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# $*$-POLYNOMIAL IDENTITIES FOR THE BLOCK UPPER TRIANGULAR MATRIX ALGEBRA $UT_2(UT_2(F))$ WITH THE TRANSPOSE-LIKE INVOLUTION*

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Abstract. Let $UT_4(F)$ be $4 \times 4$ upper triangular matrix algebra over $F$ a field of characteristic zero and let $\mathcal{A}$ be the subalgebra of $UT_4(F)$ linearly generated by $\{\mathbf{e}_{ij} : 1 \leq i \leq j \leq 4\} \setminus \mathbf{e}_{23}$, where $\mathbf{e}_{ij}$, $1 \leq i \leq j \leq 4$ is the standard basis of $UT_4(F)$. We describe the set of all $*$-polynomial identities for $\mathcal{A}$ with the transpose-like involution.

**1. Introduction.** Let $F$ be a field of $\mathrm{char}(F) \neq 2$ (characteristic different from 2). Let $\mathcal{R}$ be an unitary associative algebra over $F$. A map $* : \mathcal{R} \to \mathcal{R}$ is called an involution if it is an automorphism of the additive group $\mathcal{R}$ such that

$$(ab)^* = b^* a^* \quad \text{and} \quad (a^*)^* = a$$

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for all $a, b \in \mathcal{R}$. Let $Z(\mathcal{R})$ be the center of $\mathcal{R}$. If $a^* = a$ for all $a \in Z(\mathcal{R})$, we say that $*$ is an involution of the first kind on $\mathcal{R}$. Otherwise $*$ is called an involution of the second kind. In this paper, we will consider involutions of the first kind only.

The description of the involutions on a given algebra is an important task in ring theory. In the algebra $UT_k(F)$ of the $k \times k$ upper triangular matrix over F, we have an important involution of the first kind. For every matrix $A \in UT_k(F)$ define $A^* = JA^tJ$ where $A \mapsto A^t$ denotes the usual matrix transposition and $J$ is the following permutation matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

If $k$ is an odd integer, any other involution in $UT_k(F)$ is completely determined by $*$. The involution $*$ is called the *transpose-like involution* on $UT_k(F)$. For instance, if $k = 2$ we have that

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}$$

for all $a, b, c \in F$. When $k$ is even integer, there exist two classes of inequivalent involutions. One of them is the same transpose-like involution and the other is defined by $A^s = DA^*D$ for all $A \in UT_k(F)$. Here, $D$ is the matrix

$$\begin{pmatrix} \mathbf{I}_{k/2} & 0 \\ 0 & -\mathbf{I}_{k/2} \end{pmatrix}$$

and $\mathbf{I}_{k/2}$ is the identity matrix of the full matrix algebra $M_{k/2}(F)$. The involution $s$ is called the *symplectic involution* on $UT_k(F)$. Details about this result can be found in [1] and various others properties of involutions and involution-like maps for the upper triangular matrix can be found in [5].

Let $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$ be two disjoint countably infinite sets. Denote by $F\langle Y \cup Z\rangle$ the free unitary associative algebra freely generated by $Y \cup Z$. The elements of $F\langle Y \cup Z\rangle$ are polynomials in the associative non-commutative variables $Y \cup Z$ with scalars in $F$. Let $\mathcal{R}$ be an unitary associative algebra with involution $*$, we set

$$\mathcal{R}^+ = \{a \in \mathcal{R} : \ a^* = a\} \qquad \text{and} \qquad \mathcal{R}^- = \{a \in \mathcal{R} : \ a^* = -a\}.$$

The elements in $\mathcal{R}^+$ are called *symmetric* and elements in $\mathcal{R}^-$ are called *skew-symmetric*. A given polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is a $*$-polynomial identity for $\mathcal{R}$ if

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$$

for all $a_1, \ldots, a_n \in \mathcal{R}^+$ and all $b_1, \ldots, b_m \in \mathcal{R}^-$. Denote by $Id(\mathcal{R}, *)$ the set of all $*$-polynomial identities for $\mathcal{R}$. There is a description of $Id(UT_2(F), *)$ and $Id(UT_2(F), s)$ (see [1] when $F$ is infinite and [4] when $F$ is finite). It has also been described $Id(UT_3(F), *)$ when $F$ is a field of $\mathrm{char}(F) = 0$ (see [1]). It is an open problem to describe $Id(UT_k(F), *)$ in other cases.

Let $\mathcal{A}$ be the subset of $UT_4(F)$ consisting of all the elements

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

such that $A, B, C \in UT_2(F)$. It is easy to see that $\mathcal{A}$ is a subalgebra with the transpose-like involution. By definition, we have that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix}$$

for all $A, B, C \in UT_2(F)$. In this paper, we describe the set of all $*$-polynomial identities for $\mathcal{A}$ with transpose-like involution.

**2. Preliminaries.** Since $\mathrm{char}\, F \neq 2$, $UT_4(F)$ can be written as $UT_4(F) = UT_4(F)^+ \oplus UT_4(F)^-$. The center of the $UT_4(F)$ consists of the scalar matrices $\{\lambda \mathbf{I}_4 : \lambda \in F\}$. Given an algebra $\mathcal{R}$, the commutators in $\mathcal{R}$ are defined inductively by

$$[a_1, a_2] = a_1a_2 - a_2a_1 \qquad \text{and} \qquad [a_1, \ldots, a_n] = [[a_1, \ldots, a_{n-1}], a_n]$$

for all $a_1, \ldots, a_n \in \mathcal{R}$. Besides, $\mathcal{R}$ satisfies the Jacobi identity

$$[a_3, a_2, a_1] + [a_2, a_1, a_3] + [a_1, a_3, a_2] = 0$$

for all $a_1, a_2, a_3 \in \mathcal{R}$.

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set. Denote by $F\langle X\rangle$ the free unitary associative algebra over $F$, freely generated by $X$. We say that $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a *polynomial identity* for $\mathcal{R}$ if

$$f(a_1, \ldots, a_n) = 0$$

for all $a_1, \ldots, a_n \in \mathcal{R}$. Denote by $Id(\mathcal{R})$ the set of all polynomial identities for $\mathcal{R}$. A $T$-ideal of $F\langle X\rangle$ is an ideal closed under all algebra endomorphisms of $F\langle X\rangle$. In other words, an ideal $I$ is a $T$-ideal of $F\langle X\rangle$ if and only if

$$f(g_1, \ldots, g_n) \in I$$

for all $f(x_1, \ldots, x_n) \in I$ and $g_1, \ldots, g_n \in F\langle X\rangle$.

The following result is well-known.

**Lemma 2.1** ([2, Theorem 5.2.1]). *A linear basis for*

$$F\langle X\rangle / Id(UT_2(F))$$

*is given by the elements*

$$x_1^{r_1} \cdots x_m^{r_m} [x_{j_1}, \ldots, x_{j_n}]^t + Id(UT_2(F)),$$

*where* $r_1, \ldots, r_m \geq 0$, $t \in \{0, 1\}$ *and* $j_1 > j_2 \leq \cdots \leq j_n$.

The free algebra $F\langle Y \cup Z\rangle$, $Y = \{y_1, y_2, \ldots\}$, $Z = \{z_1, z_2, \ldots\}$, has an involution, which we denote by $*$ as well, satisfying $y_i^* = y_i$ and $z_i^* = -z_i$ for all $i \geq 1$. Endowed with this involution, $F\langle Y \cup Z\rangle$ is a free algebra of countably infinite rank in the class of unitary associative algebras with involution. The elements of $Y$ are called *symmetric variables* and the elements of $Z$ are called *skew-symmetric variables*. An endomorphism $\varphi$ of $F\langle Y \cup Z\rangle$ preserves involution if $\varphi(f^*) = (\varphi(f))^*$ for all $f \in F\langle Y \cup Z\rangle$. An ideal $I$ of $F\langle Y \cup Z\rangle$ is a $*$-ideal if $f \in I$ implies $f^* \in I$. A $T(*)$-ideal of $F\langle Y \cup Z\rangle$ is a $*$-ideal closed under all endomorphisms of $F\langle Y \cup Z\rangle$ which preserve the involution. In other words, a $*$-ideal $I$ is a $T(*)$-ideal if and only if

$$f(g_1, \ldots, g_n, h_1, \ldots, h_m) \in I$$

for all $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in I$, $g_1, \ldots, g_n \in F\langle Y \cup Z\rangle^+$ and $h_1, \ldots, h_m \in F\langle Y \cup Z\rangle^-$. Then $Id(\mathcal{R}, *)$ is a $T(*)$-ideal of $F\langle Y \cup Z\rangle$.

A polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is called $Y$*-proper* if $f$ is a linear combination of polynomials

$$z_1^{r_1} \cdots z_m^{r_m} c_1 \cdots c_t$$

where $r_1, \ldots, r_m \geq 0$ and $c_1, \ldots, c_t$ are commutators in the variables $Y \cup Z$ ($c_0 = 1$ if $t = 0$). Denote by $B$ the vector space of all $Y$-proper polynomials.

Every polynomial $f(y_1,\dots,y_n,z_1,\dots,z_m)\in F\langle Y\cup Z\rangle$ is a linear combination of polynomials

$$(1)\qquad y_1^{s_1}\cdots y_n^{s_n}f_{(s_1,\dots,s_n)},$$

where $s_1,\dots,s_n\geq 0$ and $f_{(s_1,\dots,s_n)}\in B$. When $\operatorname{char}F=0$ every $T(*)$-ideal is generated by its $Y$-proper multilinear ones (see [3, Lemma 2.1]). From now on, we only consider a field of characteristic zero ($\operatorname{char}F=0$).

Consider the following order on the variables $Y\cup Z$:

$$z_1<z_2<\cdots<z_m<\cdots<y_1<y_2<\cdots<y_n<\cdots.$$

We will use the next lemma for obtaining ordered commutators.

**Lemma 2.2.** *Let $u=[u_1,\dots,u_n]$ be a multilinear commutator in the variables $u_1,\dots,u_n\in Y\cup Z$. Then $u=v+v'$, where $v$ is a linear combination of multilinear commutators of length $n$ and $v'$ is a linear combination of multilinear products of at least two commutators in the variables $u_1,\dots,u_n$ of total length $n$. All commutators participating in $v$ and $v'$ are of the form*

$$[u_{j_1},\dots,u_{j_n}]$$

*with $u_{j_1}>u_{j_2}<\cdots<u_{j_n}$.*

Proof. See [2, Theorem 5.2.1]. □

Denote by $\mathcal{B}$ the subalgebra of $UT_4(F)$ consisting of all the matrices

$$\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}$$

such that $A,B\in UT_2(F)$. Obviously,

$$(2)\qquad \mathrm{Id}(UT_2(F))=\mathrm{Id}(\mathcal{B})$$

The next proposition describes the $*$-polynomial identities of $\mathcal{B}$.

**Proposition 2.3.** *A linear basis for $B/(B\cap Id(\mathcal{B},*))$ is given by its elements of type*

$$(3)\qquad z_{i_1}\cdots z_{i_m}[u_{j_1},\dots,u_{j_s}]^t+Id(\mathcal{B},*),$$

*where $i_1\leq\cdots\leq i_m$, $m\geq 0$, $t\in\{0,1\}$, $u_{j_1}>u_{j_2}\leq\cdots\leq u_{j_s}$ and $u_{j_1},\dots,u_{j_s}\in Y\cup Z$.*

Proof. By (2) we have that the product of two commutators in $F\langle Y \cup Z\rangle$ is contained in $Id(\mathcal{B},*)$. Next, by Lemma 2.2, we have that each $Y$-proper polynomial, modulo $Id(\mathcal{B},*)$, is a linear combination of elements in (3).

Let $f(z_1,\dots,z_m,y_1,\dots,y_n) \in F\langle Y \cup Z\rangle$ be a linear combination of elements in (3) such that $f \in \mathrm{Id}(\mathcal{B},*)$. Set

$$x_1 = z_1,\dots,x_m = z_m, x_{m+1} = y_1,\dots,x_{m+n} = y_n,$$

$f$ can be written in the form:

$$f = \sum_{r,j} \alpha_{r,j} z_1^{r_1}\cdots z_m^{r_m}[x_{j_1},\dots,x_{j_s}] + \sum_r \alpha_r z_1^{r_1}\cdots z_m^{r_m},$$

where $\alpha_{r,j},\alpha_r \in F$, $r = (r_1,\dots,r_m)$, $j = (j_1,\dots,j_s)$, $j_1 > j_2 \leq \cdots \leq j_s$. Let $A_1,\dots,A_n,B_1,\dots,B_m \in UT_2(F)$, then

$$Y_i = \begin{pmatrix} A_i & 0 \\ 0 & A_i^* \end{pmatrix} \in \mathcal{B}^+ \text{ and } Z_k = \begin{pmatrix} B_k & 0 \\ 0 & -B_k^* \end{pmatrix} \in \mathcal{B}^-.$$

By substituting in $f$, we have that $f(Z_1,\dots,Z_m,Y_1,\dots,Y_n)$

$$= \begin{pmatrix} f(B_1,\dots,B_m,A_1,\dots,A_n) & 0 \\ 0 & f(-B_1^*,\dots,-B_m^*,A_1^*,\dots,A_n^*) \end{pmatrix} = 0.$$

Thus, it follows that $f(B_1,\dots,B_m,A_1,\dots,A_n) = 0$. Since $A_i$ and $B_k$ are arbitrary, we have that $f(x_1,\dots,x_m,x_{m+1},\dots,x_{m+n}) \in Id(UT_2(F))$ seen as element of $F\langle X\rangle$. By Lemma 2.1 the proof is complete. □

**3. $*$-identities for $\mathcal{A}$.** Let $A_1,A_2,A_3 \in UT_2(F)$. It is easy to check that $[A_1,A_2](A_3-A_3^*) = [A_1,A_2,A_3]$ and that if $C \in UT_2(F)^+$ then $A_1C-CA_1^* = \lambda(A_1 - A_1^*)$ where $\lambda = 2^{-1}tr(C)$ (here $tr(C)$ is the sum of the diagonal elements of $C$).

If $Y \in \mathcal{A}^+$, then

$$Y = \begin{pmatrix} A & C \\ 0 & A^* \end{pmatrix}$$

for some $A, C \in UT_2(F)$ with $C^* = C$. If $Z \in \mathcal{A}^-$ then

$$Z = \begin{pmatrix} B & D \\ 0 & -B^* \end{pmatrix}$$

for some $B, D \in UT_2(F)$ with $D^* = -D$.

Denote by $\mathbf{e}_{ij}$ the element of $M_2(F)$ with exactly one nonzero entry at the ($i$-row and $j$-column), which is 1. We will prove some facts about $\mathcal{A}$.

**Lemma 3.1.** *Let $P_i$, $1 \le i \le 6$, be arbitrary elements of $\mathcal{A}$. Then*

(*i*) *The elements*

$$[P_1, P_2][P_3, P_4], \quad [P_1, P_2]P_3[P_4, P_5] \quad \textit{and} \quad [P_1, P_2][P_3, P_4]P_5$$

*are of the type* $\begin{pmatrix} 0 & \alpha\mathbf{e}_{12} \\ 0 & 0 \end{pmatrix}$, *for some $\alpha \in F$.*

(*ii*) $[P_1, P_2][P_3, P_4][P_5, P_6] = 0$.

Proof. (*i*) The elements $[P_1, P_2]$ and $[P_1, P_2]P_3$ are matrices of the type

$$\begin{pmatrix} \alpha\mathbf{e}_{12} & \Theta \\ 0 & \beta\mathbf{e}_{12} \end{pmatrix} \tag{4}$$

where $\alpha, \beta \in F$, $\Theta \in UT_2(F)$. The product of two matrices of type (4) is

$$\begin{pmatrix} \alpha_1\mathbf{e}_{12} & \Theta_1 \\ 0 & \beta_1\mathbf{e}_{12} \end{pmatrix} \begin{pmatrix} \alpha_2\mathbf{e}_{12} & \Theta_2 \\ 0 & \beta_2\mathbf{e}_{12} \end{pmatrix} = \begin{pmatrix} 0 & \Theta \\ 0 & 0 \end{pmatrix},$$

where $\Theta = \alpha_1\mathbf{e}_{12}\Theta_2 + \Theta_1\beta_2\mathbf{e}_{12} = \gamma\mathbf{e}_{12}$, for some $\gamma \in F$.
(*ii*) The product of three matrices of type (4) is

$$\begin{pmatrix} 0 & \gamma\mathbf{e}_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_3\mathbf{e}_{12} & \Theta_3 \\ 0 & \beta_3\mathbf{e}_{12} \end{pmatrix} = 0.$$

And this completes the proof. □

**Lemma 3.2.** *Let $Y \in \mathcal{A}^+$. Then, there is $\alpha \in F$ such that*

$$Q_1 Y Q_2 = \alpha Q_1 Q_2$$

*for all $Q_1, Q_2 \in \mathcal{A}$ such that*

$$Q_i = \begin{pmatrix} \alpha_i\mathbf{e}_{12} & \Theta_i \\ 0 & \beta_i\mathbf{e}_{12} \end{pmatrix}$$

*and $\alpha_i, \beta_i \in F$, $\Theta_i \in UT_2(F)$, $i = 1, 2$.*

Proof. Let $Y = \begin{pmatrix} A & C \\ 0 & A^* \end{pmatrix}$ where $A = a\mathbf{e}_{11} + b\mathbf{e}_{22} + c\mathbf{e}_{12}$ and $C \in UT_2(F)$. We can write $Y = Y' + Y''$ where

$$Y' = \begin{pmatrix} a\mathbf{e}_{11} + c\mathbf{e}_{12} & C \\ 0 & a\mathbf{e}_{22} + c\mathbf{e}_{12} \end{pmatrix} \quad \text{and} \quad Y'' = \begin{pmatrix} b\mathbf{e}_{22} & 0 \\ 0 & b\mathbf{e}_{11} \end{pmatrix}.$$

We have that $Q_1Y' = \begin{pmatrix} 0 & \Theta \\ 0 & \beta_1 a\mathbf{e}_{12} \end{pmatrix}$ with $\Theta$ satisfying $\Theta = \Theta\mathbf{e}_{22}$. Thus, $Q_1Y'Q_2 = 0$. It is easy to verify that $Q_1Y''Q_2 = bQ_1Q_2$. Thus, the proof is complete. □

**Proposition 3.3.** *Let $v_1, v_2, v_3$ be commutators in the variables $Y \cup Z$ and let $u \in Y \cup Z$. Then, the following polynomials*

$$v_1v_2v_3, \qquad v_1uv_2 - (v_1uv_2)^* \qquad \text{and} \qquad v_1v_2u - (v_1v_2u)^*$$

*are $*$-polynomial identities for $\mathcal{A}$. In particular, $v_1v_2 - v_2^*v_1^* \in Id(\mathcal{A}, *)$.*

Proof. The first polynomial is a $*$-identity for $\mathcal{A}$ by item $(ii)$ of the Lemma 3.1. For the other polynomials, let $P_i \in \mathcal{A}^+ \cup \mathcal{A}^-$, then by item $(i)$ of the Lemma 3.1 we obtain

$$([P_1, P_2]P_3[P_4, P_5])^* = [P_1, P_2]P_3[P_4, P_5],$$

$$([P_1, P_2][P_3, P_4]P_5)^* = [P_1, P_2][P_3, P_4]P_5.$$

This completes this proof. □

**Proposition 3.4.** *The polynomial*

$$[y_4, y_3][y_2, y_1] + [y_3, y_2][y_4, y_1] + [y_2, y_4][y_3, y_1]$$

*is $*$-polynomial identity for $\mathcal{A}$.*

Proof. Let $Y_i = \begin{pmatrix} A_i & C_i \\ 0 & A_i^* \end{pmatrix} \in \mathcal{A}^+$, $1 \leq i \leq 4$. We have that

$$[Y_j, Y_i] = \begin{pmatrix} [A_j, A_i] & \Lambda_{ji} \\ 0 & -[A_j, A_i] \end{pmatrix}$$

where $\Lambda_{ji} = A_jC_i + C_jA_i^* - A_iC_j - C_iA_j^* = \lambda_i(A_j - A_j^*) - \lambda_j(A_i - A_i^*)$ for some $\lambda_i \in F$. Thus,

$$[Y_4, Y_3][Y_2, Y_1] = \begin{pmatrix} 0 & \Theta_2 \\ 0 & 0 \end{pmatrix},$$

where $\Theta_2 = [A_4, A_3]\Lambda_{21} - \Lambda_{43}[A_2, A_1]$. After some manipulations, we obtain

$$\Theta_2 = \lambda_1[A_4, A_3, A_2] + \lambda_2[A_3, A_4, A_1] + \lambda_3[A_2, A_1, A_4] + \lambda_4[A_1, A_2, A_3].$$

Set

$$[Y_2, Y_4][Y_3, Y_1] = \begin{pmatrix} 0 & \Theta_3 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad [Y_3, Y_2][Y_4, Y_1] = \begin{pmatrix} 0 & \Theta_4 \\ 0 & 0 \end{pmatrix}.$$

By the Jacobi identity we obtain that $\Theta_2 + \Theta_3 + \Theta_4 = 0$. □

Let $f(y_1, y_2, y_3, w_1, \ldots, w_m) \in F\langle Y \cup Z\rangle$ where $w_1 \cdots w_m \in Y$ and let

$$\mathbf{Ja}_{(y_1,y_2,y_3)} f$$

denote the polynomial

$$f(y_1, y_2, y_3, w_1, \ldots, w_m) + f(y_2, y_3, y_1, w_1, \ldots, w_m) + f(y_3, y_1, y_2, w_1, \ldots, w_m).$$

**Corollary 3.5.** *Let $w_1, w_2 \in Y$. The polynomial*

$$\mathbf{Ja}_{(y_1,y_2,y_3)}[y_1, y_2, w_1][y_3, w_2]$$

*is a $*$-polynomial identity for $\mathcal{A}$.*

P r o o f. First, observe that $[y_i, y_j, w_1] = [y_i, y_j]w_1 - w_1[y_i, y_j]$. Thus, we have that

$$\begin{aligned}&\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2, w_1][y_3, w_2])\\ &= \mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2]w_1[y_3, w_2]) - w_1\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2][y_3, w_2]).\end{aligned}$$

Let $Y_1, Y_2, Y_3, W_1, W_2$, be elements of $\mathcal{A}^+$. By Lemma 3.2 there exists $\alpha \in F$ such that

$$[Y_i, Y_j]W_1[Y_k, W_2] = \alpha[Y_i, Y_j][Y_k, W_2]$$

for all $i, j, k \in \{1, 2, 3\}$. Therefore

$$\mathbf{Ja}_{(y_1,y_2,y_3)}([Y_1, Y_2]W_1[Y_3, W_2]) = \alpha\mathbf{Ja}_{(y_1,y_2,y_3)}([Y_1, Y_2][Y_3, W_2]).$$

By Proposition 3.4 we obtain that $\mathbf{Ja}_{(y_1,y_2,y_3)}([Y_1, Y_2][Y_3, W_2]) = 0$. Thus, the proof is complete. □

**Definition 3.6.** *Let $I$ denote the $T(*)$-ideal of $F\langle Y \cup Z\rangle$ generated as $T(*)$-ideal by*

$$\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2][y_3, y_4]), \quad \mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2, y_4][y_3, y_5])$$

*and by the set of polynomials*

$$v_1v_2v_3, \qquad v_1uv_2 - v_2^*u^*v_1^* \quad \textit{and} \quad v_1v_2u - u^*v_2^*v_1^*$$

*where $v_1, v_2, v_3$ are commutators in the variables $Y \cup Z$ and $u \in Y \cup Z$.*

By Proposition 3.3, Proposition 3.4 and Corollary 3.5 we have that

$$I \subseteq Id(\mathcal{A}, *). \tag{5}$$

**Lemma 3.7.** *For every commutators $v_1, v_2$ in the variables $Y \cup Z$, there exists $\alpha \in F$ such that $2z_1v_1v_2 + [v_1, z_1]v_2 + \alpha[v_2, z_1]v_1 \in I$.*

P r o o f. It is easy to verify that, given a commutator $v$, either $v^* = v$ or $v^* = -v$. Thus, there exists $\alpha \in F$ such that $v_2^* v_1^* = \alpha v_2 v_1$, $v_1v_2 - \alpha v_2 v_1 + I = I$ and $v_1 z_1 v_2 + \alpha v_2 z_1 v_1 + I = I$. Therefore, $z_1v_1v_2 + v_1z_1v_2 + \alpha[v_2, z_1]v_1 + I = I$. The proof is complete. □

**Lemma 3.8.** *Let $f$ be a polynomial of $F\langle Y \cup Z\rangle$ and let $v_1, v_2, v_3$ be commutators in the variables $Y \cup Z$. Then $v_1 f v_2 v_3 \in I$.*

P r o o f. Let $u \in Y \cup Z$. Then,

$$v_1 u v_2 v_3 + I = u v_1 v_2 v_3 + [v_1, u] v_2 v_3 + I = I.$$

Since $f = f^+ + f^-$, the proof is complete. □

**Proposition 3.9.** *Let $m \geq 1$. Let $w_1, \ldots, w_{m+1}$ be symmetric variables. Then,*

$$\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2, w_1, \ldots, w_m][y_3, w_{m+1}]) \in I.$$

P r o o f. We proceed by induction. The equality $a[b, c] = [b, ac] - [b, a]c$ is satisfied by the elements of $F\langle Y \cup Z\rangle$. First observe that

$$[y_1, y_2][y_3, w_1w_2] = w_1[y_1, y_2][y_3, w_2] + [y_1, y_2, w_1][y_3, w_2] + [y_1, y_2][y_3, w_2]w_2.$$

Then, by the definition of $I$, we have that $\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2][y_3, w_1w_2]) \in I$.

Suppose that $m \geq 2$ and that

$$\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2, w_1, \ldots, w_{k-1}][y_3, w_k w_{m+1}]) \in I$$

and

$$\mathbf{Ja}_{(y_1,y_2,y_3)}([y_1, y_2, w_1, \ldots, w_k][y_3, w_{m+1}]) \in I$$

for all $1 \leq k < m$. Observe that

$$\begin{aligned}
[y_1, y_2, w_1, \ldots, w_{m-1}][y_3, w_m w_{m+1}] &= [y_1, y_2, w_1, \ldots, w_{m-2}][y_3, w_{m-1} w_m w_{m+1}] \\
&\quad - [y_1, y_2, w_1, \ldots, w_{m-2}][y_3, w_{m-1}] w_m w_{m+1} \\
&\quad - w_{m-1}[y_1, y_2, w_1, \ldots, w_{m-2}][y_3, w_m w_{m+1}].
\end{aligned}$$

Then, by induction we have that

$$\begin{aligned}\mathbf{Ja}_{(y_1,y_2,y_3)}&[y_1,y_2,w_1,\dots,w_{m-1}][y_3,w_mw_{m+1}]+I\\&=\mathbf{Ja}_{(y_1,y_2,y_3)}[y_1,y_2,w_1,\dots,w_{m-2}][y_3,w_{m-1}w_mw_{m+1}]+I.\end{aligned}$$

Write $w_{m-1}w_m = 2^{-1}(w+z)$ where $w = w_{m-1}w_m + w_{m-1}w_m$ and $z = [w_{m-1}, w_m]$, then

$$[y_3, w_{m-1}w_mw_{m+1}] = 2^{-1}([y_3, ww_{m+1}] + z[y_3, w_{m+1}] + [y_3, z]w_{m+1}).$$

Since $w_{m-1}w_m + w_{m-1}w_m \in F\langle Y \cup Z\rangle^+$ and any product of three commutators lies in $I$, we have that

$$\begin{aligned}\mathbf{Ja}_{(y_1,y_2,y_3)}&[y_1,y_2,w_1,\dots,w_{m-2}][y_3,w_{m-1}w_mw_{m+1}]+I\\&=\mathbf{Ja}_{(y_1,y_2,y_3)}[y_1,y_2,w_1,\dots,w_{m-2}][y_3,z]w_{m+1}+I=I.\end{aligned}$$

Finally, since

$$\begin{aligned}[y_1,y_2,w_1,\dots,w_m][y_3,w_{m+1}] =&[y_1,y_2,w_1,\dots,w_{m-1}][y_3,w_mw_{m+1}]\\&-[y_1,y_2,w_1,\dots,w_{m-1}][y_3,w_m]w_{m+1}\\&-w_m[y_1,y_2,w_1,\dots,w_{m-1}][y_3,w_{m+1}]\end{aligned}$$

then

$$\mathbf{Ja}_{(y_1,y_2,y_3)}[y_1,y_2,w_1,\dots,w_m][y_3,w_{m+1}] \in I.$$ □

**Proposition 3.10.** *Let $\delta, \gamma, \varepsilon$ be integers such that $\delta \geq 0$, $\gamma, \varepsilon \geq 2$. Let $u_i \in Y \cup Z$ where $i \in \mathbb{N}$ and*

$$f = z_{r_1} \cdots z_{r_\delta}[u_{s_1}, \dots, u_{s_\gamma}][u_{t_1}, \dots, u_{t_\varepsilon}].$$

*Then, $f + I$ is a linear combination of elements the type*

$$u_{i_1} \cdots u_{i_m}[u_{j_1}, \dots, u_{j_n}][u_{k_1}, u_{k_2}] + I,$$

*where*

1) *$m \geq 0$, $u_{k_2} < u_{k_1}$, $u_{j_1} > u_{j_2} \leq \cdots \leq u_{j_n}$ and $u_{i_1} \leq \cdots \leq u_{i_m}$.*

2) *$u_{k_2} \leq u_{i_1}, u_{j_2}$.*

3) *Every $u_{i_1} \cdots u_{i_m}[u_{j_1}, \dots, u_{j_n}][u_{k_1}, u_{k_2}]$ has the same multidegree as $f$.*

P r o o f. Let

$$u_{k_2} = \min\{z_{r_1}, \ldots, z_{r_\delta}, u_{s_1}, \ldots, u_{s_\gamma}, u_{t_1}, \ldots, u_{t_\varepsilon}\}.$$

We have two cases:

**Case 1.** $\delta = 0$. In this case $f = [u_{s_1}, \ldots, u_{s_\gamma}][u_{t_1}, \ldots, u_{t_\varepsilon}]$. By definition of $I$, we can suppose that $u_{k_2}$ it is in the second commutator of $f$. In addition, by Lemma 2.2, we can suppose that $k_2 = t_2$ and $u_{t_1} > u_{t_2} \leq u_{t_3} \leq \cdots \leq u_{t_\varepsilon}$.Write

$$[u_{t_1}, \ldots, u_{t_\varepsilon}] = \sum_{m_1, m_2} m_1[u_{k_1}, u_{k_2}]m_2,$$

where, $m_1, m_2$ are monomials such that $m_1[u_{k_1}, u_{k_2}]m_2$ has the same multidegree as $[u_{t_1}, \ldots, u_{t_\varepsilon}]$. Now, for every $m_1$, we have that $[u_{s_1}, \ldots, u_{s_\gamma}]m_1$ is a linear combination of elements of the type $u_{i_1} \cdots u_{i_l}[u_{j_1}, \ldots, u_{j_n}]$ with the same multidegree as $[u_{s_1}, \ldots, u_{s_\gamma}]m_1$. Then, $f + I$ is linear combination of elements of the type

$$u_{i_1} \cdots u_{i_l}[u_{j_1}, \ldots, u_{j_n}][u_{k_1}, u_{k_2}]m_2 + I.$$

By definition of $I$ again, $f + I$ is linear combination of elements of the type

$$u_{i_1} \cdots u_{i_m}[u_{j_1}, \ldots, u_{j_n}][u_{k_1}, u_{k_2}] + I,$$

where each $u_{i_1} \cdots u_{i_m}[u_{j_1}, \ldots, u_{j_n}][u_{k_1}, u_{k_2}]$ has the same multidegree as $f$. Finally, by Lemma 2.2 we can suppose that $u_{j_1} > u_{j_2} \leq \cdots \leq u_{j_n}$ and by Lemma 3.8 we can suppose that $u_{i_1} \leq \cdots \leq u_{i_m}$.

**Case 2.** $\delta > 0$. In this case $u_{k_2} = z_r$ for some $r$. By Lemma 3.7 and Lemma 3.8 we can suppose that $u_{k_2}$ it is in

$$[u_{s_1}, \ldots, u_{s_\gamma}][u_{t_1}, \ldots, u_{t_\varepsilon}].$$

In fact, if $r = r_1$ then

$$f + I = uz_r v_1 v_2 + I = \alpha u[v_2, z_r]v_1 + \beta u[v_1, z_r]v_2 + I,$$

where $u = z_{r_2} \cdots z_{r_\delta}$, $v_1 = [u_{s_1}, \ldots, u_{s_\gamma}]$, $v_2 = [u_{t_1}, \ldots, u_{t_\varepsilon}]$ and for some $\alpha, \beta \in F$. Now, it is enough to apply the preceding case to $[u_{s_1}, \ldots, u_{s_\gamma}][u_{t_1}, \ldots, u_{t_\varepsilon}]$ and, if necessary, apply Lemma 3.8 to reorder the variables that eventually appear outside of the commutators. So, we conclude the proof. □

**Corollary 3.11.** *Let $v_1, v_2$ be commutators involving symmetric variables only. Then $v_1v_2 + I$ is a linear combination of elements of the type*

$$y_{i_1} \cdots y_{i_r}[y_{j_1}, \dots, y_{j_s}][y_{k_1}, y_{k_2}] + I$$

*where*

(*a*) $r \geq 0$, $k_2 < k_1$, $j_1 > j_2 \leq \cdots \leq j_s$ *and* $i_1 \leq \cdots \leq i_r$.

(*b*) $k_2 \leq i_1, j_2$.

(*c*) $k_1 \leq j_1$.

(*d*) *Every* $y_{i_1} \cdots y_{i_r}[y_{j_1}, \dots, y_{j_s}][y_{k_1}, y_{k_2}]$ *has the same multidegree as* $v_1v_2$.

Proof. By Proposition 3.10, there exist $\alpha_{j,k_1} \in F$ such that

$$v_1v_2 + I = \sum_{j,k_1} \alpha_{j,k_1} y_{i_1} \cdots y_{i_m}[y_{j_1}, y_{j_2}, \dots, y_{j_n}][y_{k_1}, y_{k_2}] + I, \tag{6}$$

where $k_2 < k_1$, $j = (j_1, \dots, j_n)$, $j_1 > j_2 \leq \cdots \leq j_n$, $i_1 \leq \cdots \leq i_m$, $k_2 \leq i_1$, $k_2 \leq j_2$. Suppose that one of the terms of the sum in (6) is such that $k_1 > j_1$. By Proposition 3.9, we have that $[y_{j_1}, y_{j_2}, \dots, y_{j_n}][y_{k_1}, y_{k_2}] + I = g - h + I$, where

$$g = [y_{k_1}, y_{j_2}, y_{j_3}, \dots, y_{j_n}][y_{j_1}, y_{k_2}], \;\; h = [y_{k_1}, y_{j_1}, y_{j_3}, \dots, y_{j_n}][y_{j_2}, y_{k_2}].$$

If $j_1 > j_3$, the Jacobi identity $[y_{k_1}, y_{j_1}, y_{j_3}] = [y_{k_1}, y_{j_3}, y_{j_1}] - [y_{j_1}, y_{j_3}, y_{k_1}]$ can be applied in $h$, and, modulo $I$, the variables $y_{k_1}, y_{j_2}, \dots, y_{j_n}$ in

$$[y_{j_1}, y_{j_3}, y_{k_1}, y_{j_2}, \dots, y_{j_n}]$$

can be ordered as desired. □

**Proposition 3.12.** *Let $\Omega_z$ be the subset of $F\langle Y \cup Z \rangle$ of multilinear polynomials of the type*

$$u_{i_1} \cdots u_{i_r}[u_{j_1}, \dots, u_{j_s}][u_k, z_1]$$

*where, $r \geq 0$, $u_i \in Y \cup Z$, $u_{i_1} < \cdots < u_{i_r}$ and $u_{j_1} > u_{j_2} < \cdots < u_{j_s}$. Then, $\Omega_z$ is linearly independent modulo $Id(\mathcal{A}, *)$.*

Proof. Let $f(z_1, \dots, z_m, y_1, \dots, y_n)$ be a linear combination of polynomials in $\Omega_z$ such that $f \in Id(\mathcal{A}, *)$. Without loss of generality we can suppose that $f$ is multilinear polynomial. We can rewrite $f$ as

$$f = \sum_{i=2}^{m} h_i[z_i, z_1] + \sum_{j=1}^{n} g_j[y_j, z_1],$$

where

$(i)$ Each of $h_i$ and $g_j$ is a multilinear polynomial in the variables

$$\{y_1, \ldots, y_n, z_2, \ldots, z_m\} \setminus z_i \quad \text{and} \quad \{y_1, \ldots, y_n, z_2, \ldots, z_m\} \setminus y_j,$$

respectively.

$(ii)$ Each of $h_i$ and $g_j$ is a linear combination of polynomials of the type

$$u_{i_1} \ldots u_{i_r}[u_{j_1}, \ldots, u_{j_s}]$$

with $u_i \in Y \cup Z$, $u_{i_1} < \cdots < u_{i_r}$ and $u_{j_1} > u_{j_2} < \cdots < u_{j_s}$.

Fix $2 \leq k \leq n$. Denote by $\mathbf{D} = \mathbf{e}_{11} - \mathbf{e}_{22}$ and set

$$Z_1 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \quad \text{and} \quad Z_k = \begin{pmatrix} 0 & \mathbf{D} \\ 0 & 0 \end{pmatrix}.$$

Let $A_j, B_i \in UT_2(F)$ where $1 \leq j \leq n$, $1 \leq i \leq m$, $i \neq 1, k$. Set

$$Y_j = \begin{pmatrix} A_j & 0 \\ 0 & A_j^* \end{pmatrix} \text{ and } Z_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i^* \end{pmatrix}.$$

Since $f \in Id(\mathcal{A}, *)$ we have that $f(Z_1, \ldots, Z_m, Y_1, \ldots, Y_n) = 0$. A straightforward verification shows that $[Y_j, Z_1] = 0$ for $1 \leq j \leq n$, $[Z_i, Z_1] = 0$ for all $i \neq k$ and

$$[Z_k, Z_1] = \begin{pmatrix} 0 & -2\mathbf{D} \\ 0 & 0 \end{pmatrix}.$$

Thus, by substituting these matrices in $f$, we have that $h_k[Z_k, Z_1] = 0$ and therefore

$$h_k(B_2, \ldots, \hat{B}_k, \ldots, B_m, A_1, \ldots, A_n) = 0.$$

Therefore, $h_k$ seen as an element of $F\langle X \rangle$ is a polynomial identity for $UT_2(F)$, and by Lemma 2.1 we have that $h_k = 0$. Thus

$$f = \sum_{j=1}^{n} g_j[y_j, z_1].$$

Analogously, given $l$ such that $1 \leq l \leq m$, we can be show that $g_l = 0$ by considering the following elements

$$Y_l = \begin{pmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{pmatrix}, \ Z_1 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix}$$

and the following arbitrary elements in $\mathcal{A}$

$$Y_j = \begin{pmatrix} A_j & 0 \\ 0 & A_j^* \end{pmatrix} \text{ and } Z_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i^* \end{pmatrix},$$

where $A_j, B_i \in UT_2(F)$, $1 \leq j \leq n$, $j \neq l$, $1 \leq i \leq m$, $i \neq 1$. □

**Lemma 3.13.** *Consider the following elements of* $\mathcal{A}^+$*:* $Y = \begin{pmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{pmatrix}$ *and* $Y_i = \begin{pmatrix} A_i & 0 \\ 0 & A_i^* \end{pmatrix}$*, where* $A_i \in UT_2(F)$*,* $1 \leq i \leq n$*. Then,*

1. *For all* $n \geq 3$ *we have that*

$$[Y, Y_3, \ldots, Y_n][Y_2, Y_1] = \begin{pmatrix} 0 & -[A_2, A_1, A_3, \ldots, A_n] \\ 0 & 0 \end{pmatrix}.$$

2. *For all* $n \geq 4$ *we have that*

$$[Y_3, \ldots, Y_n, Y][Y_2, Y_1] = 0.$$

Proof. We shall show the part 1 only. The proof can be done by induction, we shall omit the case $n = 3$. Suppose that $n > 3$ and write $[Y, Y_3, \ldots, Y_n][Y_2, Y_1] = P_1 - P_2$, where

$$P_1 = [Y, Y_3, \ldots, Y_{n-1}]Y_n[Y_2, Y_1] \text{ and } P_2 = Y_n[Y, Y_3, \ldots, Y_{n-1}][Y_2, Y_1].$$

Let $[Y, Y_3, \ldots, Y_{n-1}] = \begin{pmatrix} 0 & \Theta \\ 0 & 0 \end{pmatrix}$, then

$$[Y, Y_3, \ldots, Y_{n-1}][Y_2, Y_1] = \begin{pmatrix} 0 & -\Theta[A_2, A_1] \\ 0 & 0 \end{pmatrix}.$$

By induction, $\Theta[A_2, A_1] = [A_2, A_1, A_3, \ldots, A_{n-1}]$. Thus

$$P_2 = \begin{pmatrix} 0 & -A_n[A_2, A_1, A_3, \ldots, A_{n-1}] \\ 0 & 0 \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} 0 & \Theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & A_n^* \end{pmatrix} \begin{pmatrix} [A_2, A_1] & 0 \\ 0 & -[A_2, A_1] \end{pmatrix} = \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix}$$

where $\Psi = -\Theta A_n^*[A_2, A_1] = -\Theta[A_2, A_1]A_n = -[A_2, A_1, A_3, \ldots, A_{n-1}]A_n$ as desired. □

**Lemma 3.14.** *Let* $W = \begin{pmatrix} \mathbf{e}_{22} & 0 \\ 0 & \mathbf{e}_{11} \end{pmatrix}$ *and let* $Y_1, \dots Y_n$*,* $n \geq 4$*, be arbitrary elements of* $\mathcal{A}^+$*. Then,* $[Y_3, \dots, Y_n, W][Y_2, Y_1] = [Y_3, \dots, Y_n][Y_2, Y_1]$.

Proof. There exist $\alpha_i, \beta_i \in F$ and $\Theta_i \in UT_2(F)$ such that

$$[Y_3, \dots, Y_n] = \begin{pmatrix} \alpha_1 \mathbf{e}_{12} & \Theta_1 \\ 0 & \beta_1 \mathbf{e}_{12} \end{pmatrix} \text{ and } [Y_2, Y_1] = \begin{pmatrix} \alpha_2 \mathbf{e}_{12} & \Theta_2 \\ 0 & \beta_2 \mathbf{e}_{12} \end{pmatrix}.$$

Then, $[Y_3, \dots, Y_n, W] = \begin{pmatrix} \alpha_1 \mathbf{e}_{12} & \Theta_1 \mathbf{e}_{11} - \mathbf{e}_{22} \Theta_1 \\ 0 & -\beta_1 \mathbf{e}_{12} \end{pmatrix}$ and

$$[Y_3, \dots, Y_n, W] - [Y_3, \dots, Y_n] = \begin{pmatrix} 0 & \Theta_1 \mathbf{e}_{11} - \mathbf{e}_{22} \Theta_1 - \Theta_1 \\ 0 & -2\beta_1 \mathbf{e}_{12} \end{pmatrix}.$$

Since that $(\Theta_1 \mathbf{e}_{11} - \mathbf{e}_{22} \Theta_1 - \Theta_1) \mathbf{e}_{12} = -\mathbf{e}_{22} \Theta_1 \mathbf{e}_{12} = 0$, this proof is complete. □

**Proposition 3.15.** *Let* $\Omega_y \subseteq F \langle Y \cup Z \rangle$ *be the subset of multilinear polynomials of the type*

$$[y_{j_1}, \dots, y_{j_s}][y_k, y_1]$$

*where* $j_1 > j_2 < \dots < j_s$ *and* $j_1 > k$*. Then,* $\Omega_y$ *is linearly independent modulo* $Id(\mathcal{A}, *)$.

Proof. Let $n \geq 4$ and let $f(y_1, \dots, y_n)$ be a linear combination of polynomials in $\Omega_y$ such that $f \in Id(\mathcal{A}, *)$. Without loss of generality, we may suppose that $f$ is a multilinear polynomial. We write $f = f_4 + \dots + f_n$, where

$$f_t = \sum_{k=2}^{t-1} \alpha_k^{(t)} [y_t, y_{k_4}, \dots, y_{k_n}][y_k, y_1], \quad (4 \leq t \leq n) \ (\alpha_k^{(t)} \in F).$$

Observe that the indices $(k_4 < \dots < k_n)$ are uniquely determined by $k$ and $t$. Consider the following elements in $\mathcal{A}^+$:

$$Y_n = \begin{pmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{pmatrix} \text{ and } Y_j = \begin{pmatrix} A_j & 0 \\ 0 & A_j^* \end{pmatrix},$$

where $j < n$. If $n = 4$ we have that

$$f = f_4 = \alpha_2^{(4)} [y_4, y_3][y_2, y_1] + \alpha_3^{(4)} [y_4, y_2][y_3, y_1].$$

By Lemma 3.13 item 1, we obtain $\alpha_2^{(4)} [A_2, A_1, A_3] + \alpha_3^{(4)} [A_3, A_1, A_2] = 0$, therefore, $\alpha_2^{(4)} = \alpha_3^{(4)} = 0$. Let $4 \leq t < n$, then by Lemma 3.13 item 2 we have that

$[Y_t, Y_{k_4}, \dots, Y_{k_{n-1}}, Y_n][Y_k, Y_1] = 0$ for all $k < t$. Thus $f_t(Y_1, \dots, Y_n) = 0$ for all $4 \le t < n$ and

$$f(Y_1, \dots, Y_n) = f_n(Y_1, \dots, Y_n) = \sum_{k=2}^{n-1} \alpha_k^{(n)} [Y_n, Y_{k_4}, \dots, Y_{k_n}][Y_k, Y_1] = 0.$$

Now, by Lemma 3.13 item 1 again, we conclude that

$$\sum_{k=2}^{n-1} \alpha_k^{(n)} [A_k, A_1, A_{k_4}, \dots, A_{k_n}] = 0$$

for all $A_1, \dots, A_{n-1} \in UT_2(F)$. By Lemma 2.1 we have that $\alpha_k^{(n)} = 0$ for all $2 \le k < n$. Therefore, $f = f_4 + \cdots + f_{n-1}$.

Define $g = g_4 + \cdots + g_{n-1}$ where

$$g_t(y_1, \dots, y_{n-1}) = \sum_{k=2}^{t-1} \alpha_k^{(t)} [y_t, y_{k_4}, \dots, y_{k_{n-1}}][y_k, y_1]\ , \quad 4 \le t < n.$$

We claim that $g \in Id(\mathcal{A}, *)$. In fact, let $W = \begin{pmatrix} \mathbf{e}_{22} & 0 \\ 0 & \mathbf{e}_{11} \end{pmatrix}$ then by Lemma 3.14, for all $Y_1, \dots, Y_n \in \mathcal{A}^+$ and $4 \le t < n$, we have that

$$g_t(Y_1, \dots, Y_{n-1}) = f_t(Y_1, \dots, Y_{n-1}, W).$$

Since $g \in Id(\mathcal{A}, *)$ is a polynomial in $n-1$ variables, by induction, we obtain that $\alpha_k^{(t)} = 0$ for all $2 \le k < t < n$. Thus, the proof is complete. $\square$

The next Theorem is our main result.

**Theorem 3.16.** *Let $F$ be a field of characteristic zero. Let $I$ be defined as in Definition* 3.6*, then $I = Id(\mathcal{A}, *)$.*

Proof. We shall show that $Id(\mathcal{A}, *) \subseteq I$. Let $f(z_1, \dots, z_m, y_1, \dots, y_n) \in Id(\mathcal{A}, *)$ be a $Y$-proper multilinear polynomial. Since $\operatorname{char} F = 0$, its suffices to show that $f \in I$. Since the product of three or more commutators lie in $I$, we can write

$$f + I = f_1 + f_2 + I,$$

where $f_1$ is a multilinear polynomial and a linear combination of the elements

$$z_1^{r_1} \cdots z_m^{r_m} [u_{j_1}, \dots, u_{j_s}]^t, \tag{7}$$

with $0 \leq r_1, \ldots, r_m \leq 1$, $t \in \{0,1\}$, $u_{j_1}, \ldots, u_{j_s} \in Y \cup Z$, and $f_2$ is a multilinear polynomial and a linear combination of the elements

$$z_1^{r_1} \cdots z_m^{r_m}[u_{j_1}, \ldots, u_{j_s}][v_{k_1}, \ldots, v_{k_t}],$$

where $0 \leq r_1, \ldots, r_m \leq 1$, $u_{j_1}, \ldots, u_{j_s}, v_{k_1}, \ldots, v_{k_t} \in Y \cup Z$. By Lemma 2.2, we may assume that $u_{j_1} > u_{j_2} < \cdots < u_{j_s}$ in (7). Since $I \subseteq Id(\mathcal{A}, *)$ (see (5)), we have that $f_1 + f_2 \in Id(\mathcal{A}, *) \subseteq Id(\mathcal{B}, *)$. By (2) it follows that $f_2 \in Id(\mathcal{B}, *)$. Then $f_1 \in Id(\mathcal{B}, *)$. By Proposition 2.3 we have that $f_1 = 0$. Thus $f_2 \in Id(\mathcal{A}, *)$. Now, we shall show that $f_2 \in I$. For this, we consider two cases:

**Case 1.** $m \geq 1$. By Proposition 3.10, there exists a multilinear polynomial $g$ such that $f_2 + I = g + I$ and $g$ is a linear combination of the elements

$$u_{i_1} \cdots u_{i_r}[u_{j_1}, \ldots, u_{j_s}][u_k, z_1]$$

with $r \geq 0$, $u_i \in Y \cup Z$, $u_{i_1} < \cdots < u_{i_r}$, $u_{j_1} > u_{j_2} < \cdots < u_{j_s}$. Since $f_2 \in Id(\mathcal{A}, *)$, we have that $g \in Id(\mathcal{A}, *)$. By Proposition 3.12, it follows that $g = 0$.

**Case 2.** $m = 0$. By Proposition 3.11 there exists a multilinear polynomial $g = g(y_1, \ldots, y_n)$ such that $f_2 + I = g + I$ and $g$ is a linear combination of the elements

$$y_{i_1} \cdots y_{i_r}[y_{j_1}, \ldots, y_{j_s}][y_k, y_1] + I,$$

where $r \geq 0$, $i_1 < \cdots < i_r$, $j_1 > j_2 < \cdots < j_s$ and $k < j_1$. Write

$$g = \sum_i y_{i_1} \cdots y_{i_r} g_i \ , \qquad i = (i_1, \ldots, i_r),$$

where each $g_i$ is a multilinear polynomial in the variables

$$\{y_1, \ldots, y_n\} \setminus \{y_{i_1}, \ldots, y_{i_r}\}.$$

Suppose that there exists $i = (i_1, \ldots, i_r)$ such that $g_i \neq 0$. Choose $r$ as being the maximum integer with this property and put $y_{i_1} = \cdots = y_{i_r} = 1$ in $g$. Then $g_j = 0$ for all $j \neq i$. Since $g \in Id(\mathcal{A}, *)$ we have that $g_i \in Id(\mathcal{A}, *)$. By Proposition 3.15 we have a contradiction. Therefore, $g = 0$. So, $f_2 \in I$. $\square$

Now, if we combine Proposition 3.10, Corollary 3.11, Proposition 3.12, Proposition 3.15 and Theorem 3.16, we obtain the following:

**Theorem 3.17.** *Let $\Gamma_{m,n}(\mathcal{A}, *)$ be the subspace of the $Y$-proper multilinear polynomials in the variables $z_1, \ldots, z_m, y_1, \ldots, y_n$ of the relatively free algebra $F\langle Y \cup Z \rangle / Id(\mathcal{A}, *)$. A linear basis of the space $\Gamma_{m,n}(\mathcal{A}, *)$, with $m \geq 1$, is given by*

(a) $z_{i_1} \cdots z_{i_r}[u_{j_1}, \ldots, u_{j_s}]^t + Id(\mathcal{A}, *)$, *where* $i_1 < \cdots < i_r$, $u_{j_1} > u_{j_2} < u_{j_3} < \cdots < u_{j_s}$, $t \in \{0, 1\}$.

(b) $u_{i_1} \cdots u_{i_r}[u_{j_1}, \ldots, u_{j_s}][u_k, z_1] + Id(\mathcal{A}, *)$, *where* $i_1 < \cdots < i_r$, $u_{j_1} > u_{j_2} < u_{j_3} < \cdots < u_{j_s}$.

*And, a linear basis of the space* $\Gamma_{0,n}(\mathcal{A}, *)$ *is given by*

(c) $[y_{j_1}, \ldots, y_{j_s}][y_k, y_1] + Id(\mathcal{A}, *)$ *where* $j_1 > j_2 < j_3 < \cdots < j_s$, $k < j_1$.

**4. Concluding remarks.** Theorem 3.16 shows that the $T(*)$-ideal of the $*$-polynomial identities of $\mathcal{A}$ is finitely generated as $T(*)$-ideal because, as $T(*)$-ideal, a commutator in the variables $Y \cup Z$ is describe completely by either

$$[z_1, z_2] \quad \text{or} \quad [y_1, y_2] \quad \text{or} \quad [y_1, z_1].$$

Over the study of the $*$-polynomial identities of $UT_4(F)$, can be shown that

$$[y_1, z_1][y_2, z_2][y_3, z_3]$$

is not the $*$-polynomial identity for $UT_4(F)$. That is,

$$(Id(UT_4(F)), *) \subsetneq (Id(\mathcal{A}), *).$$

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