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# CLOSEDNESS OF RADICALS IN TOPOLOGICAL RINGS

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Abstract. The well-known construction of Hartman-Mycielski delivers a method of embedding of a topological ring $(R, \mathcal{T})$ into an arcwise connected, locally arcwise connected topological ring $(\tilde{R}, \tilde{\mathcal{T}})$. An important question in the theory of topological rings is: When is the radical $\rho(R)$ of a topological ring closed? We study in this paper the following question: Let $\rho$ be a radical in the sense of Kurosh in the class of associative rings and $(R, \mathcal{T})$ be a topological ring. Under which conditions is closed the radical $\rho(\tilde{R})$ of the image of $R$ under the Hartman-Mycielski functor? We give in this paper a complete answer for a few classical radicals. This question has been solved for the Jacobson radical in the master thesis of the third author.

**1. Introduction.** If some radical of a topological ring is closed it can be used to study some properties of a topological ring. From this reason the question when the Jacobson radical of a topological ring is closed was studied in the first papers on topological rings (see [4]).

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We are interested in the question: Let $(\tilde{R}, \tilde{T})$ be the image of a topological ring $(R, \mathcal{T})$ under the Hartman-Mycielski functor and $\rho$ be a radical in the sense of Kurosh in the class of associative rings (see [8, 1]). When $\rho(\tilde{R})$ is closed?

**2. Preliminaries and notation.** All rings are assumed to be associative and unitary. Topological rings are assumed to be Hausdorff. The symbol $N(R)$ stands for the regular radical of $R$, i.e. for the largest regular ideal of $R$ ([5], p.112). The letter $\mu$ stands for the Lebesgue measure on the interval $[0, 1]$.

Furthermore, $Nil^*R$ will denote the upper nilradical of a ring $R$ ([9], Definition (10.26)).

The symbol $B(R)$ stands for the Brown-McCoy radical of a ring $R$, i.e., for the intersection of all maximal ideals of $R$ ([10]).

Recall the construction of Hartman-Mycielski for topological rings [3], [2]: Let $(R, \mathcal{T})$ be a topological ring. The set $\tilde{R}$ consists of functions $f : [0, 1) \to R$ for which there exists a partition $0 = a_0 < \cdots < a_n = 1$ such that $f \upharpoonright_{[a_i, a_{i+1})}$ is constant, $i = 0, \ldots, n - 1$.

The operations on $\tilde{R}$ are defined as follows:

$$(f + g)(t) := f(t) + g(t), \quad (fg)(t) := f(t)g(t) \quad (t \in [0, 1)).$$

The family $\mathcal{B} = \{(V, \varepsilon)\}$, where $V$ runs a fundamental system of neighborhoods of zero of $(R, \mathcal{T})$ and

$$(V, \varepsilon) := \{f \in \tilde{R} | \mu\{t \in [0, 1) | f(t) \notin V\} < \varepsilon\}$$

is a fundamental system of neighborhoods of zero for $(\tilde{R}, \tilde{T})$. Put for $[\alpha, \beta) \subseteq [0, 1)$

$$\chi_{[\alpha,\beta)}(t) := \begin{cases} 1 & (t \in [\alpha, \beta)) \\ 0 & (t \notin [\alpha, \beta)) \end{cases}.$$

We notice that $\chi_{[\alpha,\beta)}$ is a central idempotent of $\tilde{R}$. Henceforth $S = \{f_r | r \in R\} \subseteq \tilde{R}$, where $f_r : [0, 1) \to R, t \mapsto r$. It is easy to check that the mapping $\lambda : (R, \mathcal{T}) \to (\tilde{R}, \tilde{\mathcal{T}}), r \mapsto f_r$ is a topological embedding. Below $g_{[\alpha,\beta),r} := \chi_{[\alpha,\beta)} f_r$.

**Remark 2.1.** The mapping $\varphi_t : \tilde{R} \to R_t = R, f \mapsto f(t)$ is a surjective ring homomorphism for any fixed $t \in [0, 1)$.

**Lemma 2.1.** *For any radical $\rho$ in the sense of Kurosh, $h \in \rho(\tilde{R})$ and $t \in [0,1)$ implies $h(t) \in \rho(R)$.*

P r o o f. Consider homomorphism $\varphi_t : \tilde{R} \to R_t = R$. Then $\varphi(\rho(\tilde{R})) \subseteq \rho(R)$ and so $h(t) \in \rho(R)$. □

**Lemma 2.2.** *Let $f \in \tilde{R}$, $0 = a_0 < \cdots < a_n = 1$ and $f = g_{[a_0,a_1),r_0} + \cdots + g_{[a_{n-1},a_n),r_{n-1}}$. Thus $f \in N(\tilde{R})$ iff $g_{[a_i,a_{i+1}),r_i} \in N(\tilde{R})$ for every $i = 0, ..., n-1$.*

P r o o f. Indeed, $\tilde{R} = \chi_{[a_0,a_1)}\tilde{R} \oplus \cdots \oplus \chi_{[a_{n-1},a_n)}\tilde{R}$, a topological direct sum of ideals. □

**Lemma 2.3.** *An element $f \in \tilde{R}$ is regular if and only if $f(t)$ is regular for all $t \in [0,1)$.*

P r o o f. Assume that $f$ is regular. Then for each $t \in [0,1)$, $f(t) = \varphi_t(f)$ is a regular element of $R$.

Conversely, assume that $f(t)$ is a regular element of $R$ for every $t \in [0,1)$. Consider the decomposition $f = g_{[a_0,a_1),r_0} + \cdots + g_{[a_{n-1},a_n),r_{n-1}}$ like in Lemma 2.2.

Let $r'_i \in R$ such that $r_i r'_i r_i = r_i, i = 0, \ldots, n-1$. Then $ff'f = f$ for $f' = g_{[a_0,a_1),r'_0} + \cdots + g_{[a_{n-1},a_n),r'_{n-1}}$. □

**Lemma 2.4.** *Let $r \in N(R)$ and $[\alpha, \beta) \subseteq [0,1)$. Then $g_{[\alpha,\beta),r} \in N(\tilde{R})$.*

P r o o f. Let $f_i, f'_i \in \tilde{R}$, $n \in \mathbb{N}$, and $f = \sum_{i=0}^{n-1} f_i g_{[\alpha,\beta),r} f'_i$. If $t \in [0,1)$, $i \in \{0, \ldots, n-1\}$, then $\varphi_t(f_i g_{[\alpha,\beta),r} f'_i) = \varphi_t(f_i) r \varphi(f'_i) \in N(R)$, hence $f \in N(R)$. Therefore $f(t) \in N(R)$. By Lemma 2.3 $f$ is regular. Therefore every element of the ideal of $\tilde{R}$ generated by $g_{[\alpha,\beta),r}$ is regular, hence $g_{[\alpha,\beta),r} \in N(\tilde{R})$. □

**Lemma 2.5.** *If $\varphi : \tilde{R} \to L$ is a surjective ring homomorphism where $L$ is a prime ring, then $\varphi(S) = L$.*

P r o o f. Let $y \in L$ and $h \in \tilde{R}$ such that $\varphi(h) = y$. Let $0 = a_0 < \cdots < a_n = 1$ such that $h([a_i, a_{i+1})) = r_i, i \in \{0, \ldots, n-1\}$.

Thus $h = g_{[a_0,a_1),r_0} + \cdots + g_{[a_{n-1},a_n),r_{n-1}}$ (see Lemma 2.2). Since $\chi_{[0,a_1)} + \cdots + \chi_{[a_{n-1},1)} = 1_{\tilde{R}}$ (the identity of $\tilde{R}$) and $\tilde{R}$ is prime, there exists $i \in \{0, \ldots, n-1\}$ such that $\varphi(\chi_{[a_i,a_{i+1})}) = 1$ and $\varphi(\chi_{[a_j,a_{j+1})}) = 0$ for $j \neq i$.

Thus

$$\begin{aligned} y &= \varphi(g) \\ &= \varphi(f_{r_0})\varphi(\chi_{[0,a_1)}) + \cdots + \varphi(f_{r_{n-1}})\varphi(\chi_{[a_{n-1},1)}) \\ &= \varphi(f_{r_i})\varphi(\chi_{[a_i,a_{i+1})}) = \varphi(f_{r_i}). \end{aligned}$$ □

## 3. Results.

**Theorem 3.1.** *The regular radical $N(R)$ of a topological ring $(R, \mathcal{T})$ is closed if and only if $N(\tilde{R})$ is closed.*

**Theorem 3.2.** *The upper nilradical $Nil^*R$ of a topological ring $R$ is closed iff $Nil^*\tilde{R}$ is closed.*

**Corollary 3.3.** *If $R$ is a commutative topological ring with 1, then the nilradical $Nil^*R$ is closed iff the nilradical $Nil^*\tilde{R}$ is closed.*

**Theorem 3.4.** *Let $R$ be a topological ring. Then $B(R)$ is closed $\Longleftrightarrow$ $B(\tilde{R})$ is closed.*

## 4. Proofs.

Proof of Theorem 3.1. $\Rightarrow$: Assume on the contrary that $N(\tilde{R})$ is not closed. Let $f \in \overline{N(\tilde{R})} \setminus N(\tilde{R})$. Let $f = g_{[a_0,a_1),r_0} + \cdots + g_{[a_{n-1},a_n),r_{n-1}}$ be the decomposition of $f$ like in Lemma 2.2. Then by Lemma 2.2 there exists $i \in \{0, \ldots, n-1\}$ such that $g_{[a_i,a_{i+1}),r_i} \notin N(R)$. Thus by Lemma 2.4, $r_i \notin N(R)$. By continuity of ring operations, $g_{[a_i,a_{i+1}),r_i} = \chi_{a_i,a_{i+1}} f \in \chi_{a_i,a_{i+1}} \overline{N(\tilde{R})} \subseteq \overline{N(\tilde{R})}$.

Since $N(R)$ is closed there exists a neighborhood $V$ of zero of $(R, \mathcal{T})$ such that $(r_i + V) \cap N(R) = \varnothing$.

We claim that $(g_{[a_i,a_{i+1}),r_i} + (V, a_{i+1} - a_i)) \cap N(\tilde{R}) = \varnothing$. Assume on the contrary that there exists $h \in (g_{[a_i,a_{i+1}),r_i} + (V, a_{i+1} - a_i)) \cap N(\tilde{R})$; then $h - g_{[a_i,a_{i+1}),r_i} \in (V, a_{i+1} - a_i)$. If were $(h - g_{[a_i,a_{i+1}),r_i})(t_1) \notin V$ for all $t_1 \in [a_i, a_{i+1})$, then $[a_i, a_{i+1}) \subseteq \{t' | (h - g_{[a_i,a_{i+1}),r_i})(t') \notin V\}$. Thus $\mu(\{t' | (h - g_{[a_i,a_{i+1}),r_i})(t') \notin V\}) \geq \mu([a_i, a_{i+1})) = a_{i+1} - a_i$, a contradiction. It follows that there exists $t'' \in [a_i, a_{i+1})$ such that $(h - g_{[a_i,a_{i+1}),r_i})(t'') \in V$ which implies $h(t'') - g_{[a_i,a_{i+1}),r_i}(t'') \in V$ and so $h(t'') \in r_i + V$. However by Lemma 2.1 $h(t'') \in N(R)$, hence $(r_i + V) \cap N(R) \neq \varnothing$, a contradiction. We have proved that $N(\tilde{R})$ is closed.

$\Leftarrow$: Assume that $N(\tilde{R})$ is closed and let $r \notin N(R)$. Then by Lemma 2.4 $f_r \notin N(\tilde{R})$, where $f_r(t) = r$ for all $t \in [0, 1)$. There exists $\varepsilon \in (0, 1)$ such that $(f_r + (V, \varepsilon) \cap N(\tilde{R}) = \varnothing$.

We claim that $(r+V)\cap N(R)=\varnothing$. Assume on the contrary that there exists $v\in V$ such that $j=r+v\in N(R)$.

Let $f_j, f_v\in\tilde{R}$, where $\forall_{t\in[0,1)}[f_j(t)=j]$ and $\forall_{t\in[0,1)}[f_t(v)=v]$. Then according to Lemma 2.4, $f_j\in N(R)$. Since $\{t\in[0,1)|f_v(t)\notin V\}=\varnothing$, $\mu(\{t\in[0,1)|f_v(t)\notin V\})=0<\varepsilon$, we obtain $f_v\in(V,\varepsilon)$. Thus $f_j=f_r+f_v\in(f_r+(V,\varepsilon))\cap N(\tilde{R})$, a contradiction. Thereby we have proved that $R\setminus N(R)$ is open and so $N(R)$ is closed. □

P r o o f o f T h e o r e m 3.2. $\Rightarrow$: Let $f\notin Nil^*\tilde{R}$ and $0=a_0<\cdots<a_n=1$ be a partition of $[0,1)$ such that $f\restriction_{[a_i,a_{i+1})}$ is constant for $i=0,\ldots,n-1$. By [9] the ideal $(f)$ of $\tilde{R}$ is not a nil ideal. We have that $f=g_{[a_0,a_1),r_0}+\cdots+g_{[a_{n-1},a_n),r_{n-1}}=f_0+\cdots+f_{n-1}$, where $f_i=r_i\chi_{[a_i,a_{i+1})}$ and $r_i\in R$, $i=0,\ldots,n-1$.

Then $(f_i)$ will be not a nil ideal for some $i\in\{0,\ldots,n-1\}$. Let $\varepsilon=a_{i+1}-a_i$ and $r=r_i=f_i[a_i,a_{i+1})$. Then $(f_r)$ is not nil. Since $(r)$ is not a nil ideal, we obtain that $r\notin Nil^*R$.

By assumption there exists a neighborhood $V$ of zero such that $(r+V)\cap Nil^*\tilde{R}=\varnothing$.

We claim that $(f+(V,\varepsilon))\cap Nil^*\tilde{R}=\varnothing$. Assume on the contrary that there exists $g\in(V,\varepsilon)$ such that $f+g\in Nil^*\tilde{R}$. We notice that $f(t)+g(t)\in N^*R$ for all $t\in[0,1)$. Since $g\in(V,\varepsilon)$ there exists $t_0\in[a_i,a_{i+1})$ such that $g(t_0)\in V$. Thus $f(t_0)+g(t_0)=r+g(t_0)\in(r+V)\cap Nil^*R$, a contradiction.

$\Leftarrow$: Assume that $Nil^*\tilde{R}$ is closed. Let $r\notin Nil^*R$. Thus $f_r\notin Nil^*\tilde{R}$. Let $V$ be a neighborhood of zero and $\varepsilon\in(0,1)$ such that $(f_r+(V,\varepsilon))\cap Nil^*\tilde{R}=\varnothing$.

We claim that $(r+V)\cap Nil^*R=\varnothing$. Assume on the contrary that there exists $v\in V$ such that $r+v\in Nil^*R$. Thus $f_{r+v}=f_r+f_v\in Nil^*\tilde{R}$. Since $\{t|f_v(t)=v\notin V\}=\varnothing, f_v\in(V,\varepsilon)$. Thus $f_r+f_v\in Nil^*\tilde{R}\cap(f_r+(V,\varepsilon))$, a contradiction. □

P r o o f o f T h e o r e m 3.3. $\Rightarrow$: Let $f\notin B(\tilde{R})$. Then there exists a maximal ideal $M$ of $\tilde{R}$ such that $f\notin M$.

Let $f=g_{[a_0,a_1),r_0}+\cdots+g_{[a_{n-1},a_n),r_{n-1}}$ be the decomposition of $f$ like in Lemma 2.2. Let $i\in\{0,\ldots,n-1\}$ such that $g_{[a_i,a_{i+1}),r_i}\notin M$.

We have an isomorphism $\lambda:R\to S, r\mapsto f_r$, where $S=\{f_r|r\in R\}$. Consider the canonical homomorphism $q:\tilde{R}\to\tilde{R}/M$. Since every simple ring is prime, by Lemma 2.5 $q(S)=\tilde{R}/M$. Thus $S\cap M$ will be a maximal ideal of $S$ and $g_{[a_i,a_{i+1}),r_i}=f_{r_i}\chi_{[a_i,a_{i+1})}=r_i\chi_{[a_i,a_{i+1})}\notin S\cap M$.

It follows that $r_i \notin \lambda^{-1}(S \cap M)$ and $\lambda^{-1}(S \cap M)$ is a maximal ideal of $R$. Thus $r_i \notin B(R)$. There exists a neighborhood $V$ of $0_R$ such that $(r_i+V)\cap B(R) = \varnothing$.

We claim that $(f + (V,\varepsilon)) \cap B(\tilde{R}) = \varnothing$, where $\varepsilon = a_{i+1} - a_i$.

Assume on the contrary that there exists $h \in (V,\varepsilon)$ such that $f + h \in B(\tilde{R})$. There exists $t_0 \in [a_i, a_{i+1})$ such that $h(t_0) \in V$. Thus $f(t_0) + h(t_0) \in r_i + V$. Furthermore, $f(t_0) + h(t_0) = \varphi_{t_0}(f + h) \in \varphi_{t_0}(B(\tilde{R})) \subseteq B(R)$. Therefore $(r_i + V) \cap B(R) \neq \varnothing$, a contradiction.

$\Leftarrow$: Let $r \notin B(R)$. Then $f_r \notin B(\tilde{R})$. Indeed, otherwise $r = \varphi_0(f_r) \in \varphi_0(B(\tilde{R})) \subseteq B(R_0) = B(R)$, a contradiction.

Let $V$ be a neighborhood of zero of $R$ and $\varepsilon \in (0,1)$ such that $(f_r + (V,\varepsilon)) \cap B(\tilde{R}) = \varnothing$. Assume that there exists $v \in V$ such that $r + v \in B(R)$. Thus $f_v \in (V,\varepsilon)$ and $f_r + f_v \in f_r + (V,\varepsilon)$.

We notice that if $x \in B(R)$, then $f_x \in B(\tilde{R})$. For, $\lambda : R \to S$ is an isomorphism. Then $\lambda(x) = f_x \in B(S)$.

If $M$ is a maximal ideal of $\tilde{R}$ and $\Psi : \tilde{R} \to \tilde{R}/M$ the canonical homomorphism, then $\Psi(S) = \tilde{R}/M$, hence $(\Psi|S)(S) = \tilde{R}/M$. Thus $(\Psi|S)(B(S)) \subseteq B(\tilde{R}/M) = 0$, hence $B(S) \subseteq B(\tilde{R})$ and so $f_x \in B(\tilde{R})$. Now $f_r + f_v = f_{r+v} \in B(\tilde{R})$, a contradiction. □

**5. Examples.** An example of a topological ring whose Jacobson radical isn't closed has been constructed in [4, pp. 158–159].

The Jacobson radical of a locally compact ring is closed ([6]). A topological ring with 1 is called a $Q$-ring if the set of all its invertible elements is open. It is well-known that the radical of a $Q$-ring is closed [11, Corollary 6.3, page 125].

We will construct a non-regular locally compact ring whose regular ideal is dense, hence, nonclosed.

Let $R_i = M(2, \mathbb{F}_2)(i = 1, 2, \ldots)$ be the matrix ring of 2-by-2 matrices over the field $\mathbb{F}_2$ consisting of 2 elements and $S_i = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 \end{bmatrix} (i = 1, 2, \ldots)$.

Furthermore, let $A = \prod\limits_{i\in\mathbb{N}}(R_i : S_i)$ be the local product of $R_i$ with respect to open subrings $S_i(i \in \mathbb{N})([11]$, page 211). The ring $A$ is locally compact. By [11], Theorem 17.2, page 211 $A$ is not regular. However we will give here a direct prooof.

We claim that the equation $\{a\}x\{a\} = \{a\}$ has no solution in $x = \{x_i\} \in A$ for $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Indeed, on the contrary there exists $i_0$ such that $x_i \in S_i$ for $i \geq i_0$. Thus $0 = ax_{i_0}a = a$, a contradiction.

The ideal $\oplus_{i\in\mathbb{N}} R_i$ is regular and dense in $A$. Therefore, the regular radical of $A$ is not closed.

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