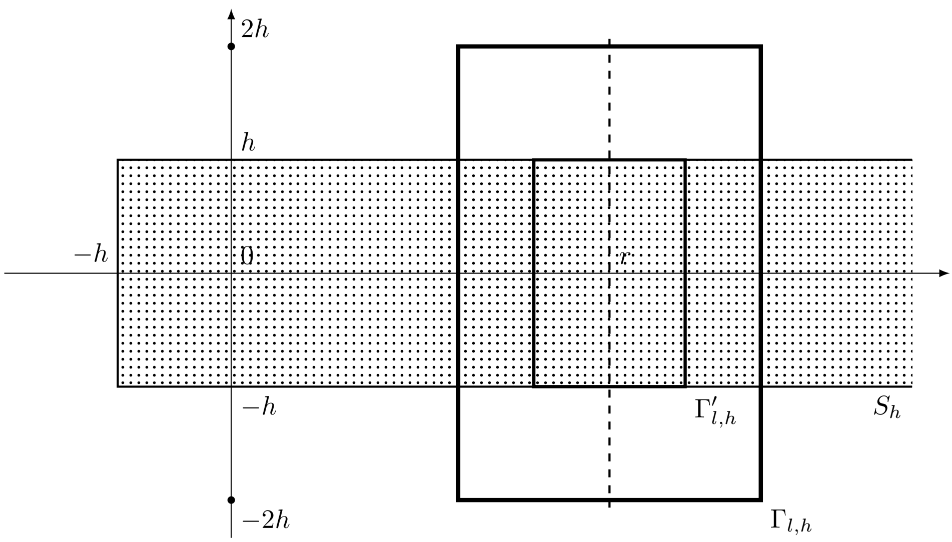

Fig. 4

Therefore, in view of Lemma 2.1 (i) we have for $|\varepsilon| < W$

$$n_{D_\varepsilon(\lambda)}(\Gamma'_{l,h}) \le C'' \left[\ln\left(\left(1 + 2\left(1 - \sqrt{M(\varepsilon,a,2p,q,l)}\right)^{-1}\right)^{4lp}\right) - \ln\left(\left(1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon,a,2p,q,h)}\right)^{-1}\right)^{4lp}\right)\right], \text{ where } C'' > 0.$$

So, for $|\varepsilon| < W$, we get

$$n_{D_\varepsilon(\lambda)}(\Gamma'_{l,h}) \le C''_\varepsilon,$$

where

$$C''_\varepsilon := 4lp\, C'' \ln \left(\frac{1 + 2\left(1 - \sqrt{M(\varepsilon,a,2p,q,l)}\right)^{-1}}{1 - \frac{2l}{h}\left(1 - \sqrt{N(\varepsilon,a,2p,q,h)}\right)^{-1}} \right).$$

Then, the number of zeros of $D_\varepsilon(\lambda)$ inside $\Gamma'_{l,h}$ does not exceed C''_ε (i.e., the number of eigenvalues of $T(\varepsilon)$ does not exceed C''_ε). Consequently, the relation remains valid for all $r \to \infty$, i.e., $n(r, T(\varepsilon)) - n(r, T_0) = O(1)$. □

Corollary 3.2. *Assume that hypotheses* $(H1)$ *and* $(H3)$ *and Eq.* (3.4) *hold. Then, for* $\beta \in \left]0, 1 + \frac{\alpha - 1}{2}\right[$ *and* $|\varepsilon|$ *enough small, the spectrum of the operator* $T(\varepsilon)$ *is constituted by isolated eigenvalues satisfying*

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1) \text{ i.e., } |n(r, T(\varepsilon)) - n(r, T_0)| < C \text{ as } r \to \infty,$$

where C *is a constant.*

Proof. Using $(H3)$ and making the same reasoning as the one developed in the proof of Proposition 3.1, we get for $|\varepsilon| < \frac{1}{q}$

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \nu_n|^2} \le \frac{|\varepsilon|^2}{(1 - |\varepsilon| q)^2} \left(\frac{a^2}{|\lambda - \nu_n|^2} + \frac{2ab\lambda_n^\beta}{|\lambda - \nu_n|^2} + \frac{b^2\lambda_n^{2\beta}}{|\lambda - \nu_n|^2} \right).$$

Now, let $\sigma = \operatorname{Re}\lambda$. Then there exists $k \in \mathbb{N}^*$ such that $\nu_{k-1} \le \sigma$ and $\nu_k > \sigma$. Further, since $\nu_n^{1-\alpha} \ge \frac{n-1}{2p} + \nu_1^{1-\alpha}$, thus $\nu_n \ge \left(\frac{n-1}{2p} + \nu_1^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$. Hence, Eq. (3.8) yields

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \nu_n|^2} < \frac{2c}{C_3^2} + \sum_{n<k-1} \frac{c}{\left(\left(\frac{k-2}{2p} + \nu_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\frac{n-1}{2p} + \nu_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}}\right)^2}$$

$$+\sum_{n>k} \frac{c}{\left(\left(\frac{n-1}{2p}+\nu_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}}-\left(\frac{k-1}{2p}+\nu_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}}\right)^2}.$$

Using an elementary calculations, we reveal that for $\beta \in]0, 1+\dfrac{\alpha-1}{2}[$ we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda-\nu_n|^2} < \infty.$$

We note here that we should distinguish the cases when $\beta \in]0, \alpha]$ and $\beta \in \left]\alpha, 1+\dfrac{\alpha-1}{2}\right[$. Further, we infer that

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{\beta}}{|\lambda-\nu_n|^2} < \infty.$$

Indeed, it is sufficient to replace β with $\dfrac{\beta}{2}$. Hence, the series $\sum_n \dfrac{2ab\lambda_n^{\beta}}{|\lambda-\nu_n|^2}$ and $\sum_n \dfrac{b^2\lambda_n^{2\beta}}{|\lambda-\nu_n|^2}$ are convergent. So, let ξ' be a positive constant such that

$$\sum_{n=1}^{\infty} \frac{2ab\lambda_n^{\beta}}{|\lambda-\nu_n|^2} + \sum_{n=1}^{\infty} \frac{b^2\lambda_n^{2\beta}}{|\lambda-\nu_n|^2} = \xi'.$$

Consequently, by a similar reasoning as the one of Theorem 3.2, we get for $|\varepsilon| < W_2$

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1),$$

where $W_2 := \min\left\{ \dfrac{1}{q+\sqrt{\xi' + 4pa^2\frac{l+1}{l^2}}}, \dfrac{1}{q+\frac{h}{h-2l}\sqrt{\xi'+2a^2\frac{p}{h}(\pi+\frac{2}{h})}} \right\}$. □

4. Applications.

4.1. Application to a Gribov operator in Bargmann space. To illustrate the applicability of the abstract results described above, we give an application to the Gribov operator whose expression is given by Eq. (1.5).

Let T_0 and H_1 be the operators defined by:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset E \longrightarrow E \\ \qquad\qquad\qquad \varphi \longrightarrow T_0\varphi(z) = (A^*A)^3\varphi(z) \\ \mathcal{D}(T_0) = \{\varphi \in E \text{ such that } T_0\varphi \in E\}, \end{cases}$$

and

$$\begin{cases} H_1 : \mathcal{D}(H_1) \subset E \longrightarrow E \\ \qquad\qquad\qquad\quad \varphi \longrightarrow H_1\varphi(z) = A^*(A+A^*)A\varphi(z) \\ \mathcal{D}(H_1) = \{\varphi \in E \text{ such that } H_1\varphi \in E\}. \end{cases}$$

Before going further, we state the following result from [8].

Proposition 4.1 ([8, Proposition 6.2])**.** *We have the following assertions:*
(i) T_0 *is a self-adjoint operator.*
(ii) The resolvent of T_0 *is compact.*
(iii) $\left\{e_n(z) = \dfrac{z^n}{\sqrt{n!}}\right\}_1^\infty$ *is a system of eigenvectors associated to the eigenvalues* $\{n^3\}_{n\geq 1}$ *of* T_0*.*

In view of Proposition 4.1 (i) and (ii), T_0 is a self-adjoint operator with compact resolvent. Then, let

$$T_0 = \sum_{n=1}^{\infty} n^3 \langle ., e_n\rangle e_n$$

be its spectral decomposition, where n^3 is the n^{th} eigenvalue of T_0 associated to the eigenvector $e_n(z) := \dfrac{z^n}{\sqrt{n!}}$. Hence, for a strictly decreasing sequence $(u_k)_{k\in\mathbb{N}}$ with strictly positive terms such that $u_0 = 1$ and $u_1 = \dfrac{1}{2}$, we define the operators $(T_0^{u_k})_{k\geq 0}$ by:

$$\begin{cases} T_0^{u_k} : \mathcal{D}(T_0^{u_k}) \subset E \longrightarrow E \\ \qquad\qquad\qquad\quad \varphi \longrightarrow T_0^{u_k}\varphi = \displaystyle\sum_{n=1}^{\infty} n^{3u_k}\langle \varphi, e_n\rangle e_n \\ \mathcal{D}(T_0^{u_k}) = \{\varphi \in E \text{ such that } \displaystyle\sum_{n=1}^{\infty} n^{6u_k}|\langle \varphi, e_n\rangle|^2 < \infty\}. \end{cases}$$

It is easy to check that for all $k \geq 0$, $\mathcal{D}(T_0^{u_k}) \subset \mathcal{D}(T_0^{u_{k+1}})$. Then,

$$\bigcap_{k\geq 2} \mathcal{D}(T_0^{u_k}) = \mathcal{D}(T_0^{u_2}).$$

Let $\mathcal{D} = \mathcal{D}(T_0^{u_2}) \cap \mathcal{D}(H_1)$, T_1, $(T_k)_{k\geq 2}$ the restrictions of H_1 and $T_0^{u_k}$ to $\mathcal{D}$, respectively. So, the operators $(T_k)_{k\geq 1}$ have the same domain $\mathcal{D}$ and we have $\mathcal{D}(T_0) \subset \mathcal{D}$.

In order to prove our theorem, we shall recall the following results.

Proposition 4.2 ([8, Proposition 6.3]). *There exist constants $a, b, q > 0$ and $\beta \in \left[\frac{1}{2}, 1\right]$ such that for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$ we have*

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}).$$

Remark 4.1. In Proposition 4.2, we take $q = 1$ and $a = b = 1 + 2\sqrt{2}$.

Proposition 4.3. *For $|\varepsilon| < 1$, the series $\sum_{k\geq 0} \varepsilon^k T_k \varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If we denote its sum by $T(\varepsilon)\varphi$, then we define a linear operator $T(\varepsilon)$ with domain $\mathcal{D}(T_0)$. Also, for $|\varepsilon| < \frac{1}{1+\beta a}$, the operator $T(\varepsilon)$ is closed.*

The objective of this subsection is formulated in the following theorem.

Theorem 4.1. *For $|\varepsilon|$ enough small and $\beta \in \left[\frac{1}{2}, \frac{5}{6}\right[$, we have*

$$n(r, T(\varepsilon)) - n(r, T_0) = O(1).$$

Proof. Let λ_n be the eigenvalue number n of $(A^*A)^3$ and $B(\varepsilon) = \sum_{k\geq 1} \varepsilon^k T_k$. We have

$$\lambda_{n+p}^{\frac{1}{3}} - \lambda_n^{\frac{1}{3}} = (n+p) - n = p \geq 1. \tag{4.1}$$

On the other hand, Proposition 4.2 implies that for all $k \geq 1$ we have

$$\|T_k\varphi\| \leq a\left(\|\varphi\| + \|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}\right), \text{ for all } \varphi \in \mathcal{D}(T_0) \tag{4.2}$$

where $a = 1 + 2\sqrt{2}$ and $\beta \in \left[\frac{1}{2}, \frac{5}{6}\right[$. Consequently, the result follows immediately from Corollary 3.2, Propositions 4.1 and 4.3 and Eqs (4.1) and (4.2). □

4.2. Application to a problem of radiation of a vibrating structure in a light fluid. We consider an elastic membrane lying in the domain $-L < x < L$ of the plane $z = 0$. The two half-spaces $z < 0$ and $z > 0$ are filled with gas. The membrane is excited by a harmonic force $F(x)e^{-i\omega t}$.

We denote by u the displacement of the membrane and p the acoustic pressure in the fluid. These functions satisfy the system:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right)u(x) = \frac{1}{D}(F(x) - P(x)) \text{ for all } x \in]-L, L[, \tag{4.3}$$

where

$$u(x) = \frac{\partial u(x)}{\partial x} = 0 \text{ for } x = -L \text{ and } x = L, \tag{4.4}$$

(4.5) $$P(x) = \lim_{\eta \to 0^+} (p(x,\eta) - p(x,-\eta))$$

and

(4.6) $p(x,z) =$

$$-\text{sgn}\ zi\frac{\rho_0}{2}\int_{-L}^{L} H_0(k\sqrt{(x-x')^2+z^2})\left(\omega^2 - \frac{D}{m}\left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}}\right)\right)u(x')dx',$$

for $z < 0$ or $z > 0$, where H_0 is the Hankel function of the first kind and order 0 stated in [14, p. 11] and given by

$$H_0(x) = \frac{-1}{\pi}\, e^{i\left(x-\frac{\pi}{4}\right)}\int_0^{+\infty} \frac{e^{-xt}}{\sqrt{t\left(1+i\frac{t}{2}\right)}}\, dt$$

and the mechanical parameters of the membrane are E the Young modulus, ν the Poisson ratio, m the surface density, h the thickness of the membrane and $D := \dfrac{Eh^3}{12(1-\nu^2)}$ the rigidity. The fluid is characterized by ρ_0 the density, c the sound speed and $k := \dfrac{\omega}{c}$ the wave number.

The systems (4.3), (4.4), (4.5) and (4.6) lead to the boundary value problem

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right)u(x) - i\rho_0\int_{-L}^{L} H_0(k|x-x'|)\left(\frac{\omega^2}{D} - \frac{1}{m}\left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}}\right)\right)u(x')dx' = \frac{F(x)}{D},$$

for all $x \in]-L, L[$ such that $u(x) = \dfrac{\partial u(x)}{\partial x} = 0$ for $x = -L$ and $x = L$.

In order to study this problem, we consider the following operator

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset L^2\big(]-L,L[\big) \longrightarrow L^2\big(]-L,L[\big) \\ \qquad\qquad\qquad \varphi \longrightarrow T_0\varphi(x) = \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(T_0) = H_0^2\big(]-L,L[\big) \cap H^4\big(]-L,L[\big). \end{cases}$$

Now, we recall the following result from [14].

Lemma 4.1 ([14, Lemmas 3.1 and 3.2])**.** *The following assertions hold:*
(i) T_0 *is a self-adjoint operator.*
(ii) The injection from $\mathcal{D}(T_0)$ *into* $L^2\big(]-L,L[\big)$ *is compact.*
(iii) The spectrum of T_0 *is constituted only of eigenvalues which are positive, countable and of which the multiplicity is one and which have no finite limit*

points and satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty.$$

Further,

$$\left(\frac{(2n+1)\pi}{4L}\right)^4 \leq \lambda_n \leq \left(\frac{(2n+3)\pi}{4L}\right)^4,$$

i.e.,

$$\lambda_n \sim_{+\infty} \left(\frac{n\pi}{2L}\right)^4.$$

Remark 4.2. Due to Lemma 4.1, T_0 is a self-adjoint operator and has a compact resolvent. Then, let

$$T_0\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle \varphi_n$$

be its spectral decomposition, where λ_n is the n^{th} eigenvalue of T_0 associated to the eigenvector $\varphi_n(x) = \mu e^{\sqrt[4]{\lambda_n}x} + \eta e^{-\sqrt[4]{\lambda_n}x} + \theta e^{i\sqrt[4]{\lambda_n}x} + \delta e^{-i\sqrt[4]{\lambda_n}x}$ (see [14, p. 7]).

Hence, for $\gamma > 0$, we define the operator T_0^γ by

$$\begin{cases} T_0^\gamma : \mathcal{D}(T_0^\gamma) \subset L^2(]-L,L[) \longrightarrow L^2(]-L,L[) \\ \qquad\qquad \varphi \longrightarrow T_0^\gamma\varphi(x) = \sum_{n=1}^{\infty} \lambda_n^\gamma \langle \varphi, \varphi_n \rangle \varphi_n \\ \mathcal{D}(T_0^\gamma) = \left\{ \varphi \in L^2(]-L,L[) \text{ such that } \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |\langle \varphi, \varphi_n \rangle|^2 < \infty \right\}. \end{cases}$$

In the sequel, we consider the following operators:

$$\begin{cases} B = T_0^{\frac{1}{2}} : \mathcal{D}(B) \subset L^2(]-L,L[) \longrightarrow L^2(]-L,L[) \\ \qquad\qquad \varphi \longrightarrow B\varphi(x) = \left(\frac{d^4\varphi}{dx^4}\right)^{\frac{1}{2}} \\ \mathcal{D}(B) = \left\{ \varphi \in L^2(]-L,L[) \text{ such that } \sum_{n=1}^{\infty} \lambda_n |\langle \varphi, \varphi_n \rangle|^2 < \infty \right\} \end{cases}$$

and

$$\begin{cases} K : L^2(]-L,L[) \longrightarrow L^2(]-L,L[) \\ \qquad\qquad \varphi \longrightarrow K\varphi(x) = \frac{i}{2} \int_{-L}^{L} H_0(k|x-x'|)\varphi(x')dx' \end{cases}$$

and the following eigenvalue problem:

Find the values $\lambda \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2(]-L,L[) \cap H^4(]-L,L[)$, $\varphi \neq 0$ for the equation

$$T_0\varphi + \varepsilon K(T_0 - B)\varphi = \lambda(I + \varepsilon K)\varphi \tag{4.7}$$

where $\lambda = \dfrac{m\omega^2}{D}$ and $\varepsilon = \dfrac{2\rho_0}{m}$.

Due to [18, chapter 9, section 4], λ is the eigenvalue and φ is the eigenmode.

Note that λ and φ each depend on the value of ε. So, we designate by $\lambda := \lambda(\varepsilon)$ and $\varphi := \varphi(\varepsilon)$.

For $|\varepsilon| < \dfrac{1}{\|K\|}$, the operator $I + \varepsilon K$ is invertible. Then, the problem (4.7) becomes:

Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2(]-L,L[) \cap H^4(]-L,L[)$, $\varphi \neq 0$ for the equation

$$(I + \varepsilon K)^{-1}T_0 + \varepsilon(I + \varepsilon K)^{-1}K(T_0 - B)\varphi = \lambda(\varepsilon)\varphi. \tag{4.8}$$

The problem (4.8) is equivalent to:

Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2(]-L,L[) \cap H^4(]-L,L[)$, $\varphi \neq 0$ for the equation

$$\left(\frac{d^4}{dx^4} - \varepsilon K\left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}} + \varepsilon^2K^2\left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}} + \cdots + (-1)^n\varepsilon^nK^n\left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}} + \cdots\right)\varphi$$
$$= \lambda(\varepsilon)\varphi.$$

We denote by $T_n := (-1)^nK^n\left(\dfrac{d^4}{dx^4}\right)^{\frac{1}{2}}$, for all $n \geq 1$.

To state the objective of this subsection, we need first to introduce some results.

Proposition 4.4 ([7, Proposition 4.1]). *There exist positive constants $a, b, q > 0$ and $\beta \in \left[\dfrac{1}{2}, 1\right]$ such that for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$*

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}).$$

Remark 4.3. It suffices to take $a = b = q = \|K\|$.

Proposition 4.5 ([7, Proposition 4.1]). *For $|\varepsilon| < \dfrac{1}{\|K\|}$, the series $\sum_{k\geq 0}\varepsilon^kT_k\varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If we designate its sum by $T(\varepsilon)\varphi$, we* *define a linear operator* $T(\varepsilon)$ *with domain* $\mathcal{D}(T_0)$. *For* $|\varepsilon| < \dfrac{1}{\|K\|(1+\beta)}$, *the operator* $T(\varepsilon)$ *is closed.*

The main result of this subsection is formulated as:

Theorem 4.2. *For* $|\varepsilon|$ *enough small and* $\beta \in \left[\dfrac{1}{2}, \dfrac{7}{8}\right[$, *we have*

$$n(r, T(\varepsilon)) - n(r, T_0) = O(1).$$

P r o o f. Let λ_n be the eigenvalue number n of T_0 and $B(\varepsilon) = \sum_{k\geq 1}^{\infty} \varepsilon^k T_k$. It follows from Lemma 4.1 that

$$\frac{(2n+1)\pi}{4L} \leq \sqrt[4]{\lambda_n} \leq \frac{(2n+3)\pi}{4L}.$$

Hence, using simple calculations, we reveal that

$$\lambda_{n+p}^{\frac{1}{4}} - \lambda_n^{\frac{1}{4}} = \frac{p\pi}{2L} \geq 1, \text{ where } p \geq \frac{2L}{\pi}.$$

Further, it follows from Proposition 4.4 that for all $k \geq 1$ and for $\beta \in \left[\dfrac{1}{2}, \dfrac{7}{8}\right[$ we have

$$\|T_k \varphi\| \leq \|K\|^k \left(\|\varphi\| + \|T_0 \varphi\|^\beta \|\varphi\|^{1-\beta} \right), \text{ for all } \varphi \in \mathcal{D}(T_0).$$

Then, for $|\varepsilon|$ enough small it follows from Corollary 3.2, Remark 4.2, Lemma 4.1 and Proposition 4.5 that

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1).$$

□

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