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ON THE CHOW RING OF FANO VARIETIES OF TYPE *S*6

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Abstract. Fatighenti and Mongardi have defined Fano varieties of type S6 as zero loci of a certain vector bundle on the Grassmannian Gr(2, 10). These varieties have 3 Hodge structures of K3 type in their cohomology. We show that the Chow ring of these varieties also displays "K3 type" behaviour.

Introduction. For a smooth projective variety X over $\mathbb{C}$, we write $A^i(X) = CH^i(X)_{\mathbb{Q}}$ for the Chow group of codimension i algebraic cycles modulo rational equivalence with $\mathbb{Q}$-coefficients, and $A^i_{hom}(X)$ for the subgroup of homologically trivial cycles. Intersection product defines a ring structure on $A^*(X) = \oplus_i A^i(X)$. In the case of K3 surfaces, this ring structure has a curious property:

Theorem 1 (Beauville–Voisin [2])**.** *Let S be a projective K3 surface. Let $R^*(S) \subset A^*(S)$ be the $\mathbb{Q}$-subalgebra generated by $A^1(S)$ and the Chow-theoretic Chern class $c_2(S)$. The cycle class map induces an injection*

$$R^*(S) \hookrightarrow H^*(S, \mathbb{Q}).$$

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This note is about the Chow ring of Fano varieties *of type S6* in the terminology of Fatighenti–Mongardi [7]. By definition, a Fano variety of type S6 is the smooth dimensionally transverse zero locus of a global section of the bundle $\mathcal{Q}^*(1)$ on the Grassmannian $\mathrm{Gr}(2,10)$ (here $\mathcal{Q}$ is the universal quotient bundle). Varieties of type S6 are 8-dimensional, and their Hodge diamond looks like

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & 1 & & & & \\
 & & & & 2 & & & & \\
 & & & 1 & 22 & 1 & & & \\
0 & \ldots & \ldots & 1 & 23 & 1 & \ldots & \ldots & 0 \\
 & & & 1 & 22 & 1 & & & \\
 & & & & 2 & & & & \\
 & & & & 1 & & & & \\
 & & & & 1 & & & &
\end{array}$$

(where all empty entries are 0). The remarkable thing is that these varieties have 3 Hodge structures of K3 type in their cohomology. As shown in [7, Proposition 3.29] (cf. also [3, Section 4]), general varieties of type S6 are related to the hyperkähler fourfolds of Debarre–Voisin [6], which somehow explains these Hodge structures.

The goal of this note is to see how the peculiar shape of this Hodge diamond translates into peculiar properties of the Chow ring. A first result is as follows:

Theorem 2. *Let X be a Fano variety of type S6. Then*

$$A^i_{hom}(X)=0 \quad \forall i\notin\{4,5,6\}.$$

Moreover, for X general intersecting with an ample divisor h induces isomorphisms

$$\begin{aligned}
\cdot h\colon\quad A^4_{hom}(X) &\xrightarrow{\cong} A^5_{hom}(X)\ ,\\
\cdot h\colon\quad A^5_{hom}(X) &\xrightarrow{\cong} A^6_{hom}(X).
\end{aligned}$$

This is in accordance with the Bloch–Beilinson conjectures [9]. (Indeed, the fact that $A^8_{hom}(X)=A^7_{hom}(X)=0$ corresponds to the fact that $h^{p,q}(X)=0$ for $p\neq q, p+q\leq 4$. The fact that $A^4_{hom}(X)\cong A^5_{hom}(X)\cong A^6_{hom}(X)$ corresponds to the fact that cupping with an ample divisor induces isomorphisms in transcendental cohomology $H^6_{tr}(X)\cong H^8_{tr}(X)\cong H^{10}_{tr}(X)$).

A second result concerns the ring structure of the Chow ring:

Theorem 3. *Let X be a Fano variety of type S6. Let $R^*(X) \subset A^*(X)$ denote the $\mathbb{Q}$-subalgebra*

$$R^*(X) := \langle A^1(X), A^2(X), c_j(X), Im\big(A^j(\mathrm{Gr}(2,10)) \to A^j(X)\big)\rangle \quad \subset A^*(X).$$

The cycle class map induces an injection

$$R^*(X) \hookrightarrow H^*(X, \mathbb{Q}).$$

For X general, we also prove that $A^2(X) \cdot A^3(X)$ injects into cohomology (Proposition 3.3). This is reminiscent of the behaviour of the Chow ring of a K3 surface (Theorem 1). Theorem 3 suggests that X might perhaps have a multiplicative Chow–Künneth decomposition in the sense of [18], which would be a manifestation of the fact that "X is close to K3 surfaces". Establishing this seems difficult however (cf. Remark 3.2 below).

Conventions. *In this note, the word variety will refer to a reduced irreducible scheme of finite type over the field of complex numbers $\mathbb{C}$.* **All Chow groups will be with $\mathbb{Q}$-coefficients, unless indicated otherwise:** *For a variety X, we will write $A_j(X) := CH_j(X)_{\mathbb{Q}}$ for the Chow group of dimension j cycles on X with rational coefficients. For X smooth of dimension n, the notations $A_j(X)$ and $A^{n-j}(X)$ will be used interchangeably. The notation $A^j_{hom}(X)$ will be used to indicate the subgroups of homologically trivial cycles.*

We will write $\mathcal{M}_{\mathrm{rat}}$ for the contravariant category of Chow motives (i.e., pure motives as in [17], [14]*).*

1. Preliminaries.

1.1. Transcendental part of the motive. We recall a classical result concerning the motive of a surface:

Theorem 1.1 (Kahn–Murre–Pedrini [11])**.** *Let S be a surface. There exists a decomposition*

$$\mathfrak{h}^2(S) = \mathfrak{t}(S) \oplus \mathfrak{h}^2_{alg}(S) \quad \textit{in } \mathcal{M}_{\mathrm{rat}},$$

such that

$$H^*(\mathfrak{t}(S), \mathbb{Q}) = H^2_{tr}(S), \quad H^*(\mathfrak{h}^2_{alg}(S), \mathbb{Q}) = NS(S)_{\mathbb{Q}}$$

(here $H^2_{tr}(S)$ is defined as the orthogonal complement of the Néron–Severi group $NS(S)_{\mathbb{Q}}$ in $H^2(S, \mathbb{Q})$), and

$$A^*(\mathfrak{t}(S)) = A^2_{AJ}(S).$$

(The motive $\mathfrak{t}(S)$ is called the transcendental part of the motive*.)*

The following provides a higher-dimensional version:

Proposition 1.2. *Let X be a smooth projective variety of dimension n, and assume X is a complete intersection in a variety with trivial Chow groups. There exists a decomposition*

$$\mathfrak{h}(X) = \mathfrak{t}(X) \oplus \bigoplus_j \mathbb{1}(j) \quad \text{in } \mathcal{M}_{\text{rat}},$$

such that

$$H^*(\mathfrak{t}(X), \mathbb{Q}) = H^*_{tr}(X)$$

(here the transcendental part $H^*_{tr}(X)$ is defined as the orthogonal complement of the algebraic part $N^*(X) := Im\big(A^*(X) \to H^*(X, \mathbb{Q})\big)$), and

$$A^*(\mathfrak{t}(X)) = A^*_{hom}(X).$$

(The motive $\mathfrak{t}(X)$ will be called the transcendental part of the motive*.)*

Proof. This is a standard construction (cf. [5, Section 2], where this decomposition is constructed for cubic fourfolds). One can apply [19, Theorem 1], where projectors π^j_{alg} on the algebraic part of cohomology are constructed for any smooth projective variety satisfying the standard conjectures. The motive $\mathfrak{t}(X)$ is then defined by the projector $\Delta_X - \sum_j \pi^j_{alg}$. □

1.2. Voevodsky motives.

Definition 1.3. *Let* DM *be the triangulated category*

$$\text{DM} := \text{DM}^{\text{eff},-}_{\text{Nis}}(\mathbb{C}, \mathbb{Z})$$

as defined in [13, *Definition 14.1*].

For any (not necessarily smooth) variety X, let $M^c(X) := z_{equi}(X, 0) \in$ DM *denote the* motive with compact support *as in* [13, *Definition 16.13*]. *Here $z_{equi}(X, 0)$ denotes the sheaf of equidimensional cycles of relative dimension* 0 [13, *Definition 16.1*], *considered as object of* DM.

Given $M \in$ DM *and $n \in \mathbb{Z}$, there are objects $M[n] \in$* DM *(this corresponds to shifting the degrees of the complex defining M), and $M(n) \in$* DM *(this corresponds to the "Tate twist" in $\mathcal{M}_{\text{rat}}$).*

Remark 1.4. There is a contravariant fully faithful functor

$$\mathcal{M}_{\text{rat}} \ \to \ \text{DM},$$

sending the motive $h(X) = (X, \Delta_X, 0)$ to the motive $M^c(X)$ and sending $\mathbb{1}(n)$ to $\mathbb{1}(n)[2n]$ [13, Chapter 20].

If X is smooth projective, there is an isomorphism $M(X) \cong M^c(X)$ [13, Example 16.2], where the motive $M(X)$ is as in [13, Definition 14.1].

For any closed immersion $Y \subset X$ with complement $U := X \setminus Y$, there is a "Gysin" distinguished triangle in DM

$$M^c(Y) \ \to \ M^c(X) \ \to \ M^c(U) \ \xrightarrow{[1]}$$

[13, Theorem 16.15].

The functor $M^c(-)$ is contravariant with respect to étale morphisms, and covariant for proper morphisms. Given X smooth and $h \in A^1(X)$ ample, one can define functorial maps $\cdot h^j \colon M^c(X) \to M^c(X)(j)[2j]$, that correspond to intersecting with h^j.

Proposition 1.5. *Let X be a smooth quasi-projective variety, and let $p\colon P \to X$ be a $\mathbb{P}^r$-bundle. Let $h \in A^1(P)$ be ample. There is an isomorphism in* DM

$$\alpha := \sum_{j=0}^{r} p_* \cdot h^j \colon \quad M^c(P) \ \xrightarrow{\cong} \ \bigoplus_{j=0}^{r} M^c(X)(j)[2j].$$

P r o o f. This is dual to the projective bundle formula for $M(X)$ given in [13, Theorem 15.12]. Using the Gysin distinguished triangle [13, Exercice 16.18], one reduces to the case $P = X \times \mathbb{P}^r$. Using the isomorphism $M^c(X \times \mathbb{P}^r) \cong M^c(X) \otimes M^c(\mathbb{P}^r)$ [13, Corollary 16.16], one reduces to the case where X is a point, which follows from $M^c(\mathbb{A}^i) = \mathbb{1}(i)[2i]$. □

Lemma 1.6. *Let X, Y be smooth projective varieties, let $j \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathrm{DM}}\big(M(X)(k)[2k], M(Y)(j)[2j+1]\big) = 0.$$

P r o o f. For any smooth projective varieties X, Y with $d := \dim Y$, and any $k \geq 0$ one has

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}}&\big(M(X)(k)[2k],\ M(Y)(j)[2j+1]\big) \\
&= \mathrm{Hom}_{\mathrm{DM}}\big(M(X \times Y)(k-d)[2k-2d], \mathbb{1}(j)[2j+1]\big) \\
&= \mathrm{Hom}_{\mathrm{DM}}\big(M(X \times Y), \mathbb{1}(j-k+d)[2(j-k+d)+1]\big) \\
&= H^{2(j-k+d)+1\,,\,j-k+d}(X, \mathbb{Z}) \\
&= A^{j-k+d}(X, -1) = 0\ .
\end{aligned}$$

Here, the first equality follows from the duality theorem [13, Theorem 16.24], the second equality is cancellation [13, Theorem 16.25], the third equality is by definition of motivic cohomology $H^{*,*}(-,\mathbb{Z})$ for the smooth variety X [13, 14.5], and the last equality follows from the relation between motivic cohomology and higher Chow groups $A^*(-,*)$ [13, Theorem 19.1]. □

2. First result. This section contains the proof of Theorem 2 stated in the introduction.

Proof of Theorem 2. The argument is based on a nice geometric relation between X and the 20-dimensional Debarre–Voisin hypersurface $X_{DV} \subset \mathrm{Gr}(3,10)$ [6]. This geometric relation is described in [7, Section 3.10] (cf. also [3, Section 4], where X is called $T = T(2,10)$). A Debarre–Voisin hypersurface is by definition a smooth hyperplane section $X_{DV} \subset \mathrm{Gr}(3,10)$ (with respect to the Plücker embedding). Starting from an X_{DV}, we construct the diagram (1) below. Here $\mathrm{Fl}(2,3,10)$ denotes the flag variety, and the morphism p_{Fl} is a $\mathbb{P}^2$-bundle (the fibres correspond to a choice of 2-dimensional subvector space in a fixed 3-dimensional vector space). The morphism p induced by restricting p_{Fl} to $Z := (p_{Fl})^{-1}(X_{DV})$ is again a $\mathbb{P}^2$-bundle. The projection ϕ_{Fl} from $\mathrm{Fl}(2,3,10)$ to $\mathrm{Gr}(2,10)$ is a $\mathbb{P}^6$-bundle. The restriction $\phi := \phi_{Fl}|_Z$ is therefore generically a $\mathbb{P}^6$-bundle. The locus over which ϕ has 7-dimensional fibres is the zero locus X of a section of the dual of $\mathcal{Q}(-1)$. Hence, if this locus X is smooth and dimensionally transverse, it is a variety of type S6.

$$
\begin{array}{ccccccccc}
 & & Z_X & \xrightarrow{\iota} & Z & \hookrightarrow & \mathrm{Fl}(2,3,10) & & \\
 & \swarrow{\scriptstyle \phi_X} & & \swarrow{\scriptstyle \phi} & & \searrow{\scriptstyle p} & & \searrow{\scriptstyle p_{Fl}} & \\
X & \hookrightarrow & \mathrm{Gr}(2,10) & & & & X_{DV} & \hookrightarrow & \mathrm{Gr}(3,10)
\end{array} \tag{1}
$$

As explained in [7], there is an isomorphism

$$H^0(\mathrm{Gr}(2,10),\mathcal{Q}(1)) \cong \wedge^3 V_{10}^{\vee},$$

and so the space parametrizing Fano varieties of type S6 is of the same dimension as the space parametrizing Debarre–Voisin hypersurfaces. It follows that a general Fano variety X of type $S6$ can be obtained from a diagram (1).

We proceed to relate X and X_{DV} on the level of motives:

Theorem 2.1. *Let X be a Fano variety of type $S6$, and assume X is related to a Debarre–Voisin hypersurface X_{DV} as in diagram* (1)*. Then there is an isomorphism of Chow motives*

$$h(X) \cong \bigoplus_{j=-7}^{-5} \mathfrak{t}(X_{DV})(j) \oplus \bigoplus \mathbb{1}(*) \quad \text{in } \mathcal{M}_{\mathrm{rat}}\ .$$

Proof. Let us write $G := \mathrm{Gr}(2,10)$ and $U := G \setminus X$ and $Z_U := Z \setminus Z_X$. Also, let us write $\phi_U \colon Z_U \to U$ for the restriction of ϕ to Z_U. The inclusion $Z_X \hookrightarrow Z$ gives rise to a distinguished triangle

$$M^c(Z_X) \to M^c(Z) \to M^c(Z_U) \xrightarrow{[1]}$$

in DM. Since $Z_U \to U$ is a $\mathbb{P}^6$-bundle, there is an isomorphism $M^c(Z_U) \cong \oplus_{j=0}^{6} M^c(U)(j)[2j]$ (Proposition 1.5). It follows there is also a distinguished triangle

$$M^c(Z_X) \ \to\ M^c(Z) \ \to\ \bigoplus_{j=0}^{6} M^c(U)(j)[2j] \ \xrightarrow{[1]},$$

and after rotating one obtains a distinguished triangle in DM

$$\bigoplus_{j=0}^{6} M^c(U)(j)[2j-1] \to M^c(Z_X) \to M^c(Z) \xrightarrow{[1]}\ . \tag{2}$$

We claim that the triangle (2) fits into a commutative diagram:

Claim 2.2. *There is a commutative diagram in* DM

$$\begin{array}{ccccccc}
\bigoplus_{j=0}^{6} M^c(U)(j)[2j-1] \to & & M^c(Z_X) & \to & M^c(Z) & & \xrightarrow{[1]} \\
& \searrow & \downarrow & & & & \\
& & \bigoplus_{j=0}^{6} M^c(X)(j)[2j] & & \vdots\downarrow & & \\
& & \downarrow & \searrow & & & \\
& & M^c(X)(7)[15] & \leftarrow\text{--} & \bigoplus_{j=0}^{6} M^c(G)(j)[2j] & & \\
& & \downarrow [1] & & & \searrow^{[1]} &
\end{array} \tag{3}$$

where the three lines with solid arrows are distinguished triangles.

Granting this claim, one readily proves Theorem 2.1: applying the octahedral axiom to the diagram (3), one obtains a distinguished triangle following the dotted arrows:

$$M^c(Z) \to \bigoplus_{j=0}^{6} M^c(G)(j)[2j] \to M^c(X)(7)[15] \xrightarrow{[1]} .$$

Applying Lemma 1.6, the second arrow in this triangle must be zero, and so there is a direct sum decomposition

$$M^c(Z) \cong M^c(X)(7)[14] \oplus \bigoplus_{j=0}^{6} M^c(G)(j)[2j] \quad \text{in DM} .$$

Since all varieties in this isomorphism are smooth projective, and $\mathcal{M}_{\mathrm{rat}} \to \mathrm{DM}$ is a full embedding, this means there is also a direct sum decomposition

$$h(Z) \cong h(X)(7) \oplus \bigoplus_{j=0}^{6} h(G)(j) \quad \text{in } \mathcal{M}_{\mathrm{rat}} . \tag{4}$$

(Alternatively, the isomorphism (4) can also be obtained directly by applying [10, Corollary 3.2], which does not use Voevodsky motives.)

The variety Z is a $\mathbb{P}^2$-bundle over X_{DV}, and so one gets an isomorphism of motives

$$\bigoplus_{i=0}^{2} h(X_{DV})(i) \cong h(X)(7) \oplus \bigoplus_{j=0}^{6} h(G)(j) \ \text{ in } \mathcal{M}_{\mathrm{rat}}.$$

Taking the transcendental parts on both sides (and remembering that the motive of the Grassmannian G is a sum of twisted Lefschetz motives), one gets an isomorphism

$$\mathfrak{t}(X) \cong \bigoplus_{i=0}^{2} \mathfrak{t}(X_{DV})(-7+i) \ \text{ in } \mathcal{M}_{\mathrm{rat}},$$

which implies Theorem 2.1.

It remains to prove Claim 2.2. The horizontal line of (3) is the distinguished triangle (2). The diagonal line is obtained from the "Gysin" distinguished triangles

$$M^c(X)(j)[2j] \ \to \ M^c(G)(j)[2j] \ \to \ M^c(U)(j)[2j] \ \xrightarrow{[1]}$$

by rotation and summing. The vertical line is a distinguished triangle because $Z_X \to X$ is a $\mathbb{P}^7$-bundle (Proposition 1.5). To check commutativity, one observes there is a commutative diagram

$$
\begin{array}{ccccccc}
M^c(Z_X) & \to & M^c(Z) & \to & M^c(Z_U) & \xrightarrow{[1]} \\
\downarrow \alpha_X & & \downarrow \alpha & & \cong\ \downarrow \alpha_U & \\
\bigoplus_{j=0}^{6} M^c(X)(j)[2j] & \to & \bigoplus_{j=0}^{6} M^c(G)(j)[2j] & \to & \bigoplus_{j=0}^{6} M(U)(j)[2j] & \xrightarrow{[1]}
\end{array} \tag{5}
$$

Here the map α is defined as a sum $\sum_{j=0}^{6} \phi_* \cdot h^j$ where h is an ample class, and the maps α_X, α_U are defined similarly by restricting h to X resp. to U. The map α_U is an isomorphism because $Z_U \to U$ is a $\mathbb{P}^6$-bundle. (The left square commutes by functoriality of proper push-forward. The commutativity of the right square follows from commutativity on the level of $z_{equi}(-,0)$, which can be checked directly.) After rotation, diagram (5) gives a commutative diagram

$$
\begin{array}{ccccccc}
M^c(Z_U)[-1] & \to & M^c(Z_X) & \to & M^c(Z) & \xrightarrow{[1]} \\
\cong\ \downarrow \alpha_U & & \downarrow \alpha_X & & \downarrow \alpha & \\
\bigoplus_{j=0}^{6} M^c(U)(j)[2j-1] & \to & \bigoplus_{j=0}^{6} M^c(X)(j)[2j] & \to & \bigoplus_{j=0}^{6} M^c(G)(j)[2j] & \xrightarrow{[1]}
\end{array} \tag{6}
$$

In the diagram (3), the horizontal line is the top horizontal line of (6) combined with the isomorphism α_U, while the diagonal line in (3) is the bottom horizontal line of (6). This proves the commutativity of Claim 2.2. □

We now pursue the proof of Theorem 2. Let $\mathcal{X} \to B$ denote the universal family of Fano varieties of type S6, where B is a Zariski open in $\mathbb{P}H^0(\mathrm{Gr}(2,10), \mathcal{Q}(1))$. As we have seen, a general Fano variety of type S6 fits into a diagram (1), and so there is a Zariski open $B_0 \subset B$ to which Theorem 2.1 applies. Taking Chow groups of the motives in Theorem 2.1, we get an isomorphism

$$A^i_{hom}(X_b) \cong A^{i+7}_{hom}(X_{DV}) \oplus A^{i+6}_{hom}(X_{DV}) \oplus A^{i+5}_{hom}(X_{DV}) \quad \forall b \in B_0 \ . \tag{7}$$

As an illustration of her celebrated method of spread, Voisin has proven the following:

Theorem 2.3 (Voisin [22]). *Let X_{DV} be a Debarre–Voisin hypersurface. Then*

$$A^i_{hom}(X_{DV}) = 0 \quad \forall i \neq 11.$$

(More precisely, Voisin [22, Theorem 2.4] proves $A^i_{hom}(X_{DV}) = 0$ for $i > 11$, but this readily implies that $A^i_{hom}(X_{DV}) = 0$ for $i < 11$ as well, as explained in [12, Proof of Theorem 2.1].) Plugging in Theorem 2.3 into isomorphism (7), we find that

$$A^i_{hom}(X_b) = 0 \ \ \forall i \notin \{4,5,6\} \quad \forall b \in B_0. \tag{8}$$

We now proceed to extend this vanishing from B_0 to all of B. To do this, we observe that the vanishing (8) implies, via the Bloch–Srinivas "decomposition of the diagonal" method [4], that for any $b \in B_0$ there is a decomposition of the diagonal

$$\Delta_{X_b} = x \times X_b + C_b \times D_b + \Gamma_b \quad \text{in } A^8(X_b \times X_b), \tag{9}$$

where $x \in X_b$, C_b and D_b are a curve resp. a divisor, and Γ_b is a cycle supported on $X_b \times W_b$ where $W_b \subset X_b$ is a codimension 2 closed subvariety. Using the Hilbert schemes argument of [20, Proposition 3.7], these data can be spread out over the base B. That is, we can find $x \in A^8(\mathcal{X})$, $\mathcal{C} \in A^7(\mathcal{X})$, $\mathcal{D} \in A^1(\mathcal{X})$, a codimension 2 subvariety $\mathcal{W} \subset \mathcal{X}$ and a cycle Γ supported on $\mathcal{X} \times_B \mathcal{W}$ such that restricting to a fibre $b \in B_0$ we get

$$\Delta_{X_b} = x|_b \times X_b + \mathcal{C}|_b \times \mathcal{D}|_b + \Gamma|_b \quad \text{in } A^8(X_b \times X_b).$$

Applying [21, Lemma 3.2], we find that this decomposition is actually true for any b in the larger base B. What's more, given any $b_0 \in B$, this construction can be done in such a way that $\mathcal{C}, \mathcal{D}, \mathcal{W}$ are in general position with respect to the fibre X_{b_0}. That is, we have obtained a decomposition (9) for any $b_0 \in B$. Letting this decomposition act on Chow groups, it follows that the vanishing (8) is true for all $b \in B$. This proves the first part of Theorem 2.

We now prove the "moreover" part of Theorem 2. So let us consider a Fano variety $X = X_b$ for $b \in B_0$, which means that we can assume X fits into a diagram (1) with some 22-dimensional variety Z and a Debarre–Voisin hypersurface X_{DV}. The isomorphism (4) implies that there are isomorphisms

$$\iota_*(\phi_X)^*\colon \quad A^i_{hom}(X) \xrightarrow{\cong} A^{i+7}_{hom}(Z) \quad (i \in \{4,5,6\}). \tag{10}$$

The Picard number of X being 1, any ample divisor $h \in A^1(X)$ comes from a divisor $h_G \in A^1(\mathrm{Gr}(2,10))$. Let $h_Z := \phi^*(h_G) \in A^1(Z)$. In view of the isomorphisms (10), to prove the "moreover" statement it suffices to prove there are isomorphisms

$$(11) \qquad \begin{aligned} \cdot h_Z\colon \quad A^{11}_{hom}(Z) &\xrightarrow{\cong} A^{12}_{hom}(Z)\ , \\ \cdot h_Z\colon \quad A^{12}_{hom}(Z) &\xrightarrow{\cong} A^{13}_{hom}(Z)\ . \end{aligned}$$

Since $p\colon Z \to X_{DV}$ is a $\mathbb{P}^2$-bundle, the divisor $h_Z \in A^1(Z)$ can be written

$$h_Z = p^*(c) + \lambda\,\xi \quad \text{in } A^1(Z),$$

where $\lambda \in \mathbb{Q}$ and ξ is a relatively ample class for the fibration p. We claim that λ must be non-zero. (Indeed, suppose λ were zero. Then the intersection of $(h_Z)^2$ with a fibre F of p would be zero. But the image $\phi(F) \subset \mathrm{Gr}(2,10)$ is a 2-dimensional subvariety and so $(h_G)^2 \cdot \phi(F) \neq 0$; contradiction.)

The fact that p is a $\mathbb{P}^2$-bundle, plus the fact that $A^j_{hom}(X_{DV}) = 0$ for $j \neq 11$, gives us isomorphisms

$$\begin{aligned} A^{11}_{hom}(Z) &\cong p^* A^{11}_{hom}(X_{DV})\ , \\ A^{12}_{hom}(Z) &\cong p^* A^{11}_{hom}(X_{DV}) \cdot \xi\ , \\ A^{13}_{hom}(Z) &\cong p^* A^{11}_{hom}(X_{DV}) \cdot \xi^2 . \end{aligned}$$

Thus, we see that $\cdot p^*(c)$ acts as zero on $A^{11}_{hom}(Z)$ and on $A^{12}_{hom}(Z)$ (indeed, this map factors over $A^{12}_{hom}(X_{DV}) = 0$). It follows that intersecting with h_Z is the same as intersecting with $\lambda\,\xi$ on $A^j_{hom}(Z)$, and this induces the desired isomorphisms (11). We have now proven the "moreover" statement for X_b with $b \in B_0$. □

3. Second result. In this section we prove Theorem 3 stated in the introduction. In order to prove Theorem 3, we first establish a "Franchetta property" type of statement (for more on the generalized Franchetta conjecture, cf. [15], [16], [8]):

Theorem 3.1. *Let $\mathcal{X} \to B$ denote the universal family of Fano varieties of type S6 (as above). Let $\Psi \in A^j(\mathcal{X})$ be such that*

$$\Psi|_{X_b} = 0 \quad \textit{in } H^{2j}(X_b, \mathbb{Q}) \quad \forall b \in B.$$

Then

$$\Psi|_{X_b} = 0 \quad \textit{in } A^j(X_b) \quad \forall b \in B.$$

P r o o f. Invoking the spread lemma [21, Lemma 3.2], it will suffice to prove the theorem over the Zariski open $B_0 \subset B$ of Theorem 2.1. The construction of diagram (1) being geometric in nature, this diagram also exists as a diagram of B_0-schemes. Writing $\mathcal{X} \to B_0$ for the universal family of varieties of type S6 (as before), and $\mathcal{X}_{DV} \to B_0$ for the universal Debarre–Voisin hypersurface, this means that there exists a relative correspondence $\Gamma \in A^*(\mathcal{X} \times_{B_0} \mathcal{X}_{DV}) \oplus A^*(\mathcal{X})$ inducing the fibrewise injections (given by Theorem 2.1)

$$(\Gamma|_b)_*\colon \quad A^j(X_b) \ \hookrightarrow \ A^{11}((X_{DV})_b) \oplus \bigoplus \mathbb{Q} \ .$$

Thus, Theorem 3.1 is implied by the Franchetta property for $\mathcal{X}_{DV}$, which is [12, Theorem 3.2]. □

It remains to prove Theorem 3:

P r o o f o f T h e o r e m 3. Clearly, the Chern classes $c_j(X) := c_j(T_X)$ are universally defined: for any $b \in B$, we have

$$c_j(T_{X_b}) = c_j(T_{\mathcal{X}/B})|_{X_b}.$$

Also, the image

$$\mathrm{Im}\big(A^j(\mathrm{Gr}(2,10)) \to A^j(X_b)\big)$$

consists of universally defined cycles (for a given $a \in A^j(\mathrm{Gr}(2,10))$, the relative cycle

$$(a \times B)|_{\mathcal{X}} \ \ \in \ A^j(\mathcal{X})$$

does the job).

Since $A^1(X_b)$ is generated by a hyperplane section, clearly $A^1(X_b)$ is universally defined. Similarly, the fact that $A^2_{hom}(X_b) = 0$, combined with weak Lefschetz in cohomology, implies that

$$A^2(X_b) = \mathrm{Im}\big(A^2(\mathrm{Gr}(2,10)) \to A^2(X_b)\big),$$

and so $A^2(X_b)$ also consists of universally defined cycles.

Intersections of universally defined cycles are universally defined, since $A^*(\mathcal{X}) \to A^*(X_b)$ is a ring homomorphism. In conclusion, we have shown that $R^*(X_b)$ consists of universally defined cycles, and so Theorem 3 is a corollary of Theorem 3.1. □

Remark 3.2. Theorem 3 is an indication that maybe varieties X of type S6 have a *multiplicative Chow–Künneth decomposition*, in the sense of [18, Chapter 8]. Unfortunately, establishing this seems difficult; one would need something like theorem 3.1 for

$$A^{16}(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}).$$

Presumably, one can also add $A^3(X)$ to the subring $R^*(X)$ of Theorem 3. (Indeed, $A^3_{hom}(X) = 0$, so *provided* X has a multiplicative Chow–Künneth decomposition, one would have $A^3(X) = A^3_{(0)}(X)$ where $A^*_{(*)}(X)$ indicates the bigrading induced by the multiplicative Chow–Künneth decomposition.) While I cannot prove this, I can prove at least a weaker result:

Proposition 3.3. *Let X be a general Fano variety of type S6. Then*

$$A^2(X) \cdot A^3(X) \ \subset\ A^5(X)$$

injects into $H^{10}(X, \mathbb{Q})$ under the cycle class map.

Proof. Assume $X = X_b$ with $b \in B_0$, so that X is related to a Debarre–Voisin hypersurface X_{DV} as in diagram (1). We want to prove that

$$\Big(A^2(X) \cdot A^3(X)\Big) \cap A^5_{hom}(X) = 0.$$

Since $\phi_X \colon Z_X \to X$ is a $\mathbb{P}^7$-bundle, it will suffice to prove that

$$\Big((\phi_X)^* A^2(X) \cdot (\phi_X)^* A^3(X)\Big) \cap A^5_{hom}(Z_X) = 0.$$

Restriction induces an isomorphism $\iota^* \colon A^2(Z) \cong A^2(Z_X)$. Moreover, we know (isomorphism (10)) that

$$\iota_*(\phi_X)^* \colon \quad A^5_{hom}(X) \ \to\ A^{12}_{hom}(Z)$$

is injective. Thus, it suffices to prove that

$$\Big(A^2(Z) \cdot A^{10}(Z)\Big) \cap A^{12}_{hom}(Z) = 0.$$

But this follows from the $\mathbb{P}^2$-bundle structure of $p \colon Z \to X_{DV}$: indeed, any $a \in A^2(Z)$ and $b \in A^{10}(Z)$ can be written

$$\begin{aligned} a &= p^*(a_2) + p^*(a_1) \cdot \xi + p^*(a_0) \cdot \xi^2 \quad \text{in } A^2(Z), \\ b &= p^*(b_{10}) + p^*(b_9) \cdot \xi + p^*(b_8) \cdot \xi^2 \quad \text{in } A^{10}(Z), \end{aligned}$$

where ξ is a relatively ample class, and $a_j, b_j \in A^j(X_{DV})$. The intersection $a \cdot b$ can be written

$$a \cdot b = p^*(a_2 \cdot b_{10}) + p^*(a_1 \cdot b_{10} + a_2 \cdot b_9) \cdot \xi + p^*(a_2 \cdot b_8 + a_1 \cdot b_9 + a_0 \cdot b_{10}) \cdot \xi^2 \quad \text{in } A^{12}(Z).$$

As the intersection $a \cdot b$ is assumed to be homologically trivial, this means that

$$a_2 \cdot b_{10} \ , \quad a_1 \cdot b_{10} + a_2 \cdot b_9 \ , \quad a_2 \cdot b_8 + a_1 \cdot b_9 + a_0 \cdot b_{10}$$

are homologically trivial on X_{DV}. But $A^{12}_{hom}(X_{DV}) = A^{10}_{hom}(X_{DV}) = 0$, and so

$$a_2 \cdot b_{10} = a_2 \cdot b_8 + a_1 \cdot b_9 + a_0 \cdot b_{10} = 0 \quad \text{in } A^*(X_{DV}).$$

As for the remaining term, it is proven in [12, Theorem 3.1] that

$$A^1(X_{DV}) \cdot A^{10}(X_{DV}) + A^2(X_{DV}) \cdot A^9(X_{DV}) \quad \subset \quad A^{11}(X_{DV})$$

injects into cohomology, and so also

$$a_1 \cdot b_{10} + a_2 \cdot b_9 = 0 \quad \text{in } A^{11}(X_{DV}).$$

It follows that $a \cdot b = 0$, and the proposition is proven. □

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