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CI-PROPERTY FOR CYCLOTOMIC S-RINGS OVER $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$

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Abstract. An S-ring $\mathcal{A}$ over a group H is called cyclotomic if it is a transitive module of a group $K \leq \mathrm{Aut}(H)$. In this paper we prove that every cyclotomic S-ring over $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$ is a CI-S-ring where p, q, r are pairwise different primes.

1. Introduction. Let H be a finite group and S be a subset of H. The Cayley digraph $\mathrm{Cay}(H, S)$ is the digraph that has a vertex set H, and an arc set $\{(x, y) : y \cdot x^{-1} \in S\}$. It follows from the definition that $\mathrm{Cay}(H, S)$ is loopless if the identity element $1 \notin S$, and it is regarded as an undirected graph when S is an inverse-closed set, i.e., $S = S^{-1} = \{x^{-1} : x \in S\}$.

Two Cayley digraphs $\mathrm{Cay}(H, S)$ and $\mathrm{Cay}(H, T)$ are called Cayley isomorphic if $T = S^{\phi}$ for some automorphism $\phi \in \mathrm{Aut}(H)$. It is trivial to show that Cayley isomorphic Cayley digraphs are isomorphic as digraphs. The converse, however, does not hold in general. A subset $S \subseteq H$ is called a CI-subset if

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for any $T \subseteq H$, the isomorphism $\mathrm{Cay}(H,T) \cong \mathrm{Cay}(H,S)$ implies that $T = S^{\phi}$ for some $\phi \in \mathrm{Aut}(H)$. The group H is a DCI-group if each of its subsets is a CI-subset, and a CI-group if each of its inverse-closed subsets are CI-subsets.

Motivated by a problem posed by Adam [1], Babai and Frankl [2] began to investigate arbitrary DCI-groups and asked for a complete classification of DCI-groups. During the last few years this problem was intensively studied by Nowitz, Li, Conder, Praeger, Xu, Meng and Palfy [5, 6].

In order to finish the classification of DCI-groups one has to answer two basic questions: Which groups are DCI-groups and when a coprime product of two DCI-groups is a DCI-group. The first question was answered affirmatively for many groups: $\mathbb{Z}_{2p}$ (Babai) [3], $\mathbb{Z}_n : (n, \varphi(n)) = 1$ (Li-Palfy) [4], $\mathbb{Z}_p^e : e \leq 3$ (Dobson) [5], $\mathbb{Z}_p^4$ (Hirasaka-Muzychuk) [10], $\mathbb{Z}_p^2 \times \mathbb{Z}_p$ (I. Kovacs and M. Muzychuk) [11], $\mathbb{Z}_p^5$ (Yan Feng and I. Kovacs 2017) [9], $\mathbb{Z}_k$ and $\mathbb{Z}_{2k}$ and $\mathbb{Z}_{4k}$ where k is square-free odd (Muzychuk) [7]. The proofs of the fact that the group $H = \mathbb{Z}_p^n$ is a DCI-group for $n \in \{4, 5\}$ and odd prime p are based on the method of S-rings. In fact, in these proofs it was checked that every Schurian S-ring over H is a CI-S-ring. Due to the result of Hirasaka and Muzychuk, this is sufficient for the proof that H is a DCI-group.

An S-ring $\mathcal{A}$ over a group H is called cyclotomic S-ring if it is a transitive module of a group $K \leq \mathrm{Aut}(H)$. In this paper we prove the following:

Theorem 1.1. *Every cyclotomic S-ring over* $(\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r$ *is a CI-S-ring where* p, q, r *are pairwise different primes.*

The text of the paper is organized in the following way. Section 2 contains a background of S-rings. In Section 3 we prove Theorem 1.1.

2. Schur rings Let $\mathbb{Z}H$ be the integer group ring of the group H. Denote the identity element of H by 1. If $T \subseteq H$ then denote the sum $\sum_{x \in H} a_x \underline{x}$ by $\underline{T}$, with: $a_x = 1$ if $x \in T$, and $a_x = 0$ otherwise. The set $\{t^{-1} : t \in T\}$ is denoted by T^{-1}. A subring $\mathcal{A} \subseteq \mathbb{Z}H$ is called an S-ring over H if there exists a partition $\delta(\mathcal{A})$ of H such that:

1. $\{1\} \in \delta(\mathcal{A})$;
2. If $T \in \delta(\mathcal{A})$ then $T^{-1} \in \delta(\mathcal{A})$;
3. $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{T} : T \in \delta(\mathcal{A})\}$.

The elements of $\delta(\mathcal{A})$ are called the basic sets of $\mathcal{A}$ and denoted by $\mathrm{Bsets}(\mathcal{A})$, and the number $|\delta(\mathcal{A})|$ is called the rank of $\mathcal{A}$. An S-ring $\mathcal{A}' \subseteq \mathbb{Z}H$ is called an S-subring of $\mathcal{A}$, if every element $z \in \mathcal{A}'$ is equal to sum of elements from $\mathcal{A}$.

Let $\mathcal{A}$ be an S-ring over a group H. Following Tamaschke [14], a subgroup $F \leq H$ for which $\underline{F} \in \mathcal{A}$ is called an $\mathcal{A}$-subgroup. There are two trivial $\mathcal{A}$-subgroups: $\{1\}$ and H. If $\mathrm{Bsets}(\mathcal{A}) = \{\{1\}, H\backslash\{1\}\}$ then $\mathcal{A}$ is called the trivial S-ring over H.

For F being an $\mathcal{A}$-subgroup, define $\mathcal{A}_F := \mathcal{A} \cap \mathbb{Z}F$. It is easy to check that $\mathcal{A}_F$ is an S-ring over the group F and that $\mathrm{Bsets}(\mathcal{A}_F) = \{T \in \mathrm{Bsets}(\mathcal{A}) : \ T \subset F\}$. Such S-rings $\mathcal{A}_F$ are called induced S-subrings of $\mathcal{A}$. If F is an $\mathcal{A}$-subgroup which is normal in H then the natural homomorphism $\pi : H \to H/F$ can be canonically extended to a homomorphism $\mathbb{Z}H \to \mathbb{Z}H/F$ which we shall also denote by π. We introduce the following notation: $T/F := \pi(T) = \{\pi(t) : t \in T\}$ for $T \subset H$, $\mathcal{A}/F := \pi(\mathcal{A}) = \{\pi(x) : x \in \mathcal{A}\}$. We call $\mathcal{A}/F$ a quotient S-ring (over the factor group H/F), and from [14], $\mathcal{A}/F$ is an S-ring over H/F with basic sets given by $\mathrm{Bsets}(\mathcal{A}/F) = \{T/F : t \in \mathrm{Bsets}(\mathcal{A})\}$.

The thin radical of an S-ring $\mathcal{A}$ is defined by the set $O_\theta(\mathcal{A}) = \{h \in H : \{h\} \in \mathrm{Bsets}(\mathcal{A})\}$. It is easy to see that $O_\theta(\mathcal{A})$ is an $\mathcal{A}$-subgroup.

Let G be a subgroup of $\mathrm{Sym}(H)$ containing the group of right translations H_R. Let G_1 stand for the stabilizer of 1 in G and $Orb(H, G_1)$ stand for the set of all orbits of G_1 on H. Schur proved that the $\mathbb{Z}$-submodule $\mathcal{V}(H, G_1) = \mathrm{Span}_{\mathbb{Z}}\{T : T \in \mathrm{Orb}(H, G_1)\}$, is an S-ring over H [15]. An S-ring $\mathcal{A}$ over H is called Schurian if $\mathcal{A} = \mathcal{V}(H, G_1)$ for some G such that $H_R \leq G \leq \mathrm{Sym}(H)$. It should be mentioned that not every S-ring is Schurian. The first example of nonschurian S-ring was found by Wielandt in [15].

An S-ring $\mathcal{A}$ over H is said to be cyclotomic if $\mathrm{Bsets}(\mathcal{A}) = \mathrm{Orb}(K, H)$ for $K \leq \mathrm{Aut}(H)$. In this case we write $\mathcal{A} = \mathrm{Cyc}(K, H)$ and it is easy to see, $\mathcal{A} = \mathcal{V}(K \cdot H_R, H)$. So every cyclotomic S-ring is Schurian.

We say that an S-ring $\mathcal{A}$ over a group H is a p-S-ring if H is a p-group, and all basic sets $T \in \mathrm{Bsets}(\mathcal{A})$ have p-power size. Following [10], if $\mathcal{A}$ is a p-S-ring over elementary abelian group $\mathbb{Z}_p^n$ then we have: If $n = 1$ then $\mathcal{A} = \mathbb{Z}H$; if $n = 2$ then $\mathcal{A} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$ or $\mathcal{A} = \mathbb{Z}C_p^2$. Every p-S-ring over $\mathbb{Z}_p^n$ where $n = 1, 2, 3$ is cyclotomic.

2.1. Isomorphisms of S-rings. Denote by $\mathrm{Iso}(\mathcal{A})$ the set of all isomorphisms $f \in \mathrm{Sym}(H)$ from $\mathcal{A}$ to S-rings over H, and let $\mathrm{Iso}_1(\mathcal{A}) = \{f \in \mathrm{Iso}(\mathcal{A}) : 1^f = 1\}$. Note that, $\mathrm{Iso}(\mathcal{A}) \subseteq \mathrm{Sym}(H)$, but it is not necessarily a subgroup. It follows from the definition that for any $f \in \mathrm{Aut}(\mathcal{A})$ and $g \in \mathrm{Aut}(H)$, their product fg is an isomorphism from $\mathcal{A}$ to an S-ring over H. Therefore, $\mathrm{Aut}(\mathcal{A}) \cdot Aut(H) \subseteq \mathrm{Iso}(\mathcal{A})$. Now, we say that $\mathcal{A}$ is a CI-S-ring, if $\mathrm{Iso}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \cdot \mathrm{Aut}(H)$. This definition was given by Hirasaka and Muzychuk in [10] where the following theorem is proved.

Theorem 2.1. *Let H be an abellian group, then H is DCI-group if and only if every Schurian S-ring over H is a CI-S-ring.*

If $K_1,\ K_2 \leq \mathrm{Aut}(H)$, then K_1 and K_2 are called Cayley equivalent if $\mathrm{Orb}(K_1, H) = \mathrm{Orb}(K_2, H)$, and then we write $K_1 \underset{\mathrm{Cay}}{\approx} K_2$. If $\mathcal{A} = \mathrm{Cyc}(K, H)$ for some $K \leq \mathrm{Aut}(H)$ with $H_R \leq K$, then $\mathrm{Aut}_H(\mathcal{A})$ is the largest group which is Cayley equivalent to K. So A cyclcotomic S-ring $\mathcal{A}$ over H is called Cayley minimal if

$$\{K \leq \mathrm{Aut}(H) : K \underset{\mathrm{Cay}}{\approx} \mathrm{Aut}_H(\mathcal{A})\} = \{\mathrm{Aut}_H(\mathcal{A})\}$$

It is easy to see that the trivial S-ring $\mathbb{Z}H$ is Cayley minimal, and every cyclotomic S-ring over $\mathbb{Z}_n$ is Cayley minimal. On other hand If $n \in \{p, p \cdot q, p^2 \cdot q\}$, then every cyclotomic S-ring $\mathcal{A}$ over $\mathbb{Z}_n$ is CI-S-ring. In this case, $\mathrm{Aut}(\mathcal{A}) \trianglelefteq \mathrm{Aut}(\mathbb{Z}_n)$.

Definition 2.2. *Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $E \leq F$ and E is normal in H. Then $\mathcal{A}$ is a generalized wreath product (or F/E-wreath product), $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$ if for every $T \in \mathrm{Bsets}(\mathcal{A})$ we have that $T \subseteq F$ or T is a union of E-cosets. And $\mathcal{A}$ is non-trivial generalized wreath product if $E \neq 1$ and $F \neq H$. In this case $S = F/E$ is called $\mathcal{A}$-section.*

Let $\mathcal{A}$ is be an S-ring over a group H. Then $\mathcal{A}$ is called a decomposable S-ring if it is the nontrivial E/F-wreath product for some $\mathcal{A}$-subgroups E, F in H.

Now, let H and H' be finite groups. For a bijection $f : H \to H'$ and a set $X \subseteq H$, the induced bijection from X onto X^f is denoted by f^X. For a set $\triangle \subseteq \mathrm{Sym}(H)$ and a section S of H we set $\triangle^S = \{f^S : f \in \triangle, S^f = S\}$. If $\mathcal{A}$ is an S-ring over H then we define $\mathrm{Aut}_H(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(H)$. If S is an $\mathcal{A}$-section then, $\mathrm{Aut}_H(\mathcal{A})^S \leq \mathrm{Aut}_S(\mathcal{A}_S)$. Here we have the next theorem.

Theorem 2.3. *[8] Let H be an elementary abelian group and let $\mathcal{A}$ be an S-ring over H, such that $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$ for subgroups E, F of H. Suppose*

$\mathcal{A}_F$ and $\mathcal{A}_{H/E}$ are CI-S-rings and

$$\mathrm{Aut}_F((\mathcal{A}_F)^{F/E}) = \mathrm{Aut}_{F/E}((\mathcal{A}_{F/E})) = \mathrm{Aut}_{H/E}((\mathcal{A}_{H/E})^{F/E}),$$

then $\mathcal{A}$ is a CI-S-ring.

This theorem can be applied to an abelian group such that its Sylow subgroups are elementary abelian. Therefore, let H be an abelian group such that the Sylow subgroups of H are elementary abelian, and suppose $\mathcal{A}$ be an S-ring over H, such that $\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$, and $\mathcal{A}_F$, $\mathcal{A}_{H/E}$ are CI-S-rings. Then we have

Lemma 2.4. *If $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$ then $\mathcal{A}$ is a CI-S ring.*

Proof. Since $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$ then $\mathrm{Aut}(\mathcal{A}_{F/E})$ is trivial. But $\mathrm{Aut}_F((A_F)^{F/E}) \leq \mathrm{Aut}_{F/E}((\mathcal{A}_{F/E}))$. So $\mathrm{Aut}_F((A_F)^{F/E}) = \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. By Theorem 2.3, $\mathcal{A}$ is CI-S-ring. □

Lemma 2.5. *If $\mathcal{A}_{F/E}$ is Cayley minimal then $\mathcal{A}$ is a CI-S ring.*

Proof. We have $\mathrm{Aut}_F(\mathcal{A}_F)^{F/E} \leq \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. So, If $\mathcal{A}_{F/E}$ is Cayley minimal, then $\mathrm{Aut}_F(\mathcal{A}_F)^{F/E} = \mathrm{Aut}_{F/E}(\mathcal{A}_{F/E})$. Therefore, by Theorem 2.3, $\mathcal{A}$ is a CI-S-ring. □

Lemma 2.6. *If $\mathcal{A}_{F/E} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$ then $\mathcal{A}$ is a CI-S ring.*

Proof. If $\mathcal{A}_{F/E} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$, then $\mathcal{A}_{F/E}$ is cyclotomic, and $|O_\theta(\mathcal{A}_{F/E})| = p$. By Proposition 4.3 of [8],we have $\mathcal{A}_{F/E}$ is Cayley minimal and by Lemma 2.5, $\mathcal{A}$ is a CI-S ring. □

3. S-rings over $\boldsymbol{H = P \times Q}$ where $\boldsymbol{P}$ is an abelian group and $\boldsymbol{Q = \mathbb{Z}_q}$. Let $\mathcal{A}$ be an S-ring over a group $H = P \times Q$ where P is an abelian group and $Q = \mathbb{Z}_q$ such that $q \nmid |P|$. Then we have the following:

Lemma 3.1 ([13, Lemma 6.2]). *Let P_1 be the maximal $\mathcal{A}$-group contained in P. Suppose that $P_1 \neq P$. Then one of the following statements holds:*

1. $\mathcal{A} = \mathcal{A}_{P_1} \wr \mathcal{A}_{H/P_1}$, *where* $\mathrm{rank}(\mathcal{A}_{H/P_1}) = 2$.

2. *$\mathcal{A}$ is an F/E-wreath product where $F = P_1E$ and $Q \leq E < H$.*

Lemma 3.2 ([12]). *Let $\mathcal{A}$ be a non-trivial S-ring over H. Assume that P_1 is the maximal $\mathcal{A}$-subgroup contained in P, while Q_1 is the minimal $\mathcal{A}$-subgroup which contains Q. If $\mathcal{A}/P_1$ has rank two or $\mathcal{A}/P_1 = \mathbb{Z}C_q$ then $\mathcal{A} = \mathcal{A}_{P_1} \star \mathcal{A}_{Q_1}$.*

4. Proof of Theorem 1.1. Let $H = (\mathbb{Z}_p)^2 \times \mathbb{Z}_q \times \mathbb{Z}_r = P \times Q$ where $Q = \mathbb{Z}_d$ with $d \in \{q, r\}$. We can see that every proper subgroup of H is a DCI-group, and so every S-ring over a proper subgroup of H is a CI-S-ring. Suppose $\mathcal{A} = \mathcal{V}(H, K)$ where $K \leq \mathrm{Aut}(H)$ and P_1 is the maximal $\mathcal{A}$-subgroup contained in P. If $P_1 \neq P$ then by Lemma 3.1, we have two cases: either $\mathcal{A} = \mathcal{A}_{P_1} \wr \mathcal{A}_{H/P_1}$, where $\mathrm{rank}(\mathcal{A}_{H/P_1}) = 2$, and in this case $\mathcal{A}$ is a CI-S-ring. Or $\mathcal{A}$ is an E/F-wreath product for the $\mathcal{A}$-subgroups F,E of H. Then $|E/F| = 1, p, q, r$ or p^2 or pq , pr , qr. So we have:

Case(1): If E/F is of prime order, then $\mathcal{A}_{F/E} = \mathbb{Z}(F/E)$ and, by Lemma 2.4, $\mathcal{A}$ is a CI-S-ring. Or $|E/F| = p^2$ and $\mathcal{A}_{F/E} = \mathbb{Z}C_p \wr \mathbb{Z}C_p$. In this case $\mathcal{A}_{F/E}$ is Cayley minimal and by Lemma 2.5 $\mathcal{A}$ is a CI-S-ring.

Case(2): If $|E/F| = pq$, pr or qr, then in this case $\mathcal{A}_S$ is a cyclotomic S-ring, so $\mathrm{Aut}_S(\mathcal{A}_S) = \mathrm{Aut}(\mathcal{A}_S) \leq \mathrm{Aut}(H)$. Therefore $|\,\mathrm{Aut}(\mathcal{A}_S)| \,|\, (x-1)(y-1)$, where $x \neq y \in \{p, q, r\}$. By the structure of the Schur rings over a group of order $x \cdot y$, we see that $\mathcal{A}_S = \mathcal{V}(S, \mathrm{Aut}(\mathcal{A}_S))$ with $|\,\mathrm{Aut}(\mathcal{A}_S)| \,|\, (x-1)(y-1)$ is Cayly minimal, because $\mathrm{Aut}(\mathcal{A}_S) \leq \{x \longmapsto ax : a \in \mathbb{Z}^*_{x\cdot y}, x \in \mathbb{Z}_{x\cdot y}\}$. By Lemma 2.5 $\mathcal{A}$ is CI-S-ring.

If $|E/F| = 1$ then $\mathcal{A} = \mathcal{A}_1 \wr \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are S-rings over proper subgroups of H. Since every proper subgroup of H is a CI-group, so $\mathcal{A}_1$ and $\mathcal{A}_2$ are CI-S-rings.

Assume now that $P_1 = P$. Then by Lemma 3.2, we have that $\mathcal{A}/P_1$ has rank two or $\mathcal{A}/P_1 = \mathbb{Z}C_d$, with $d \in \{q, r\}$, and in both cases $\mathcal{A} = \mathcal{A}_{P_1} \star \mathcal{A}_{Q_1}$ where Q_1 is the minimal $\mathcal{A}$-subgroup which contains $\mathbb{Z}_d$. Therefore, $\mathcal{A}_{P_1}$, and $\mathcal{A}_{Q_1}$ are CI-S-rings and so $\mathcal{A}$ is a CI-S-ring.

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