# Serdica
## Mathematical Journal
# Сердика
## Математическо списание


---

Serdica Math. J. **45** (2019), 259–288

**Serdica**
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

# WEINGARTEN HYPERSURFACES OF THE SPHERICAL TYPE IN SPACE FORMS

Edwin O. S. Reyes, Carlos M. C. Riveros

*Communicated by O. Mushkarov*

Abstract. In this paper, we generalize the parametrization obtained by Machado in [13] in the $n$-dimensional Euclidean space for hypersurfaces $\Sigma$ in space forms $\overline{M}^{n+1}(c)$, $c = 0, \pm 1$. Using this parametrization we study the class of Weingarten hypersurfaces of the spherical type in $\overline{M}^{n+1}(c)$, $n \geq 2$, this class of hypersurfaces includes the surfaces of the spherical type (Laguerre minimal surfaces). We generalize the results and definitions of Weingarten hypersurfaces of the spherical type and we classify the Weingarten hypersurfaces of the spherical type of rotation in forms space.

**1. Introduction.** The surfaces $M \subset \mathbb{R}^3$ satisfying a functional relation of the form $W(H, K) = 0$, where $H$ and $K$ are the mean and Gaussian curvatures of the surface $M$, respectively, are called *Weingarten surfaces*. Examples of Weingarten surfaces are the surfaces of revolution and the surfaces of constant

2010 *Mathematics Subject Classification:* 53C42, 53A35.
*Key words:* Ribaucour transformations, lines of curvature, Rotation spherical hypersurface, Weingarten hypersurfaces.

mean or Gaussian curvature. In [12], the authors study an important class of surfaces satisfying a linear relation of the form

$$aH + bK + c = 0,$$

where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. These surfaces are called *linear Weingarten surfaces.* The paper [11], is devoted to the integrability of linear Weingarten surfaces.

In [6], Corro presented a way of parameterizing surfaces as envelopes of a congruence of spheres in which an envelope is contained in a plane and with radius function $h$ associated with a hydrodynamic type system. As an application, it studies the surfaces in hyperbolic space $\mathbb{H}^3$ satisfying the relation

$$2ach^{\frac{2(c-1)}{c}}(H-1) + (a + b - ach^{\frac{2(c-1)}{c}})K_I = 0,$$

where $a, b, c \in \mathbb{R}$, $a+b \neq 0$, $c \neq 0$, $H$ is the mean curvature and $K_I$ is the Gaussian curvature. This class of surfaces includes the Bryant surfaces and the flat surfaces of the hyperbolic space and are called *generalized Weingarten surfaces of Bryant type.*

In [7] the authors study the surfaces $M$ in the hyperbolic space $\mathbb{H}^3$ satisfying the relation

$$2(H-1)e^{2\mu} + K_I(1 - e^{2\mu}) = 0,$$

where $\mu$ is a harmonic function with respect to the quadratic form $\sigma = -K_I I + 2(H-1)II$, $I$ and $II$ are the first and the second quadratic form of $M$. These surfaces are called *Generalized Weingarten surfaces of harmonic type.*

A oriented surface $\psi : M \to \mathbb{R}^3$ with non-zero Gaussian curvature $K$ and mean curvature $H$ is called a *Laguerre minimal surface* if

$$\Delta_{III}\left(\frac{H}{K}\right) = 0,$$

where $\Delta_{III}$ is the Laplacian with respect to the third fundamental form $III$ of $\psi$. The study of these surfaces was done by W. Blaschke [2, 3, 4, 5], where such surfaces appear as critical points of the functional

$$L(\Psi) = \int \frac{H^2 - K}{K} dM,$$

where $dM$ is the area element.

In [14], the authors study Laguerre's minimal surfaces as graphs of biharmonic functions in the isotropic model of Laguerre Geometry. In particular, they study the surfaces of the spherical type (Laguerre minimal surfaces), namely the surfaces $M$ of $\mathbb{R}^3$ such that the set of spheres with center $p + \dfrac{H(p)}{K(p)}N(p)$, $p \in M$ are tangent to a fixed oriented plane.

In [10], the authors study a class of oriented hypersurfaces $M^n$ in hyperbolic space $(n+1)$-dimensional that satisfy a Weingarten relation in the form

$$\sum_{r=0}^{n}(c - n + 2r)\binom{n}{r}H_r = 0,$$

where $c$ is a real constant and $H_r$ is the rth mean curvature of the hypersurface $M^n$. They show that this class of hypersurfaces is characterized by a harmonic application derived from the two hyperbolic Gauss map. Looking these hypersurfaces as orthogonal to a congruence of geodesics, they also show the relation of such hypersurfaces with solutions of the equation $\Delta u + ku^{\frac{n+2}{n-2}} = 0$, where $k \in \{-1, 0, 1\}$.

In [13], the author present a way to parameterize hypersurfaces as congruence of spheres in which an envelope is contained in a hyperplane. Using this parametrization is presented a generalization of the surfaces of the spherical type (Laguerre minimal surfaces) studied in [14], namely the *Weingarten hypersurfaces of the spherical type*, i.e. the oriented hypersurfaces of the Euclidean space $M \subset \mathbb{R}^{n+1}$ satisfying a Weingarten relation of the form

$$\sum_{r=1}^{n}(-1)^{r+1} r f^{r-1}\binom{n}{r}H_r = 0,$$

where $f \in C^\infty(M;\mathbb{R})$ and $H_r$ is the rth mean curvature of $M$.

In this paper, we generalize the results obtained in [13], for hypersurfaces in space forms, namely, we generalize the parametrization obtained by Machado in [13] in the $n$-dimensional Euclidean space for hypersurfaces $\Sigma$ in space forms $\overline{M}^{n+1}(c)$, $c = 0, \pm 1$. Using this parametrization we study the class of Weingarten hypersurfaces of the spherical type in $\overline{M}^{n+1}(c)$, $n \geq 2$, this class of hypersurfaces includes the surfaces of the spherical type (Laguerre minimal surfaces). We generalize the results and definitions of Weingarten hypersurfaces of the spherical type and we classify the Weingarten hypersurfaces of the spherical type of rotation in forms space.

## 2. Preliminaries.

Let $\overline{M}^{n+1}(c)$ be, the simply connected space form of sectional curvature $c = -1, 1, 0$. $\overline{M}^{n+1}(c)$ will denote the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, if $c = -1$, the Euclidean space $\mathbb{R}^{n+1}$ when $c = 0$ or the sphere $\mathbb{S}^{n+1}$, if $c = 1$. Let $U \subset \mathbb{R}^n$ an open set of $\mathbb{R}^n$ such that $u = (u_1, u_2, \ldots, u_n) \in U$. The partial derivatives of $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, with respect to $u_i$, $1 \leq i \leq n$, will be denoted by $f_{,i}$.

We denote by $\mathbb{L}^{n+2}$ the space of $(n+2)$-tuples $u = (u_1, u_2, \ldots, u_{n+2}) \in \mathbb{R}^{n+2}$ with the Lorentzian metric $\langle u, v\rangle = \sum_{i=1}^{n+1} u_i v_i - u_{n+2}v_{n+2}$, where $v = (v_1, v_2, \ldots, v_{n+2})$ and we consider the hyperbolic space $\mathbb{H}^{n+1}$ as a hypersurface of $\mathbb{L}^{n+2}$, namely,

$$\mathbb{H}^{n+1} = \left\{u \in \mathbb{L}^{n+2}; \langle u, u\rangle = -1, u_{n+2} > 0\right\}.$$

Also, we consider the sphere $\mathbb{S}^{n+1}$ as a hypersurface of $\mathbb{R}^{n+2}$ with the Euclidean metric, namely,

$$\mathbb{S}^{n+1} = \left\{u \in \mathbb{R}^{n+2}; \langle u, u\rangle = 1\right\}.$$

**Definition 2.1.** *Let $M$ be a hypersurface of $\overline{M}^{n+1}(c)$. We say that $M$ is* orientable, *if there exist a unit vector field $N$ normal to $T_pM$, for all $p \in M$. $N$ is known as* Gauss map *of $M$. In local coordinates,*

$$N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j},\ 1 \leq i \leq n,$$

*where $X$ is a parametrization of $M$. The matrix $W = (W_{ij})$ is known as* Weingarten matrix *of $M$.*

**Definition 2.2.** *The* mean curvature *and the* Gauss-Kronecker curvature *of $M$ are given by*

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i\ ,\ K = \prod_{i=1}^{n} k_i,$$

*where $k_1, \ldots, k_n$ are the principal curvatures of $M$.*

**Definition 2.3.** *The* rth-mean curvature *$H_r$ of $M$ is defined by*

$$H_r = \frac{S_r(W)}{\binom{n}{r}},$$

*where, for intergers $0 \leq r \leq n$, $S_r(W)$, is defined by*

$$S_0(W) = 1,$$

$$S_r(W) = \sum_{1 \le i_1 < \cdots < i_r \le n} k_{i_1} \cdots k_{i_r}.$$

The following definitions were given in [15].

**Definition 2.4.** *A* congruence of geodesic spheres in $\overline{M}^{n+1}(c)$ *is a family of $n$-parameter geodesic spheres in $\overline{M}^{n+1}(c)$, such that the set of centers of the geodesic spheres is a hypersurface of $\overline{M}^{n+1}(c)$ and the radii of the geodesic spheres are given by a differentiable function on the hypersurface.*

*An* involute *of a congruence of geodesic spheres is an $n$-dimensional submanifold $M$ of $\overline{M}^{n+1}(c)$, such that each point $p \in M$ is tangent to a geodesic sphere of the congruence of geodesic spheres.*

*Let $M_1$ and $M_2$ be the hypersurfaces in $\overline{M}^{n+1}(c)$. We say that $M_1$ and $M_2$ are* associated by a congruence of geodesic spheres, *if there exists a diffeomorfism $\Psi : M_1 \to M_2$ such that, at the corresponding points $p$ and $\Psi(p)$, $M_1$ and $M_2$ are tangent to a same geodesic sphere of the congruence of geodesic spheres.*

**Definition 2.5.** *Let $M_1$ be an orientable hypersurface in $\overline{M}^{n+1}(c)$. An orientable hypersurface $M_2 \subset \overline{M}^{n+1}(c)$ is associated to $M_1$ by a* Ribaucour transformation, *if there exist a differentiable funtion $h : M_1 \to \mathbb{R}$ and a diffeomorphism $\Psi : M_1 \to M_2$, such that*

(1) $\exp_p(h(p)N_1(p)) = \exp_{\Psi(p)}(h(p)N_2(\Psi(p))$, *for all $p \in M_1$, where* exp *is the exponential map of $\overline{M}^{n+1}(c)$ and $N_1$, $N_2$ are the unit normal vector fields of $M_1$ and $M_2$, respectively.*

(2) *The subset $S = \{\exp_p(h(p)N_1(p));\ p \in M_1\}$ is a $n$-dimensional submanifold of $\overline{M}^{n+1}(c)$.*

(3) *$\Psi$ preserves lines of curvature.*

*We say that $M_1$ and $M_2$ are* locally associated by a Ribaucour transformation, *if for all $p \in M_1$ there exists a neighborhood of $p \in M_1$ which is associated by a Ribaucour transformation to an open subset of $M_2$.*

**Remark 2.6.** Let $M$ be a hypersurface of $\overline{M}^{n+1}(c)$ and $N$ its unit normal vector field on $\overline{M}^{n+1}(c)$. If $c = \pm 1$, then $\langle N(p), p \rangle = 0$, $\forall p \in M$. In fact, suppose that $X : U \subset \mathbb{R}^{n+1} \to \overline{M}^{n+1}(c)$ is a local orthogonal parametrization of $p \in \overline{M}^{n+1}(c)$. Therefore, $\langle X(u), X(u) \rangle = \pm 1$ and $\langle X_{,i}(u), X(u) \rangle = 0$, $\forall u \in U$, $1 \le i \le n+1$. If $p \in M$, there exists $1 \le i \le n+1$ and $u \in U$, such that $N(p) = X_{,i}(u)$ with $X(u) = p$.

Considering $\overline{M}^{n+1}(c)$ as $\mathbb{R}^{n+1}$, if $c=0$, $\mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ if $c=1$ or $\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$, if $c=-1$, we can rewrite the condition (1) of the Definition 2.5 as $p+h(p)N_1(p)=\Psi(p)+h(p)N_2(\Psi(p))$, $\forall\, p\in M_1$, where

$$h(p)=\begin{cases}\tan(\phi(p)),\ \phi:M_1\to\left(0,\dfrac{\pi}{2}\right),\ \text{if } c=1,\\ \tanh(\phi(p)),\ \phi:M_1\to\mathbb{R},\ \text{if } c=-1.\end{cases}$$

In fact, the points of $\mathbb{S}^{n+1}$ which belong to the geodesic that passes through of $p\in\mathbb{S}^{n+1}$ in the direction $N_1(p)$ can be parametrized as

$$\cos(\phi(p))p+\sin(\phi(p))N_1(p),\ \phi:\mathbb{S}^{n+1}\to(0,\pi).$$

As $p\in M_1$ and $\Psi(p)\in M_2$ are tangents to the same geodesic sphere, hence, $\phi(p)\in\left(0,\dfrac{\pi}{2}\right)$.

Similarly, the points of $\mathbb{H}^{n+1}$ which belong to the geodesic that passes through of $p\in\mathbb{H}^{n+1}$ in the direction $N_1(p)$ can be parametrized as

$$\cosh(\phi(p))p+\sinh(\phi(p))N_1(p),\ \phi:\mathbb{H}^{n+1}\to\mathbb{R}.$$

Thus,

$$\phi(p)=\begin{cases}\tan^{-1}(h(p)),\ h:M_1\to(0,\infty),\ \text{if } c=1,\\ \tanh^{-1}(h(p)),\ h:M_1\to(-1,1),\ \text{if } c=-1.\end{cases}$$

From now on, we will consider $M^n(c)$ a hypersurface of $\overline{M}^{n+1}(c)$, such that $M^n(c)=\mathbb{H}^n$, if $c=-1$, $M^n(c)=\mathbb{R}^n$ if $c=0$ or $M^n(c)=\mathbb{S}^n$, if $c=1$, with unit normal vector field $N(p)=e_c$, $\forall\, p\in M^n(c)$ given by

$$e_c=\begin{cases}(0,0,\ldots,0,1,0)\in\mathbb{L}^{n+2},\ \text{if } c=-1,\\ (0,0,\ldots,0,1)\in\mathbb{R}^{n+1},\ \text{if } c=0,\\ (0,0,\ldots,0,1)\in\mathbb{R}^{n+2},\ \text{if } c=1.\end{cases}$$

Let $Y:U\to M^n(c)$ be a local orthogonal parametrization of $M^n(c)$. If $L_{ij}=\langle Y_{,i},Y_{,j}\rangle$, $1\le i,j\le n$, then $L_{ii}\neq 0$ and $L_{ij}=0$ for $i\neq j$.
The Christoffel symbols of $L_{ij}$ are given by

$$\begin{aligned}&\Gamma_{ij}^m=0,\ \text{for distinct } i,j,m,\\ &\Gamma_{ij}^j=\frac{L_{jj,i}}{2L_{jj}},\ \text{for all } i,j,\\ &\Gamma_{ii}^j=-\frac{L_{ii,j}}{2L_{jj}},\ \text{for } i\neq j.\end{aligned}\tag{2.1}$$

We consider the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, $e_{n+2} = (0,0,\dots,0,1)$ and $-e_{n+2} = (0,0,\dots,0,-1)$ the north pole and south pole of $\mathbb{S}^{n+1}$, respectively. The stereographic projection $P_- : \mathbb{S}^{n+1} - \{-e_{n+2}\} \to \mathbb{R}^{n+1}$ and $P_+ : \mathbb{S}^{n+1} - \{e_{n+2}\} \to \mathbb{R}^{n+1}$ are diffeomorphism given by

$$P_-(q) = \frac{q - \langle q, e_{n+2}\rangle e_{n+2}}{1 + \langle q, e_{n+2}\rangle},\ P_+(q) = \frac{q - \langle q, e_{n+2}\rangle e_{n+2}}{1 - \langle q, e_{n+2}\rangle},\ q \in \mathbb{S}^{n+1}. \tag{2.2}$$

Therefore, the inverse mapping $P_-^{-1}$ and $P_+^{-1}$, are given by

$$P_-^{-1}(p) = \frac{(2p, 1 - \langle p,p\rangle)}{1 + \langle p,p\rangle},\ P_+^{-1}(p) = \frac{(2p, \langle p,p\rangle - 1)}{1 + \langle p,p\rangle},\ p \in \mathbb{R}^{n+1}. \tag{2.3}$$

We consider $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ and we define

$$\begin{aligned} P\ :\ & \mathbb{H}^{n+1} \to \mathbb{R}^{n+1} \\ & u \to P(u), \end{aligned} \tag{2.4}$$

where $P(u)$ is the intersection of the hyperplane

$$\mathbb{R}^{n+1} = \left\{(u_1, u_2, \dots, u_{n+1}, u_{n+2}) \subset \mathbb{R}^{n+2}; u_{n+2} = 0\right\}$$

with the line that passes through the points $u$ and $(0,0,\dots,0,-1) \in \mathbb{R}^{n+2}$. $P$ is known as the *hyperbolic stereographic projection.* (See [1]).

**Proposition 2.7.** *Let $P : \mathbb{H}^{n+1} \to \mathbb{R}^{n+1}$ be given by*

$$P(u) = \left(\frac{u_1}{1+u_{n+2}}, \frac{u_2}{1+u_{n+2}}, \dots, \frac{u_{n+1}}{1+u_{n+2}}\right),\ u = (u_1, u_2, \dots, u_{n+2}) \in \mathbb{H}^{n+1}. \tag{2.5}$$

*Then $P$ is a diffeomorphism of $\mathbb{H}^{n+1}$ on $B^{n+1}(1) = \left\{u \in \mathbb{R}^{n+1}; |u| < 1\right\}$.*
*Therefore, $P^{-1} : B^{n+1}(1) \to \mathbb{H}^{n+1}$ given by*

$$P^{-1}(u) = \frac{1}{1 - \langle u,u\rangle^2}\left(2u, 1 + \langle u,u\rangle^2\right),\ u \in B^{n+1}(1), \tag{2.6}$$

*is a parametrization of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$.*

We will define rotation hypersurfaces in $\overline{M}^{n+1}(c)$. An orthogonal transformation of $\mathbb{L}^{n+2}$ is a linear map that preserves the Lorentzian metric; the orthogonal transformations induce, by restriction, all the isometries of $\mathbb{H}^{n+1}$. We will denote by $P^k$ a $k$-dimensional subspace of $\mathbb{L}^{n+2}$ passing through the origin and by $O(P^2)$ set of orthogonal transformations of $\mathbb{L}^{n+2}$ with positive determinant that leave $P^2$ pointwise fixed. We will use $[v_1, v_2, \ldots, v_k]$ to denote the subspace generated by the vectors $v_1, v_2, \ldots, v_k$. We say that $P^k$ is Lorentzian, if the restriction of the Lorentzian metric to $P^k$, is a Lorentzian metric; $P^k$ is Riemannian, if the restriction of the Lorentzian metric to $P^k$, is a Riemannian metric; $P^k$ is degenerate, if the restriction of the Lorentzian metric to $P^k$, is a degenerate quadratic form.

The following definition can be found in [9]

**Definition 2.8.** *Choose $P^2$ and $P^3 \supset P^2$, such that $P^3 \cap \mathbb{H}^{n+1} \neq \emptyset$. Let $C$ be a regular curve in $P^3 \cap \mathbb{H}^{n+1} = \mathbb{H}^3$ that does not meet $P^2$. The orbit of $C$ under the action of $O(P^2)$ is called a* rotation hypersurface *$M^n(-1) \subset \mathbb{H}^{n+1}$ generated by $C$ around $P^2$. If $P^2$ is Lorenztian, We say that is a* rotation spherical, *if $P^2$ is Riemannian, we say that is a* rotation hyperbolic *and if $P^2$ is degenerate, we say that is a* rotation parabolic.

Changing $\mathbb{H}^{n+1}$ by $\mathbb{S}^{n+1}$ and $\mathbb{L}^{n+2}$ by $\mathbb{R}^{n+2}$ on the definition above, we get a definition for rotation hypersurfaces in $\mathbb{S}^{n+1}$.

## 3. Hypersurfaces of $\overline{M}^{n+1}(c)$ associated to $M^n(c)$ by a congruence of geodesic spheres.

The following Theorem generalizes the result obtained by Machado in [13] for hypersurfaces in space forms.

**Theorem 3.1.** *Let $\Sigma$ be an orientable hypersurface in $\overline{M}^{n+1}(c)$ and $N$ the unit normal vector field of $\Sigma$ in $\overline{M}^{n+1}(c)$, such that $N(p) \neq e_c$, $\forall\ p \in \Sigma$. Then,*

$$v + h(p) dN_p(v) \neq 0,\ \forall\ p \in \Sigma \text{ and } v \in T_p\Sigma,\ v \neq 0, \tag{3.1}$$

*if and only if, $\Sigma$ and $M^n(c)$ are locally associated by a congruence of geodesic spheres in $\overline{M}^{n+1}(c)$, where $h : \Sigma \to \mathbb{R}$ is a differenciable function given by*

$$h(p) = \frac{\langle p, e_c \rangle}{1 - \langle N(p), e_c \rangle},\ p \in \Sigma. \tag{3.2}$$

Proof. Let $h$ be a differentiable function defined by (3.2) and $\Psi$ a differentiable map given by

$$\Psi(p) = p + h(p) \left[ N(p) - e_c \right],\ p \in \Sigma. \tag{3.3}$$

Therefore, $\Psi$ is a local diffeomorphism, such that $p + h(p)N(p) = \Psi(p) + h(p)e_c$, $p \in \Sigma$ and $\Psi(\Sigma) \subset M^n(c)$. In fact, from (3.3), we get

$$d\Psi_p(v) = v + dh_p(v)\,[N(p) - e_c] + h(p)dN_p(v),\ p \in \Sigma,\ v \in T_p\Sigma, \tag{3.4}$$

hence,

$$\langle d\Psi_p(v), N(p)\rangle = dh_p(v)\,[1 - \langle N(p), e_c\rangle]\,,\ p \in \Sigma,\ v \in T_p\Sigma,$$

and

$$dh_p(v) = \frac{\langle d\Psi_p(v), N(p)\rangle}{1 - \langle N(p), e_c\rangle},\ p \in \Sigma,\ v \in T_p\Sigma. \tag{3.5}$$

If there exists $p \in \Sigma$ and $v \in T_p\Sigma$, $v \neq 0$, such that $d\Psi_p(v) = 0$, from (3.4) and (3.5), we get that $v + h(p)dN_p(v) = 0$. This is a contradiction with (3.1). On the other hand, from (3.2) and (3.3), we get

$$\begin{aligned} \langle \Psi(p), e_c\rangle &= \langle p + h(p)\,[N(p) - e_c]\,, e_c\rangle \\ &= \langle p, e_c\rangle - h(p)\,[1 - \langle N(p), e_c\rangle] = 0,\ \forall\, p \in \Sigma. \end{aligned} \tag{3.6}$$

If $c = 0$, then $\Psi(\Sigma) \subset \mathbb{R}^n$.
If $c = \pm 1$, from Remark 2.6, we get $\langle N(p), p\rangle = 0$, $\forall\, p \in \Sigma$, thus,

$$\begin{aligned} \langle \Psi(p), \Psi(p)\rangle &= \langle p + h(p)\,[N(p) - e_c]\,, p + h(p)\,[N(p) - e_c]\rangle \\ &= \langle p, p\rangle + 2h(p)\,\langle p, N(p) - e_c\rangle + h(p)^2\,\langle N(p) - e_c, N(p) - e_c\rangle \\ &= \pm 1 - 2h(p)\,\langle p, e_c\rangle + 2h(p)^2(1 - \langle N(p), e_c\rangle) = \pm 1,\ \forall\, p \in \Sigma. \end{aligned}$$

Consider the central manifold $X^0 = \{p + h(p)N(p); p \in \Sigma\} \subset \overline{M}^{n+1}(c)$, we will show that $X^0$ is a $n$-dimensional submanifold of $\overline{M}^{n+1}(c)$. Consider $p \in \Sigma$ and $v \in T_p\Sigma$, $v \neq 0$, then $v + h(p)dN_p(v) \neq 0$. Therefore,

$$\begin{aligned} \left\langle dX^0_p(v), dX^0_p(v)\right\rangle &= \langle v{+}h(p)dN_p(v){+}dh_p(v)N(p), v{+}h(p)dN_p(v){+}dh_p(v)N(p)\rangle \\ &= |v + h(p)dN_p(v)|^2 + |dh_p(v)|^2 > 0,\ \forall p \in \Sigma,\ v \in T_p\Sigma,\ v \neq 0, \end{aligned}$$

it follows that $X^0$ is a submanifold of $\overline{M}^{n+1}(c)$. Consequently, $\Sigma$ is locally associated to $M^n(c)$ by a congruence of geodesic sphere in $\overline{M}^{n+1}(c)$.

Conversely, from (3.4) and (3.6), we get

$$\begin{aligned} (3.7)\ 0 &= \langle d\Psi_p(v), e_c\rangle \\ &= \langle v + dh_p(v)\,[N(p) - e_c] + h(p)dN_p(v), e_c\rangle \end{aligned}$$

$$= \langle v + h(p)dN_p(v), e_c\rangle + dh_p(v)(\langle N(p), e_c\rangle - 1),\ p \in \Sigma,\ v \in T_p\Sigma.$$

Consider $X^0 = \{p + h(p)N(p);\ p \in \Sigma\} \subset \overline{M}^{n+1}(c)$, then

$$dX^0_p(v) = v + h(p)dN_p(v) + dh_p(v)N(p),\ p \in \Sigma,\ v \in T_p\Sigma.$$

Suppose that $v + h(p)dN_p(v) = 0$ at a point $p \in \Sigma$ and $v \in T_p\Sigma$, $v \neq 0$, we get from (3.7) that $dh_p(v)(\langle N(p), e_c\rangle - 1) = 0$ and therefore, $dh_p(v) = 0$, otherwise, $\langle N(p), e_c\rangle = 1$, implies that $N(p) = e_c$, which is a contradiction. Hence, we have $dh_p(v) = 0$ and $dX^0_p(v) = 0$, this is a contradiction since $X^0$ is a $n$-dimensional submanifold of $\overline{M}^{n+1}(c)$. Thus, $v + h(p)dN_p(v) \neq 0$, $\forall\, p \in \Sigma$, $v \in T_p\Sigma$, $v \neq 0$. □

The following result characterizes hypersurfaces in space forms locally associated to $M^n(c)$ by a Ribaucour transformation.

**Corollary 3.2.** *Let $\Sigma$ be an orientable hypersurface of $\overline{M}^{n+1}(c)$, $N$ the unit normal vector field of $\Sigma$ in $\overline{M}^{n+1}(c)$, such that $N(p) \neq e_c$, $\forall\, p \in \Sigma$, $\{e_i\}_{i=1}^n$ orthonormal principal vector fields of $\Sigma$ in $p$, $h$ and $\Psi$ given by (3.2) and (3.3), respectively. Then, $1 + h(p)k_i \neq 0$, $\forall p \in \Sigma$, $1 \leq i \leq n$, where $k_i$ are the principal curvatures of $\Sigma$ in $\overline{M}^{n+1}(c)$, if and only if, $\Sigma$ and $M^n(c)$ are locally associated by a Ribaucour transformation in $\overline{M}^{n+1}(c)$.*

Proof. We will show that $v + h(p)dN_p(v) \neq 0$, $\forall\, p \in \Sigma$, $v \in T_p\Sigma$, $v \neq 0$ and $\langle d\Psi_p(e_i), d\Psi_p(e_j)\rangle = 0$, $\forall\, p \in \Sigma$, $1 \leq i \neq j \leq n$.

Let $v \in T_p\Sigma, v \neq 0$, then there exist $v_i \in \mathbb{R}$, $1 \leq i \leq n$, such that $v = \sum_{i=1}^{n} v_i e_i$, where $v_i \neq 0$, for some $1 \leq i \leq n$.

We assume that $v + h(p)dN_p(v) = 0$, thus,

$$0 = v + h(p)dN_p(v) = \sum_{i=1}^{n} v_i e_i + h(p)dN_p\left(\sum_{i=1}^{n} v_i e_i\right) = \sum_{i=1}^{n} (1 + h(p)k_i)v_i e_i,$$

therefore, since $\{e_i\}_{i=1}^n$ is a linearly independent set, we get

$$(1 + h(p)k_i)v_i = 0,\ 1 \leq i \leq n.$$

However, as there is some $v_i \neq 0$, then, $1 + h(p)k_i = 0$ for some $1 \leq i \leq n$, This is a contradiction. From Theorem 3.1, we have that $\Sigma$ and $M^n(c)$ are associated by a congruence of geodesic spheres.

Conversely, from (3.4) and (3.5), we get

$$d\Psi_p(e_i) = (1 + h(p)k_i)e_i + dh_p(e_i)\,[N(p) - e_c] \tag{3.8}$$

and

$$dh_p(e_i) = \frac{(1 + h(p)k_i)\langle e_i, e_c\rangle}{1 - \langle N(p), e_c\rangle},\ p \in \Sigma,\ 1 \le i \le n. \tag{3.9}$$

Thus,

$$\langle d\Psi_p(e_i), d\Psi_p(e_j)\rangle = (1 + h(p)k_i)(1 + h(p)k_j)\delta_{ij} - (1 + h(p)k_i)dh_p(e_j)\langle e_i, e_c\rangle - (1 + h(p)k_j)dh_p(e_i)\langle e_j, e_c\rangle + 2dh_p(e_i)dh_p(e_j)(1 - \langle N(p), e_c\rangle).$$

Using (3.9) in the equation above, we get $\langle d\Psi_p(e_i), d\Psi_p(e_j)\rangle = 0$ for $1 \le i \ne j \le n$. Then, $\{d\Psi(e_i)\}_{i=1}^n$ is a orthogonal principal vector fields of $M^n(c)$ in $\overline{M}^{n+1}(c)$. Thus, $\Sigma$ and $M^n(c)$ are locally associated by a Ribaucour transformation in $\overline{M}^{n+1}(c)$. $\square$

The following Theorem generalizes the result obtained by Machado in [13] for hypersurfaces in space forms.

**Theorem 3.3.** *Consider $\Sigma$ an orientable hypersurface of $\overline{M}^{n+1}(c)$, $N$ the unit normal vector field of $\Sigma$ in $\overline{M}^{n+1}(c)$ such that $N(p) \ne e_c$, $\forall\, p \in \Sigma$ and $X : U \to \Sigma$ a local parametrization of $p \in \Sigma$. Suppose that (3.1) is true. Then, there exist a local parametrization $Y : U \to M^n(c)$, such that*

$$X(u) = Y(u) + h(u)\left[e_c - N(u)\right],\ u \in U. \tag{3.10}$$

*If $Y$ is a local orthogonal parametrization of $M^n(c)$, then*

$$X = Y - \frac{2h}{S}\left(\sum_{i=1}^n \frac{h_{,i}}{L_{ii}} Y_{,i} - e_c + chY\right), \tag{3.11}$$

$$N = \frac{2}{S}\left(\sum_{i=1}^n \frac{h_{,i}}{L_{ii}} Y_{,i} - e_c + chY\right) + e_c, \tag{3.12}$$

*where*

$$S = \sum_{i=1}^n \frac{h_{,i}^2}{L_{ii}} + ch^2 + 1 \ne 0. \tag{3.13}$$

*The first, second and third fundamental forms of $\Sigma$ in $\overline{M}^{n+1}(c)$, are given by*

$$I = \langle X_{,i}, X_{,j}\rangle = L_{ij} - \frac{2h}{S}(V_{ji}L_{ii} + V_{ij}L_{jj}) + \frac{4h^2}{S^2}\sum_{k=1}^n V_{ik}V_{jk}L_{kk}, \tag{3.14}$$

$$II = -\langle N_{,i}, X_{,j}\rangle = \frac{4h}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk} - \frac{2}{S}V_{ji}L_{ii}, \tag{3.15}$$

$$III = \langle N_{,i}, N_{,j}\rangle = \frac{4}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk}, \tag{3.16}$$

*respectively, where*

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \sum_{l=1}^{n}\Gamma_{ij}^{l}h_{,l}\right) + ch\delta_{ij}, \quad 1 \le i, j \le n, \tag{3.17}$$

*and* $\Gamma_{ij}^l$ *are the Christoffel symbols of the metric* $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$, $1 \le i, j \le n$.
*The Weingarten matrix* $W = (W_{ij})$ *is given by*

$$W = 2V(SI_n - 2hV)^{-1}, \tag{3.18}$$

*where* $I_n$ *is the identity matrix and* $V = (V_{ij})$.
*The condition of regularity of* $X$ *is given by*

$$\det(SI_n - 2hV) \neq 0. \tag{3.19}$$

*Conversely, given a local orthogonal parametrization* $Y : U \to M^n(c) \subset \overline{M}^{n+1}(c)$, *where* $U$ *is a simply connected domain of* $\mathbb{R}^n$ *and a differentiable function* $h : U \to \mathbb{R}$. *Then* (3.11) *is a hypersurface of* $\overline{M}^{n+1}(c)$ *with Gauss map given by* (3.12) *and* (3.13)–(3.19) *are satisfied.*

Proof. Consider $Y : U \to M^n(c)$ a local orthogonal parametrization of $M^n(c)$, such that (3.10) is satisfied and $N$ the unit normal vector field of $\Sigma$ in $\overline{M}^{n+1}(c)$ given by

$$N = \sum_{i=1}^{n} b_i Y_{,i} + b_{n+1}e_c + b_{n+2}Y, \text{ if } c = \pm 1, \tag{3.20}$$

and

$$N = \sum_{i=1}^{n} b_i Y_{,i} + b_{n+1}e_c, \text{ if } c = 0, \tag{3.21}$$

such that

$$\langle N, N\rangle = \sum_{i=1}^{n} b_i^2 L_{ii} + b_{n+1}^2 + cb_{n+2}^2 = 1. \tag{3.22}$$

From (3.10), we get

$$X_{,i} = Y_{,i} + h_{,i}(e_c - N) - hN_{,i},\ 1 \le i \le n. \tag{3.23}$$

Hence, from (3.20), (3.21) and (3.23), we get

$$0 = \langle N, X_{,i}\rangle = b_i L_{ii} + h_{,i}(b_{n+1} - 1),\ 1 \le i \le n,$$

which implies that

$$b_i = \frac{h_{,i}(1 - b_{n+1})}{L_{ii}},\ 1 \le i \le n. \tag{3.24}$$

If $c = \pm 1$, from (3.10) and the Remark 2.6, we get

$$0 = \langle X, N\rangle = \langle Y + h(e_c - N), N\rangle = \langle Y, N\rangle + h(\langle e_c, N\rangle - 1),$$

which implies that

$$b_{n+2} = ch(1 - b_{n+1}). \tag{3.25}$$

Thus, using (3.24), (3.25) in (3.22), we have

$$\sum_{i=1}^{n}\left[\left(\frac{h_{,i}}{L_{ii}}\right)^2 (1 - b_{n+1})^2 L_{ii}\right] + b_{n+1}^2 + ch^2(1 - b_{n+1})^2 = 1,$$

which implies that

$$\left(\sum_{i=1}^{n}\frac{h_{,i}^2}{L_{ii}} + ch^2 - 1\right) - 2\left(\sum_{i=1}^{n}\frac{h_{,i}^2}{L_{ii}} + ch^2\right) b_{n+1} + \left(\sum_{i=1}^{n}\frac{h_{,i}^2}{L_{ii}} + ch^2 + 1\right) b_{n+1}^2 = 0.$$

Using (3.13) in the above equation, we get

$$(S - 2) - 2(S - 1)b_{n+1} + Sb_{n+1}^2 = 0. \tag{3.26}$$

The solutions of (3.26) are given by

$$b_{n+1} = 1 \text{ or } b_{n+1} = 1 - \frac{2}{S}.$$

If $b_{n+1} = 1$, then $N = e_c$, this is a contradiction.
Thus,

$$b_{n+1} = 1 - \frac{2}{S}. \tag{3.27}$$

Using (3.24), (3.25) and (3.27) in (3.20) and (3.21), we get (3.12). Also, using (3.12) in (3.10), we get (3.11).
On the other hand, from (3.13), we have

$$\begin{aligned}
S_{,i} &= \sum_{j=1}^{n} \frac{2h_{,j}h_{,ji}L_{jj} - L_{jj,i}h_{,j}^2}{L_{jj}^2} + 2chh_{,i}, \\
&= \sum_{j=1}^{n} \frac{2}{L_{jj}} \left( h_{,ji} - \frac{L_{jj,i}h_{,j}}{2L_{jj}} \right) h_{,j} + 2chh_{,i} \\
&\quad + \left( \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{L_{ii}L_{jj}} h_{,i}h_{,j} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{L_{ii}L_{jj}} h_{,i}h_{,j} \right), \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2}{L_{jj}} \left( h_{,ji} - \frac{L_{jj,i}h_{,j}}{2L_{jj}} \right) h_{,j} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{L_{ii}L_{jj}} h_{,i}h_{,j} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{L_{ii}L_{jj}} h_{,i}h_{,j} \\
&\quad + \frac{2}{L_{ii}} \left( h_{,ii} - \frac{L_{ii,i}h_{,i}}{2L_{ii}} \right) h_{,i} + 2chh_{,i}, \\
&= \frac{2}{L_{ii}} \left( h_{,ii} - \Gamma_{ii}^{i} h_{,i} - \sum_{\substack{l=1 \\ l\neq i}}^{n} \Gamma_{ii}^{l} h_{,l} \right) h_{,i} \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2}{L_{jj}} \left( h_{,ji} - \sum_{l=1}^{n} \Gamma_{ij}^{l} h_{,l} \right) h_{,j} + 2chh_{,i}, \\
&= 2 \sum_{j=1}^{n} \left[ \frac{1}{L_{jj}} \left( h_{,ji} - \sum_{l=1}^{n} \Gamma_{ij}^{l} h_{,l} \right) + ch\delta_{ij} \right] h_{,j}, \; 1 \leq i \leq n.
\end{aligned}$$

Using (3.17) in the above equation, we get

$$S_{,i} = 2 \sum_{j=1}^{n} V_{ij} h_{,j}, \; 1 \leq i \leq n. \tag{3.28}$$

We observe that

$$(3.29)\qquad Y_{,ij}=\sum_{k=1}^{n}\Gamma_{ij}^{k}Y_{,k}+\overline{b_{n+1}}e_c+\overline{b_{n+2}}Y,\ \text{if } c=\pm 1,\qquad 1\le i,j\le n,$$

and

$$(3.30)\qquad Y_{,ij}=\sum_{k=1}^{n}\Gamma_{ij}^{k}Y_{,k}+\overline{b_{n+1}}e_c,\ \text{if } c=0,\qquad 1\le i,j\le n.$$

If $c=\pm 1$, then $\langle Y,Y_{,j}\rangle=0$, $1\le j\le n$, thus, $\langle Y_{,i},Y_{,j}\rangle+\langle Y,Y_{,ji}\rangle=0$, therefore, $\overline{b_{n+2}}=-cL_{ij}$.
On the other hand, since $\langle Y_{,j},e_c\rangle=0$, $1\le j\le n$, then $\langle Y_{,ji},e_c\rangle=0$, hence, $\overline{b_{n+1}}=0$.
Using $\overline{b_{n+1}}=0$ and $\overline{b_{n+2}}=-cL_{ij}$ in (3.29) and (3.30), we get $Y_{,ij}=\sum_{k=1}^{n}\Gamma_{ij}^{k}Y_{,k}-cL_{ij}Y$, $1\le i,j\le n$. Thus,

$$\begin{aligned}\sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}Y_{,ji} &= \sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}\left[\sum_{k=1}^{n}\Gamma_{ji}^{k}Y_{,k}-cL_{ji}Y\right],\\ &= \sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}\Gamma_{ji}^{j}Y_{,j}+\sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}\sum_{\substack{k=1\\k\neq j}}^{n}\Gamma_{ji}^{k}Y_{,k}-c\sum_{j=1}^{n}\frac{h_{,j}L_{ji}}{L_{jj}}Y,\\ &= \sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}\Gamma_{ji}^{j}Y_{,j}+\sum_{\substack{k=1\\k\neq i}}^{n}\frac{h_{,i}}{L_{ii}}\Gamma_{ii}^{k}Y_{,k}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{h_{,j}}{L_{jj}}\sum_{\substack{k=1\\k\neq j}}^{n}\Gamma_{ji}^{k}Y_{,k}-ch_{,i}Y,\\ &= \sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}\Gamma_{ji}^{j}Y_{,j}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{h_{,i}}{L_{ii}}\Gamma_{ii}^{j}Y_{,j}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{h_{,j}}{L_{jj}}\Gamma_{ji}^{i}Y_{,i}-ch_{,i}Y.\end{aligned}$$

Using (2.1) in the above equation, we get

$$\sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}Y_{,ji}=\sum_{j=1}^{n}\frac{h_{,j}L_{jj,i}}{2L_{jj}^{2}}Y_{,j}-\sum_{\substack{j=1\\j\neq i}}^{n}\frac{h_{,i}L_{ii,j}}{2L_{jj}L_{ii}}Y_{,j}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{h_{,j}L_{ii,j}}{2L_{ii}L_{jj}}Y_{,i}-ch_{,i}Y,$$

therefore,

$$(3.31)\qquad \sum_{j=1}^{n}\frac{h_{,j}}{L_{jj}}Y_{,ji}=\sum_{j=1}^{n}\frac{h_{,j}L_{jj,i}}{2L_{jj}^{2}}Y_{,j}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{L_{ii,j}}{2L_{ii}L_{jj}}\left(h_{,j}Y_{,i}-h_{,i}Y_{,j}\right)-ch_{,i}Y.$$

We define

$$C = Y + he_c \ \text{ and } \ D = \sum_{i=1}^{n} \frac{h_{,i}}{L_{ii}} Y_{,i} + chY + \left(\frac{S}{2} - 1\right) e_c. \tag{3.32}$$

Differentiating, we obtain

$$C_{,i} = Y_{,i} + h_{,i}e_c, \ 1 \leq i \leq n, \tag{3.33}$$

$$\begin{aligned}
D_{,i} &= \sum_{j=1}^{n} \left[ \left( \frac{h_{,ji}}{L_{jj}} - \frac{L_{jj,i}h_{,j}}{L_{jj}^2} \right) Y_{,j} + \frac{h_{,j}}{L_{jj}} Y_{,ji} \right] + ch_{,i}Y + chY_{,i} + chh_{,i}e_c \\
&\quad + \sum_{j=1}^{n} \left( \frac{2h_{,j}h_{,ji}L_{jj} - L_{jj,i}h_{,j}^2}{2L_{jj}^2} \right) e_c, \\
&= \sum_{j=1}^{n} \left[ \left( \frac{h_{,ji}}{L_{jj}} - \frac{L_{jj,i}h_{,j}}{L_{jj}^2} \right) Y_{,j} + \frac{h_{,j}}{L_{jj}} Y_{,ji} + \left( \frac{h_{,j}h_{,ji}}{L_{jj}} - \frac{L_{jj,i}h_{,j}^2}{2L_{jj}^2} \right) e_c \right] + chC_{,i} \\
&\quad + ch_{,i}Y.
\end{aligned}$$

Using (3.31) in the above equation

$$\begin{aligned}
D_{,i} &= \sum_{j=1}^{n} \left( \frac{h_{,ji}}{L_{jj}} Y_{,j} - \frac{L_{jj,i}h_{,j}}{2L_{jj}^2} Y_{,j} + \frac{h_{,j}h_{,ji}}{L_{jj}} e_c - \frac{L_{jj,i}h_{,j}^2}{2L_{jj}^2} e_c \right) + chC_{,i} \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{2L_{ii}L_{jj}} (h_{,j}Y_{,i} - h_{,i}Y_{,j}), \\
&= \sum_{j=1}^{n} \frac{1}{L_{jj}} \left( h_{,ji} - \frac{L_{jj,i}h_{,j}}{2L_{jj}} \right) C_{,j} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}}{2L_{ii}L_{jj}} (h_{,j}Y_{,i} - h_{,i}Y_{,j}) + chC_{,i} \\
&\quad + \left( \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}h_{,j}h_{,i}}{2L_{ii}L_{jj}} e_c - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}h_{,j}h_{,i}}{2L_{ii}L_{jj}} e_c \right), \\
&= \frac{h_{,ii}}{L_{ii}} C_{,i} - \frac{L_{ii,i}h_{,i}}{2L_{ii}L_{ii}} C_{,i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{L_{jj}} \left( h_{,ji} - \frac{L_{jj,i}h_{,j}}{2L_{jj}} \right) C_{,j} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}h_{,j}}{2L_{ii}L_{jj}} C_{,i} \\
&\quad - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{L_{ii,j}h_{,i}}{2L_{ii}L_{jj}} C_{,j} + chC_{,i},
\end{aligned}$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{L_{jj}} \left(h_{,ji} - \Gamma_{ij}^{j} h_{,j} - \Gamma_{ij}^{i} h_{,i}\right) C_{,j} + \frac{1}{L_{ii}} \left(h_{,ii} - \Gamma_{ii}^{i} h_{,i} - \sum_{\substack{j=1\\ j\neq i}}^{n} \Gamma_{ii}^{j} h_{,j}\right) C_{,i}$$
$$+chC_{,i},$$
$$= \sum_{j=1}^{n} \left[\frac{1}{L_{jj}} \left(h_{,ji} - \sum_{l=1}^{n} \Gamma_{ij}^{l} h_{,l}\right) + ch\delta_{ij}\right] C_{,j}.$$

Using (3.17) in this expression, we have

$$D_{,i} = \sum_{j=1}^{n} V_{ij} C_{,j},\ 1 \leq i \leq n. \tag{3.34}$$

We observe that

$$X = C - \frac{2Dh}{S} \quad \text{and} \quad N = \frac{2D}{S}. \tag{3.35}$$

Thus, from (3.35) and using (3.28), (3.33) and (3.34), we get

$$\begin{aligned} X_{,i} &= C_{,i} - \frac{2h_{,i}D}{S} + \left(2\sum_{j=1}^{n} V_{ij} h_{,j}\right) \frac{2hD}{S^2} - \frac{2h}{S} \sum_{j=1}^{n} V_{ij} C_{,j}, \\ &= C_{,i} - h_{,i}N + \left(\sum_{j=1}^{n} V_{ij} h_{,j}\right) \frac{2hN}{S} - \frac{2h}{S} \sum_{j=1}^{n} V_{ij} C_{,j}, \\ &= C_{,i} - h_{,i}N + \frac{2h}{S} \sum_{j=1}^{n} V_{ij} \left(h_{,j}N - C_{,j}\right),\ 1 \leq i \leq n. \end{aligned} \tag{3.36}$$

and

$$N_{,i} = \frac{2}{S} \left(\sum_{j=1}^{n} V_{ij} C_{,j} - \left(\sum_{j=1}^{n} V_{ij} h_{,j}\right) \frac{2D}{S}\right) = \frac{2}{S} \sum_{j=1}^{n} V_{ij} \left(C_{,j} - h_{,j}N\right). \tag{3.37}$$

Since, $X_{,i} + h_{,i}N + hN_{,i} = C_{,i}$, for $1 \leq i \leq n$, we obtain

$$C_{,i} - h_{,i}N = X_{,i} + hN_i,\ \ 1 \leq i \leq n. \tag{3.38}$$

Using (3.38) in (3.37) and $N_{,i} = \sum_{k=1}^{n} W_{ik} X_{,k}$, we have

$$S \sum_{k=1}^{n} W_{ik} X_{,k} - 2h \sum_{j=1}^{n} V_{ij} \sum_{k=1}^{n} W_{jk} X_{,k} = 2 \sum_{k=1}^{n} V_{ik} X_{,k},$$

hence,

$$\sum_{k=1}^{n}\left[SW_{ik}-2h\sum_{j=1}^{n}V_{ij}W_{jk}\right]X_{,k}=2\sum_{k=1}^{n}V_{ik}X_{,k}.$$

Thus,

$$SW_{ik}-2h\sum_{j=1}^{n}V_{ij}W_{jk}=2V_{ik},$$

this equation can be written as

$$(SI_n-2hV)W=2V,$$

from this expression it follows (3.18).
On the other hand, $\langle C_{,k},C_{,l}\rangle=L_{kl}+h_{,k}h_{,l}$ and $\langle C_{,k},N\rangle=h_{,k}$, $1\leq k,l\leq n$.
Thus, using (3.36) the first fundamental form is given by

$$\begin{aligned}
I=\langle X_{,i},X_{,j}\rangle &= \left\langle C_{,i}-h_{,i}N+\frac{2h}{S}\sum_{k=1}^{n}V_{ik}\left(h_{,k}N-C_{,k}\right),C_{,j}-h_{,j}N\right.\\
&\quad\left.+\frac{2h}{S}\sum_{l=1}^{n}V_{jl}\left(h_{,l}N-C_{,l}\right)\right\rangle,\\
&= L_{ij}-\frac{2h}{S}\sum_{l=1}^{n}V_{jl}L_{il}-\frac{2h}{S}\sum_{k=1}^{n}V_{ik}L_{jk}+\frac{4h^2}{S^2}\sum_{k,l=1}^{n}V_{ik}V_{jl}L_{kl},\\
&= L_{ij}-\frac{2h}{S}V_{ji}L_{ii}-\frac{2h}{S}V_{ij}L_{jj}+\frac{4h^2}{S^2}\sum_{k=1}^{n}V_{ik}V_{jk}L_{kk},\ 1\leq i,j\leq n,
\end{aligned}$$

this equation is equivalent to (3.14).
From (3.36) and (3.37) the second fundamental form is given by

$$\begin{aligned}
II&=-\langle X_{,i},N_{,j}\rangle\\
&=\left\langle C_{,i}-h_{,i}N+\frac{2h}{S}\sum_{k=1}^{n}V_{ik}\left(h_{,k}N-C_{,k}\right),-\frac{2}{S}\sum_{l=1}^{n}V_{jl}(C_{,l}-h_{,l}N)\right\rangle,\\
&=-\frac{2}{S}\sum_{l=1}^{n}V_{jl}L_{il}+\frac{4h}{S^2}\sum_{k,l=1}^{n}V_{ik}V_{jl}L_{kl},\ 1\leq i,j\leq n,
\end{aligned}$$

this equation is equivalent to (3.15).
Finally, from (3.37) the third fundamental form is given by

$$III=\langle N_{,i},N_{,j}\rangle = \frac{4}{S^2}\left\langle\sum_{k=1}^{n}V_{ik}\left(C_{,k}-h_{,k}N\right),\sum_{l=1}^{n}V_{jl}\left(C_{,l}-h_{,l}N\right)\right\rangle,$$

$$= \frac{4}{S^2}\sum_{k,l=1}^{n} V_{ik}V_{jl}L_{kl}, \ \ 1 \leq i, j \leq n,$$

this equation is equivalent to (3.16). □

## 4. Applications.

**Proposition 4.1.** *Let $X : U \subset \mathbb{R}^n \to \Sigma \subset \overline{M}^{n+1}(c)$ be the parametrization of a hypersurface $\Sigma$ given by (3.11). Then the Gauss-Kronecker curvature $K$ of $\Sigma$ is given by*

$$K = \frac{2^n}{P}\det(V), \tag{4.1}$$

*where $P = \det(SI_n - 2hV)$.*

Proof. The Weingarten matrix $W$ of $\Sigma$ is given by (3.18), therefore,

$$K = \det(W) = \det(2V\,[SI_n - 2hV]^{-1}) = 2^n \det(V)\det([SI_n - 2hV]^{-1}). \quad \square$$

**Proposition 4.2.** *Let $X : U \subset \mathbb{R}^n \to \Sigma \subset \overline{M}^{n+1}(c)$ be the parametrization of a hypersurface $\Sigma$ given by* (3.11). *The following statements are equivalent*

(1) *$X$ is parametrized by lines of curvature.*

(2) *$V_{ij} = 0$, for $1 \leq i \neq j \leq n$.*

(3) *$N_{,i} = -k_{,i}X_{,i}$, for all $1 \leq i \leq n$, where*

$$k_i = \frac{2V_{ii}}{2hV_{ii} - S}, \quad 1 \leq i \leq n, \tag{4.2}$$

*are the principal curvatures of $X$.*

Proof. From (3.18), the Weingarten matrix $W$ is a diagonal matrix, if and only if, the matrix $V$ is a diagonal matrix. Therefore, $V$ is a diagonal matrix, if and only if, $X$ is parametrized by lines of curvature.
If $V_{ij} = 0$ for $1 \leq i \neq j \leq n$, then from (3.37) and (3.38), we get

$$N_{,i} = \frac{2}{S}V_{ii}(X_{,i} + hN_{,i}),$$

thus,

$$N_{,i} = \left( \frac{2V_{ii}}{S - 2hV_{ii}} \right) X_{,i}.$$

As $V$ is a diagonal matrix, $V_{ii}$ are the eigenvalues of $V$.
From (3.18), we have

$$-k_i = \frac{2V_{ii}}{S - 2hV_{ii}}, \quad 1 \leq i \leq n. \qquad \square$$

**Remark 4.3.** From (4.2), the eigenvalues $\sigma_i$ of $V$ are given by

$$\sigma_i = \frac{Sk_i}{2(hk_i - 1)}, \quad 1 \leq i \leq n, \tag{4.3}$$

where $k_i$ are the eigenvalues of the Weingarten matrix $W$.

**Example 4.4.** Let $Y : U \subset \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{S}^{n+1}$ be a local orthogonal parametrization of $\mathbb{S}^n$ given by

$$\begin{cases} u_1 = \cos(\phi_1), \\ u_2 = \sin(\phi_1)\cos(\phi_2), \\ u_3 = \sin(\phi_1)\sin(\phi_2)\cos(\phi_3), \\ \vdots \\ u_n = \sin(\phi_1)\sin(\phi_2)\cdots\cos(\phi_n), \\ u_{n+1} = \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_n), \\ u_{n+2} = 0. \end{cases} \tag{4.4}$$

Therefore,

$$\begin{aligned} Y_{,1} &= (-\sin(\phi_1), \cos(\phi_1)\cos(\phi_2), \cos(\phi_1)\sin(\phi_2)\cos(\phi_3), \ldots, \\ &\quad \cos(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\cos(\phi_n), \\ &\quad \cos(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-1})\sin(\phi_n), 0), \\ Y_{,2} &= (0, -\sin(\phi_1)\sin(\phi_2), -\sin(\phi_1)\cos(\phi_2)\cos(\phi_3), \ldots, \\ &\quad \sin(\phi_1)\cos(\phi_2)\cdots\sin(\phi_{n-1})\cos(\phi_n), \\ &\quad \sin(\phi_1)\cos(\phi_2)\cdots\sin(\phi_{n-1})\sin(\phi_n), 0), \\ Y_{,3} &= (0, 0, -\sin(\phi_1)\sin(\phi_2)\sin(\phi_3), \ldots, \\ &\quad \sin(\phi_1)\sin(\phi_2)\cos(\phi_3)\cdots\sin(\phi_{n-1})\cos(\phi_n), \\ &\quad \sin(\phi_1)\sin(\phi_2)\cos(\phi_3)\cdots\sin(\phi_{n-1})\sin(\phi_n), 0), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} Y_{,n-1} &= (0,0,\ldots,0,\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1})\cos(\phi_n), \\ &\quad \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1})\sin(\phi_n),0)\,, \\ Y_{,n} &= (0,0,\ldots,0,-\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1})\sin(\phi_n), \\ &\quad \sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1})\cos(\phi_n),0)\,. \end{aligned}$$

We observe that $L_{ij} = 0,\ i \neq j$ and for $1 \leq i \leq n$

(4.5) $L_{11} = 1,\ L_{22} = \sin^2(\phi_1), \ldots, L_{ii} = \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{i-1}).$

Thus, for $1 \leq i < j \leq n$, the Christoffel symbols are given by

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{L_{ii,j}}{2L_{ii}} = 0 \text{ and } \Gamma^j_{ij} = \Gamma^j_{ji} = \frac{L_{jj,i}}{2L_{jj}} = \cot(\phi_i),$$

using these expressions in (3.17), we get

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \cot(\phi_i)h_{,j}\right),\ 1 \leq i \neq j \leq n.$$

**Proposition 4.5.** *If $h \neq 0$, is a solution of the linear system of partial differential equations given by $h_{,ij} - \cot(\phi_i)h_{,j} = 0$, $1 \leq i \neq j \leq n$. Then the hypersurface $\Sigma$ of $\mathbb{S}^{n+1}$ given by* (3.11) *is parametrized by lines of curvature.*

Proof. It follows from Proposition 4.2. □

**Remark 4.6.** Let $Y : (0,\pi) \times (0,2\pi) \subset \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{S}^3$ be a local orthogonal parametrization of $\mathbb{S}^2$ given by (4.4). If $h(\phi_1,\phi_2) = \sin(\phi_1)f(\phi_2) + g(\phi_1)$, where $f$, $g$ are real-valued functions, then the hypersurface given by (3.11) is parametrized by lines of curvature.

**Example 4.7.** Let $Y : U \subset \mathbb{R}^n \to \mathbb{H}^n \subset \mathbb{H}^{n+1}$ be a local orthogonal parametrization of $\mathbb{H}^n$ given by

$$(4.6)\qquad \begin{cases} u_1 = \sinh(\phi_1), \\ u_2 = \cosh(\phi_1)\sinh(\phi_2), \\ u_3 = \cosh(\phi_1)\cosh(\phi_2)\sinh(\phi_3), \\ \vdots \\ u_n = \cosh(\phi_1)\cosh(\phi_2)\cdots\sinh(\phi_n), \\ u_{n+1} = 0, \\ u_{n+2} = \cosh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_n). \end{cases}$$

Hence,

$$\begin{aligned}
Y_{,1} &= (\cosh(\phi_1), \sinh(\phi_1)\sinh(\phi_2), \sinh(\phi_1)\cosh(\phi_2)\sinh(\phi_3), \dots, \\
&\quad \sinh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_{n-1})\sinh(\phi_n), 0, \sinh(\phi_1)\cosh(\phi_2)\cdots \\
&\quad \cdots\cosh(\phi_{n-1})\cosh(\phi_n)), \\
Y_{,2} &= (0, \cosh(\phi_1)\cosh(\phi_2), \cosh(\phi_1)\sinh(\phi_2)\sinh(\phi_3), \dots, \\
&\quad \cosh(\phi_1)\sinh(\phi_2)\cdots\cosh(\phi_{n-1})\sinh(\phi_n), 0, \cosh(\phi_1)\sinh(\phi_2)\cdots \\
&\quad \cdots\cosh(\phi_{n-1})\cosh(\phi_n)), \\
Y_{,3} &= (0, 0, \cosh(\phi_1)\cosh(\phi_2)\cosh(\phi_3), \dots, \cosh(\phi_1)\cosh(\phi_2)\sinh(\phi_3)\cdots \\
&\quad \cdots\cosh(\phi_{n-1})\sinh(\phi_n), 0, \cosh(\phi_1)\cosh(\phi_2)\sinh(\phi_3)\cdots \\
&\quad \cdots\cosh(\phi_{n-1})\cosh(\phi_n)), \\
&\ \ \vdots \\
Y_{,n-1} &= (0, 0, \dots, 0, \cosh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_{n-2})\sinh(\phi_{n-1})\sinh(\phi_n), \\
&\quad 0, \cosh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_{n-2})\sinh(\phi_{n-1})\cosh(\phi_n)), \\
Y_{,n} &= (0, 0, \dots, 0, \cosh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_{n-2})\cosh(\phi_{n-1})\cosh(\phi_n), \\
&\quad 0, \cosh(\phi_1)\cosh(\phi_2)\cdots\cosh(\phi_{n-2})\cosh(\phi_{n-1})\sinh(\phi_n)).
\end{aligned}$$

We get that $L_{ij} = 0$, $i \neq j$ and for $1 \leq i \leq n$

$$L_{11} = 1,\ L_{22} = \cosh^2(\phi_1), \dots, L_{ii} = \cosh^2(\phi_1)\cosh^2(\phi_2)\cdots\cosh^2(\phi_{i-1}). \tag{4.7}$$

Thus, for $1 \leq i < j \leq n$, the Christoffel symbols are given by

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{L_{ii,j}}{2L_{ii}} = 0 \quad \text{and} \quad \Gamma^j_{ij} = \Gamma^j_{ji} = \frac{L_{jj,i}}{2L_{jj}} = \tanh(\phi_i),$$

using these expressions in (3.17), we have

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \tanh(\phi_i)h_{,j}\right), \qquad 1 \leq i \neq j \leq n.$$

**Proposition 4.8.** *If $h \neq 0$, is a solution of the linear system of partial differential equations given by $h_{,ij} - \tanh(\phi_i)h_{,j} = 0$, $1 \leq i \neq j \leq n$. Then the hypersurface $\Sigma$ of $\mathbb{H}^{n+1}$ given by (3.11) is parametrized by lines of curvature.*

Proof. It follows from Proposition 4.2. □

**Remark 4.9.** Let $Y : U \subset \mathbb{R}^2 \to \mathbb{H}^2 \subset \mathbb{H}^3$ be a local orthogonal parametrization of $\mathbb{H}^2$ given by (4.6). If $h(\phi_1, \phi_2) = \cosh(\phi_1)f(\phi_2) + g(\phi_1)$, where $f$, $g$ are real-valued functions, then the hypersurface given by (3.11) is parametrized by lines of curvature.

As a consequence of the Remark 4.3, for $n = 2$, we get

$$V_{11} + V_{22} = \sigma_1 + \sigma_2 = \frac{Sk_1}{2(hk_1 - 1)} + \frac{Sk_2}{2(hk_2 - 1)} = \frac{S(hK - H)}{(hk_1 - 1)(hk_2 - 1)}, \tag{4.8}$$

where $H, K$ are the mean curvature and the Gaussian curvature of $\Sigma$ in $\overline{M}^3(c)$, respectively, $h$ is the radius function and $S$ is given by (3.13). Thus, $V_{11}+V_{22} = 0$, if and only if, $h = \frac{H}{K}$, with $K \neq 0$.

**Definition 4.10.** *A surface $\Sigma \subset \overline{M}^3(c)$ is a* surface of the spherical type *in $\overline{M}^3(c)$, if there exist a congruence of geodesic spheres between $\Sigma$ and $M^2(c)$ with radius function $h = \frac{H}{K}$, where $H, K$ are the mean curvature and Gaussian curvature of $\Sigma$ in $\overline{M}^3(c)$.*

**Definition 4.11.** *Let $\Sigma$ be a hypersurface of $\overline{M}^{n+1}(c)$, $n \geq 2$, $\Sigma$ is a* Weingarten hypersurface of the spherical type *in $\overline{M}^{n+1}(c)$, if the rth mean curvatures of $\Sigma$ in $\overline{M}^{n+1}(c)$ satisfy the equation*

$$\sum_{r=1}^{n} (-1)^{r-1} r f^{r-1} H_r = 0,$$

*for some function $f \in C^{\infty}(\Sigma, \mathbb{R})$.*

The following Lemma was obtained in [13].

**Lemma 4.12.** *Define $P_i$, for $1 \leq i \leq n$ and $n \geq 2$,*

$$P_i = (1 - hk_1)(1 - hk_2) \cdots \overbrace{(1 - hk_i)} \cdots (1 - hk_{n-1})(1 - hk_n),$$

*here $\overbrace{(1 - hk_i)}$ means that the factor is absent in the expression and $h, k_i : U \subset \mathbb{R}^n \to \mathbb{R}$, where $U$ is an open of $\mathbb{R}^n$. Then*

$$P_i = 1 - h(S_1 - k_i) + h^2 \left( S_2 - \sum_{\substack{1 \leq j_1 \leq n \\ j_1 \neq i}} k_{j_1} k_i \right) - h^3 \left( S_3 - \sum_{\substack{1 \leq j_1 < j_2 \leq n \\ j_1, j_2 \neq i}} k_{j_1} k_{j_2} k_i \right)$$

$$+ \quad \cdots + (-1)^{n-1}h^{n-1}\left(S_{n-1} - \sum_{\substack{1\le j_1<\cdots<j_{n-2}\le n\\ j_1,\ldots,j_{n-2}\ne i}} k_{j_1}k_{j_2}\cdots k_{j_{n-2}}k_i\right),$$

*where*

$$S_r = \sum_{1\le j_1<\cdots<j_r\le n} k_{j_1}k_{j_2}\cdots k_{j_r},\ 1\le r\le n.$$

The following result characterizes the Weingarten hypersurfaces of the spherical type in terms of the trace of the matrix $V$.

**Proposition 4.13.** *Let $\Sigma$ be an orientable hypersurface of $\overline{M}^{n+1}(c)$, $n\ge 2$, given by the Theorem* 3.3. *Then $\Sigma$ is a Weingarten hypersurface of the spherical type in $\overline{M}^{n+1}(c)$, if and only if, $\sum_{i=1}^{n} V_{ii} = 0$.*

Proof. From Remark 4.3, we get

$$\begin{aligned}\sum_{i=1}^{n} V_{ii} &= \frac{S}{2}\left[\frac{k_1}{(hk_1-1)} + \frac{k_2}{(hk_2-1)} + \cdots + \frac{k_n}{(hk_n-1)}\right]\\ &= -\frac{S}{2}\left[\frac{k_1P_1 + k_2P_2 + \cdots + k_nP_n}{(1-hk_1)(1-hk_2)\cdots(1-hk_n)}\right],\end{aligned}$$

therefore, $\sum_{i=1}^{n} V_{ii} = 0$, if and only if, $k_1P_1 + k_2P_2 + \cdots + k_nP_n = 0$.
From Lemma 4.12, we have that

$$\begin{aligned}0 = {}& k_1 - k_1hS_1 + k_1hk_1 + k_1h^2S_2 - k_1h^2\sum_{\substack{1\le j_1\le n\\ j_1\ne i}} k_{j_1}k_1 + \cdots\\ &+ (-1)^{n-1}k_1h^{n-1}S_{n-1} - (-1)^{n-1}k_1h^{n-1}\sum_{\substack{1\le j_1<\cdots<j_{n-2}\le n\\ j_1,\ldots,j_{n-2}\ne i}} k_{j_1}\cdots k_{j_{n-2}}k_1 + \cdots\\ &+ k_n - k_nhS_1 + k_nhk_n + k_nh^2S_2 - k_nh^2\sum_{\substack{1\le j_1\le n\\ j_1\ne i}} k_{j_1}k_n + \cdots + (-1)^{n-1}k_nh^{n-1}S_{n-1}\\ &- (-1)^{n-1}k_nh^{n-1}\sum_{\substack{1\le j_1<\cdots j_{n-2}\le n\\ j_1,\ldots,j_{n-2}\ne i}} k_{j_1}\cdots k_{j_{n-2}}k_n,\end{aligned}$$

thus,

$$0 = (k_1 + k_2 + \cdots + k_n) - h\left(k_1S_1 - k_1^2 + k_2S_1 - k_2^2 + \cdots + k_nS_1 - k_n^2\right)$$

$$+ h^2 \left( k_1 S_2 - k_1 \sum_{\substack{1\le j_1\le n\\ j_1\ne i}} k_{j_1}k_1 + k_2S_2 - k_2 \sum_{\substack{1\le j_1\le n\\ j_1\ne i}} k_{j_1}k_2 + \cdots + k_nS_2 \right.$$

$$\left. - k_n \sum_{\substack{1\le j_1\le n\\ j_1\ne i}} k_{j_1}k_n \right)$$

$$+ \cdots (-1)^{n-1}h^{n-1} \left( k_1S_{n-1} - k_1 \sum_{\substack{1\le j_1<\cdots j_{n-2}\le n\\ j_1,\ldots,j_{n-}\ne i}} k_{j_1}\cdots k_{j_{n-2}}k_1 + \cdots + k_nS_{n-1} \right.$$

$$\left. - k_n \sum_{\substack{1\le j_1<\cdots j_{n-2}\le n\\ j_1,\ldots,j_{n-2}\ne i}} k_{j_1}\cdots k_{j_{n-2}}k_n \right)$$

$$0 = S_1 - 2hS_2 + 3h^2S_3 + \cdots + (-1)^{n-1}h^{n-1}nS_n.$$

From Definition 4.11, it follows the result. □

**Remark 4.14.** Let $Y$ be a local orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$, given by

$$Y = \begin{cases} P_-^{-1}, P_+^{-1} : \mathbb{R}^n \to \mathbb{S}^n, \text{ if } c = 1, \\ I : \mathbb{R}^n \to \mathbb{R}^n, \text{ if } c = 0, \\ P^{-1} : B^n(1) \to \mathbb{H}^n, \text{ if } c = -1, \end{cases} \tag{4.9}$$

where $P_-^{-1}, P_+^{-1}$ are given by (2.3), $I$ is the identity function of $\mathbb{R}^n$ and $P^{-1}$ is given by (2.6). Let $\overline{J}_c$ be the function defined by

$$\overline{J}_c(u) = \begin{cases} \dfrac{4}{(1+\langle u,u\rangle)^2}, \quad u \in \mathbb{R}^n, & \text{if } c = 1, \\ 1, \ u \in \mathbb{R}^n, & \text{if } c = 0, \\ \dfrac{4}{(1-\langle u,u\rangle)^2}, \quad u \in B^n(1), & \text{if } c = -1. \end{cases} \tag{4.10}$$

From (2.1), the Christoffel symbols associated to $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$ are given by

$$\Gamma^i_{ii} = \frac{\overline{J}_{c,i}}{2\overline{J}_c} \quad \text{and} \quad \Gamma^i_{ij} = \frac{\overline{J}_{c,j}}{2\overline{J}_c} = -\Gamma^j_{ii}, \quad 1 \le i \ne j \le n. \tag{4.11}$$

**Proposition 4.15.** *Let $\Sigma$ be an orientable hypersurface of $\overline{M}^{n+1}(c)$, $n \geq 2$ given by Theorem 3.3. Then $\Sigma$ is a Weingarten hypersurface of the spherical type in $\overline{M}^{n+1}(c)$, if and only if, $h$ is a solution of the equation given by*

$$\frac{\triangle h}{\overline{J}_c} + \frac{(n-2)}{2\overline{J}_c^2}\left\langle \nabla \overline{J}_c, \nabla h \right\rangle + nch = 0, \tag{4.12}$$

*where $\overline{J}_c$ is defined by* (4.10).

Proof. By Proposition 4.13, we will show that $\sum_{i=1}^{n} V_{ii} = 0$. From (3.17) and Remark 4.14, we have that

$$V_{ii} = \frac{1}{\overline{J}_c}\left(h_{,ii} + \frac{\overline{J}_{c,1}}{2\overline{J}_c}h_{,1} + \cdots + \frac{\overline{J}_{c,i-1}}{2\overline{J}_c}h_{,i-1} - \frac{\overline{J}_{c,i}}{2\overline{J}_c}h_{,i} + \frac{\overline{J}_{c,i+1}}{2\overline{J}_c}h_{,i+1} + \cdots + \frac{\overline{J}_{c,n}}{2\overline{J}_c}h_{,n}\right) + ch,$$

therefore,

$$\sum_{i=1}^{n} V_{ii} = \frac{1}{\overline{J}_c}\left(h_{,11} + \cdots + h_{,nn}\right) + \frac{(n-2)}{2\overline{J}_c^2}\left(\overline{J}_{c,1}h_{,1} + \cdots + \overline{J}_{c,n}h_{,n}\right) + nch,$$

hence, it follows the result. □

**Remark 4.16.** From Proposition 4.15, we have:

1. For $n = 2$, the equation (4.12) is reduced to

$$\triangle h + 2c\overline{J}_c h = 0, \tag{4.13}$$

   this equation is known as the Helmholtz equation, solutions for this equation were found in [8]. Using these solutions we find explicit examples of Weingarten surfaces of the spherical type in form spaces.

2. For $c = 0$, $\Sigma \subset \mathbb{R}^{n+1}$ is a Weingarten hypersurface of the spherical type if, and only if, $h$ is harmonic, which coincides with the result obtained in [13].

**Theorem 4.17.** *Let $\Sigma$ be an orientable hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 3.3. Then $\Sigma$ is a rotation spherical hypersurface of $\overline{M}^{n+1}(c)$ (see [9]) if, and only if, $h$ is a radial function.*

Proof. The proof for $c = 0$, can be seen at [13].
For $c = -1, 1$, without loss of generality we assume that the plane of rotation

is given by $[e_c, Y]$. Let $\Sigma$ be a rotation spherical hypersurface. Therefore, the orthogonal sections of $\Sigma$ with parallels hyperplane to $[Y_{,1}, Y_{,2}, \ldots, Y_{,n}]$, determine in $\Sigma$ $(n-1)$-dimensional spheres centered in $[e_c, Y]$. Let $\overline{\Sigma}$ be, the intersection between $\Sigma$ and a parallels hyperplane to $[Y_{,1}, Y_{,2}, \ldots, Y_{,n}]$, thus, from (3.11), $\overline{\Sigma}$ can be parametrized by

$$\overline{X} = -\frac{2h}{S}\left(\sum_{i=1}^{n} \frac{h_{,i}}{L_{ii}} Y_{,i}\right).$$

Thus,

$$\left|\overline{X}\right|^2 = \langle \overline{X}, \overline{X} \rangle = \left(\frac{2h}{S}\right)^2 \left(\sum_{i=1}^{n} \frac{h_{,i}^2}{L_{ii}}\right) = h^2 \left(\frac{4}{S^2}\sum_{i=1}^{n} \frac{h_{,i}^2}{L_{ii}}\right) = A,$$

where $A \geq 0$ is constant.

If $A = 0$ and $h \neq 0$, then, $\sum_{i=1}^{n} h_{,i}^2 = 0$, hence, $h_{,i} = 0$ for all $1 \leq i \leq n$, thus, $h$ is constant.

If $A > 0$, we have that $h^2\left(\frac{4}{S^2}\sum_{i=1}^{n} \frac{h_{,i}^2}{L_{ii}}\right)$ is constant on $(n-1)$-dimensional spheres, since $\frac{1}{h^2} \neq \frac{4}{S^2}\sum_{i=1}^{n}\frac{h_{,i}^2}{L_{ii}}$, this implies that $h$ is constant on $(n-1)$-dimensional spheres. Consequently, $h$ is a radial function.

Conversely, let $Y$ be the parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$ given by Remark 4.14, then, $L_{ii} = \overline{J}_c$ for all $1 \leq i \leq n$, where $\overline{J}_c$ is the radial function given by (4.10). Suppose that $h$ is a radial function, then

$$S = \sum_{i=1}^{n} \frac{h_{,i}^2}{L_{ii}} + ch^2 + 1 = \sum_{i=1}^{n} \frac{(2h'u_i)^2}{\overline{J}_c} + ch^2 + 1 = \frac{4(h')^2}{\overline{J}_c}|u|^2 + ch^2 + 1,$$

therefore, $S$ is a radial function.

On the other hand,

$$\langle \overline{X}, \overline{X} \rangle = h^2\left(\frac{4}{S^2}\sum_{i=1}^{n}\frac{h_{,i}^2}{L_{ii}}\right) = h^2\left(\frac{4}{S^2}\sum_{i=1}^{n}\frac{4(h')^2u_i^2}{\overline{J}_c}\right) = \frac{16h^2(h')^2}{S^2\overline{J}_c}|u|^2.$$

Thus, $\langle \overline{X}, \overline{X} \rangle$ is constant on $(n-1)$-dimensional spheres. It implies that $\Sigma$ is a rotation spherical hypersurface centered on $[e_c, Y]$. □

**Remark 4.18.** For $n \geq 2$, suppose that $h$ is a radial function i.e. $h(u_1, \ldots, u_n) = f(r)$, $r = u_1^2 + \cdots + u_n^2$.
For the case $c = 0$, we obtain that the equation (4.12) is satisfied if, and only if, $h$ is harmonic and explicit solutions of $h$ are obtained (see [13]).
For the case $c = \pm 1$, using (4.10), the equation (4.12) is reduced to

$$2rf'' + \frac{f'(n + cr(4-n))}{1+cr} + \frac{2cnf}{(1+cr)^2} = 0. \tag{4.14}$$

i) For $n = 2$, we have

$$rf'' + f' + \frac{2cf}{(1+cr)^2} = 0. \tag{4.15}$$

The solutions of the equation (4.15) are given by
If $c = 1$, $\overline{M}^3(c) = \mathbb{S}^3$, thus,

$$f(r) = \frac{(r-1)C_1 + C_2(\ln(r)(r-1) - 4)}{1+r}. \tag{4.16}$$

If $c = -1$, $\overline{M}^3(c) = \mathbb{H}^3$, thus,

$$f(r) = \frac{(r+1)C_1 + C_2(\ln(r)(r+1) + 4)}{r-1}. \tag{4.17}$$

ii) For $n \geq 3$, the solutions to the equation (4.14) are given by
(4.18)

$$f(r) = \frac{2(c^2r - c)^{\frac{n}{2}} \text{AppellF1}\left[\frac{2-n}{2}, -n, \frac{n}{2}, \frac{4-n}{2}, \frac{2}{1-cr}, \frac{1}{1-cr}\right] C_2}{c^2(n-2)(1+cr)} + \frac{(cr-1)C_1}{c(1+cr)},$$

where $C_1, C_2 \in \mathbb{R}$,, AppellF1 is the Appell hypergeometric function. With these solutions we classify the Weingarten hypersurfaces of the spherical type of rotation in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$.

The next result classify the Weingarten hypersurface of the spherical type of rotation in form spaces and generalizes the result obtained in [13].

**Theorem 4.19.** *Let $Y$ be a orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$, given by* (4.9)*, $h : U \to \mathbb{R}$ a differentiable function and $X : U \to M^n(c)$ the immersion given by* (3.11) *with Gauss map $N$ given by* (3.12)*. Under these conditions $X(U)$ is a Weingarten hypersurface of the spherical type of rotation*

*if, and only if, h is given by*
*For* $c = 0$,

$$h(u) = \begin{cases} C\ln(u_1^2 + u_2^2) + D, & \text{if} \quad n = 2, \\ \dfrac{2C}{2-n}(u_1^2 + \cdots + u_n^2)^{\frac{2-n}{2}} + D, & \text{if} \quad n \geq 3, \end{cases}$$

*where* $C$ *and* $D$ *are constants,* $C > 0$.
*For* $c = \pm 1$, $h$ *is given by* (4.16), (4.17) *and* (4.18).

Proof. The proof for $c = 0$, can be seen at [13] and for $c = \pm 1$, it follows from Remark 4.18. □

## REFERENCES

[1] A. L. Barbosa. Possibilidade de confinamento no modelo SU(2)-Cor, Dissertation, Universidade Estadual Paulista (UNESP), 1994.

[2] W. Blaschke. Über die geometrie von Laguerre: I. grundformeln der flächentheorie. *Abh. Math. Sem. Univ. Hamburg.* **3**, *1* (1924), 176–194.

[3] W. Blaschke. Über die geometrie von Laguerre: II. flächentheorie in ebenenkoordinaten. *Abh. Math. Sem. Univ. Hamburg.* **3**, *1* (1924), 195–212.

[4] W. Blaschke. Über die geometrie von Laguerre: III. beiträge zur flächentheorie. *Abh. Math. Sem. Univ. Hamburg.* **4** (1925), 1–12.

[5] W. Blaschke. Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. B. 3, Bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.

[6] A. V. Corro. Generalized Weingarten surfaces of bryant type in hyperbolic 3-space. *Mat. Comtemp.* **30** (2006), 71–89.

[7] A. V. Corro, K. V. Fernandes, C. M. C. Riveros. Generalized Weingarten surfaces of harmonic type in hyperbolic 3-space, *Differential Geom. Appl.* **58** (2018), 202–226.

[8] A. V. Corro, C. M. C. Riveros. Generalized Helmholtz equation. *Selecciones Matemáticas* **6**, *1* (2019), 19–25, `http://dx.doi.org/10.17268/sel.mat.2019.01.03`.

[9] M. do Carmo, M. Dacjczer. Rotation hypersurfaces in spaces of constant curvature. In: Manfredo P. do Carmo – Selected Papers (Ed. Tenenblat K.) Berlin, Heidelberg, Springer, 2012, 195–219.

[10] W. Ferreira, P. Roitman. Hypersurfaces in hyperbolic space associated with the conformal scalar curvature equation $\Delta u + ku^{\frac{n+2}{n-2}} = 0$. *Differential Geom. Appl.* **27**, *2* (2009), 279–295.

[11] A. S. Fokas, I. M. Gelfand. Surfaces on Lie groups, on Lie algebras, and their integrability. *Comm. Math. Phys.* **177**, *1* (1996), 203–220.

[12] J. A. Gálvez, A. Martínez, F. Milán. Complete linear Weingarten surfaces of bryant type. a plateau problem at infinity. *Trans. Amer. Math. Soc.* **356**, *9* (2004), 3405–3428.

[13] C. D. F. Machado. Hipersuperfícies Weingarten de tipo esférico. Thesis, Universidade de Brasíla, 2018.

[14] H. Pottmann, P. Grohs, N. J. Mitra. Laguerre minimal surfaces, isotropic geometry and linear elasticity. *Adv. Comput. Math.* **31**, *4* (2009), 391–419.

[15] K. Tenenblat, Q. Wang. Ribaucour transformations for hypersurfaces in space forms. *Ann. Global Anal. Geom.* **29**, *2* (2006), 157–185.

*Edwin O. S. Reyes*
*Centro das Ciências Exatas e das Tecnologias*
*Universidade Federal do Oeste da Bahia*
*47808-021 Barreiras-BA, Brazil*
*e-mail:* edwin.reyes@ufob.edu.br

*Carlos M. C. Riveros*
*Departamento de Matemática*
*Universidade de Brasília*
*70910-900, Brasília-DF, Brazil*
*e-mail:* carlos@mat.unb.br

*Received September 9, 2019*