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UNIT GROUPS OF SEMISIMPLE GROUP ALGEBRAS OF CERTAIN DIHEDRAL GROUPS

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Abstract. Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements. Let D_n denote the dihedral group of order $2n$. In this article, we have obtained the structure of the unit groups of the semisimple group algebras FD_{11}, FD_{13}, FD_{17}, FD_{19} and FD_{23}.

1. Introduction. Let $U(FG)$ be the unit group of the group algebra FG of a finite group G over a finite field F of characteristic p. We denote characteristic of the field F by $\mathrm{Char}(F)$. Recently, some techniques have been developed to find the decomposition of FG and hence the structure of $U(FG)$, when $p \nmid |G|$. Let D_n be the Dihedral group of order $2n$, presented as $D_n = \langle r, s \mid r^n = s^2 = rsrs = 1\rangle$. In [9], Makhijani, Sharma and Srivastava, obtained the structure of the unit group of FD_n for any odd $n \geq 3$ and $\mathrm{Char}(F) = 2$. This is an extension of [7] and [8] in which they have studied the unit group of

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FD_p, where p is a prime number. In [10], the structure of the unitary subgroup $U_*(FD_{p^n})$ is established for an odd prime p and $\mathrm{Char}(F) = p$. Also the structure of the center of the maximal p-subgroup of $U(FD_{p^n})$ is given. Unitary units of some modular group algebras have also been studied in [1, 2]. In [11], the unit group of FD_{15} is described. In [5], Gildea has obtained the order of $U(FD_{p^m})$, where p is an odd prime and $\mathrm{Char}(F) = p$.

In this paper, we are interested in the structure of $U(FD_n)$, $n = 11, 13, 17, 19$ and 23, such that $p \nmid |D_n|$, i.e., the semisimple case. This is a continuation of our work in [13, 14]. For our work, we use the well known Witt-Berman theorem [6, Chapter 17, Theorem 5.3]. Our notations are standard. For a finite subset H of G, $\hat{H} = \sum_{h \in H} h$. Also $M(n, F)$ is the algebra of all $n \times n$ matrices over F and $GL(n, F)$ is the general linear group of degree n over F. Further, F_n is the extension field of F of degree n, $F^* = F \backslash \{0\}$ and F^n is the direct summand of n copies of F, i.e,

$$F^n = \underbrace{F \oplus F \oplus \cdots \oplus F}_{n\text{-copies}}.$$

Similarly C_n is the cyclic group of order n and C_n^k is the direct product of k copies of C_n. An element $g \in G$ is p-regular, if $(p, o(g)) = 1$, m is the l.c.m. of the orders of p-regular elements of G, η is a primitive mth root of unity, T is the multiplicative group of integers t modulo m for which $\eta \to \eta^t$ is an automorphism of $F(\eta)$ over F. Two p-regular elements $x, y \in G$ are F-conjugate if $y^t = g^{-1}xg$ for some $g \in G$ and $t \in T$. This is an equivalence relation and partitions the p-regular elements of G into F-conjugacy classes. Witt-Berman theorem states that the number of F-conjugacy classes of p-regular elements of G is equal to the number of non-isomorphic simple FG-modules.

By Wedderburn-Artin theorem [12, Theorem 2.6.18], a ring R is semisimple if and only if it is a direct sum of matrix algebras over division rings K_i, i.e.,

$$R \cong M(n_1, K_1) \oplus \cdots \oplus M(n_s, K_s).$$

By [12, Proposition 3.6.11], if FG is a semisimple group algebra and G' is the commutator subgroup of G, then $FG \cong F(G/G') \oplus \Delta(G, G')$, where $F(G/G')$ is the sum of all the commutative simple components of FG and $\Delta(G, G')$ is the sum of all the other components.

2. Structure of $U(FD_{11})$.

Theorem 2.1. *Let F be a finite field of characteristic p having $q = p^n$ elements. If $p \nmid |D_{11}|$, then*

$$U(FD_{11}) \cong \begin{cases} GL(2,F)^5 \times C_{q-1}^2, & \text{if } p \equiv \pm 1 \bmod 11; \\ GL(2,F_5) \times C_{q-1}^2, & \text{if } p \equiv \pm 2, \pm 3, \pm 4, \pm 5 \bmod 11. \end{cases}$$

Proof. Since $p \nmid |D_{11}|$,

$$FD_{11} \cong M(n_1, K_1) \oplus M(n_2, K_2) \oplus \cdots \oplus M(n_t, K_t),$$

where K_i's are finite fields. Since FD_{11} is non-commutative, at least one $n_k > 1$. Obviously, $F(D_{11}/D'_{11}) \cong FC_2 \cong F \oplus F$ by [4, Proposition 1.2] and $\dim_F Z(FD_{11}) = 7$. Due to dimension constraints $\sum_{i=1}^{t}[K_i : F] = 7$ and $n_k \leq 2$ for all k. Thus we have the following possibilities:

$$\begin{aligned} FD_{11} &\cong M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus M(2,F_2) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F) \oplus M(2,F_2) \oplus M(2,F_2) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F) \oplus M(2,F) \oplus M(2,F_3) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F_2) \oplus M(2,F_3) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F) \oplus M(2,F_4) \oplus F \oplus F, \text{ or} \\ &\cong M(2,F_5) \oplus F \oplus F. \end{aligned}$$

By [3], $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^{\pm 1}\}$, $\mathcal{C}_3 = \{r^{\pm 2}\}$, $\mathcal{C}_4 = \{r^{\pm 3}\}$, $\mathcal{C}_5 = \{r^{\pm 4}\}$, $\mathcal{C}_6 = \{r^{\pm 5}\}$ and $\mathcal{C}_7 = \{s, rs, \ldots, r^{10}s\}$ are all the conjugacy classes of D_{11}.

Now by [12, Theorem 3.6.2],

$$Z(FD_{11}) = F\widehat{\mathcal{C}_1} + F\widehat{\mathcal{C}_2} + F\widehat{\mathcal{C}_3} + F\widehat{\mathcal{C}_4} + F\widehat{\mathcal{C}_5} + F\widehat{\mathcal{C}_6} + F\widehat{\mathcal{C}_7}.$$

If $p \equiv \pm 1 \bmod 11$, then $p^n \equiv \pm 1 \bmod 11$ for all n. So, for all $1 \leq i \leq 7$, $\widehat{\mathcal{C}_i}^{p^n} = \widehat{\mathcal{C}_i}$. Thus $x^{p^n} = x$, for all $x \in Z(FD_{11})$ and

$$FD_{11} \cong M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus F \oplus F.$$

If $p \equiv \pm 2$, ± 3, ± 4 or ± 5 mod 11, then $p^{5n} \equiv \pm 1$ mod 11 for all n. So, for all $1 \le i \le 7$, $\widehat{\mathcal{C}_i}^{p^{5n}} = \widehat{\mathcal{C}_i}$. Thus $x^{p^{5n}} = x$, for any $x \in Z(FD_{11})$ and

$$FD_{11} \cong M(2, F_5) \oplus F \oplus F.$$

Hence,

$$FD_{11} = \begin{cases} M(2,F)^5 \oplus F \oplus F, & \text{if } \ p \equiv \pm 1 \text{ mod } 11; \\ M(2,F_5) \oplus F \oplus F, & \text{if } \ p \equiv \pm 2, \pm 3, \pm 4, \pm 5 \text{ mod } 11. \end{cases} \qquad \square$$

3. Structure of $U(FD_{13})$.

Theorem 3.1. *Let F be a finite field of characteristic p having $q = p^n$ elements. If $p \nmid |D_{13}|$, then*

$$U(FD_{13}) \cong \begin{cases} GL(2,F)^6 \times C_{q-1}^2, & \textit{if } \ q \equiv \pm 1 \text{ mod } 13; \\ GL(2,F_6) \times C_{q-1}^2, & \textit{if } \ q \equiv \pm 2, \pm 6 \text{ mod } 13; \\ GL(2,F_3)^2 \times C_{q-1}^2, & \textit{if } \ q \equiv \pm 3, \pm 4 \text{ mod } 13; \\ GL(2,F_2)^3 \times C_{q-1}^2, & \textit{if } \ q \equiv \pm 5 \text{ mod } 13. \end{cases}$$

P r o o f. Since, $F(D_{13}/D'_{13}) \cong FC_2$, we have

$$FD_{13} \cong F \oplus F \oplus \left(\oplus_{i=1}^{k} M(n_i, K_i) \right),$$

where $n_i \ge 2$ and K_i's are finite fields. Hence,

$$Z(FD_{13}) \cong F \oplus F \oplus \left(\oplus_{i=1}^{k} K_i \right).$$

As $\dim_F Z(FD_{13}) = 8$, so $\sum_{i=1}^{k} [K_i : F] = 6$.

By [3], $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^{\pm 1}\}$, $\mathcal{C}_3 = \{r^{\pm 2}\}$, $\mathcal{C}_4 = \{r^{\pm 3}\}$, $\mathcal{C}_5 = \{r^{\pm 4}\}$, $\mathcal{C}_6 = \{r^{\pm 5}\}$, $\mathcal{C}_7 = \{r^{\pm 6}\}$ and $\mathcal{C}_8 = \{s, rs, \ldots, r^{12}s\}$ are all the conjugacy classes of D_{13}.

Now for any $l \in \mathbb{N}$, we have $x^{q^l} = x$ for all $x \in Z(FD_{13})$ if and only if $\widehat{\mathcal{C}_i}^{q^l} = \widehat{\mathcal{C}_i}$ for all $i \in \{1, 2, \ldots, 8\}$. This happens, if and only if $r^{q^l} = r$ or r^{-1}, i.e., if and only if $13 | (q^l - 1)$ or $13 | (q^l + 1)$.

For each $i \in \{1, 2, \ldots, k\}$, let $K_i^* = \langle y_i \rangle$. Then, $x^{q^l} = x$ for all $x \in Z(FD_{13})$ if and only if $y_i^{q^l} = y_i$. This is possible if and only if $[K_i : F] | l$ for all $i = 1, \ldots, k$. Thus the least number t such that $13|(q^t - 1)$ or $13|(q^t + 1)$ is $t = l.c.m.\{[K_i : F] : 1 \leq i \leq k\}$.

If,

1. $q \equiv \pm 1 \bmod 13$, then $t = 1$.
2. $q \equiv \pm 2, \pm 6 \bmod 13$, then $t = 6$.
3. $q \equiv \pm 3, \pm 4 \bmod 13$, then $t = 3$.
4. $q \equiv \pm 5 \bmod 13$, then $t = 2$.

Clearly $m = 26$. Let a be the number of simple components in the Wedderburn decomposition of FD_{13}. Then, for

1. $q \equiv 1 \bmod 13$.
   $T = \{1\} \bmod 26$. Thus $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 8\}$ are the p-regular F-conjugacy classes. Hence $a = 8$.

2. $q \equiv -1 \bmod 13$.
   $T = \{1, -1\} \bmod 26$. Thus $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 8\}$ are the p-regular F-conjugacy classes. Hence $a = 8$.

3. $q \equiv \pm 2$ or $\pm 6 \bmod 13$.
   $T = \{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\} \bmod 26$. Since $(r^2)^3 = r^{-7}$, $r^9 = r^{-4}$, $r^7 = r^{-6}$. Thus the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, r^{\pm 4}, r^{\pm 5}, r^{\pm 6}\}$ and $\{s, rs, \ldots, r^{12}s\}$. Hence $a = 3$.

4. $q \equiv 3$ or $-4 \bmod 13$.
   $T = \{1, 3, 9\} \bmod 26$. Since $r^9 = r^{-4}$, $(r^2)^3 = r^{-7}$, $(r^2)^9 = r^5$. Thus the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 4}\}$, $\{r^{\pm 2}, r^{\pm 5}, r^{\pm 6}\}$ and $\{s, rs, \ldots, r^{12}s\}$. Hence $a = 4$.

5. $q \equiv 4$ or $-3 \bmod 13$.
   $T = \{1, 3, 9, 17, 23, 25\} \bmod 26$. Since $r^9 = r^{-4}$, $r^{25} = r^{-1}$, $r^{17} = r^4$, $r^{23} = r^{-3}$, $(r^2)^3 = r^{-7}$, $(r^2)^9 = r^5$. Thus the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 4}\}$, $\{r^{\pm 2}, r^{\pm 5}, r^{\pm 6}\}$ and $\{s, rs, \ldots, r^{12}s\}$. Hence $a = 4$.

6. $q \equiv \pm 5 \bmod 13$.
   $T = \{1, 5, 21, 25\} \bmod 26$. Since $r^{21} = r^{-5}$, $r^{25} = r^{-1}$, $(r^2)^5 = r^{-3}$, $(r^4)^5 = r^{-6}$. Thus the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 5}\}$, $\{r^{\pm 2}, r^{\pm 3}\}$, $\{r^{\pm 4}, r^{\pm 6}\}$ and $\{s, rs, \ldots, r^{12}s\}$. Hence $a = 5$.

Now, we have the following possibilities for $[K_i : F]_{i=1}^{k}$ depending on q.

1. $q \equiv \pm 1 \bmod 13$, then $[K_i : F]_{i=1}^{k} = (1,1,1,1,1,1)$.
2. $q \equiv \pm 2, \pm 6 \bmod 13$, then $[K_i : F]_{i=1}^{k} = (6)$.
3. $q \equiv \pm 3, \pm 4 \bmod 13$, then $[K_i : F]_{i=1}^{k} = (3,3)$.
4. $q \equiv \pm 5 \bmod 13$, then $[K_i : F]_{i=1}^{k} = (2,2,2)$.

Due to dimension constraints, $n_i > 2$ is impossible for any $1 \le i \le k$. Therefore,

$$FD_{13} = \begin{cases} M(2,F)^6 \oplus F \oplus F, & \text{if } \ q \equiv \pm 1 \bmod 13; \\ M(2,F_6) \oplus F \oplus F, & \text{if } \ q \equiv \pm 2, \pm 6 \bmod 13; \\ M(2,F_3)^2 \oplus F \oplus F, & \text{if } \ q \equiv \pm 3, \pm 4 \bmod 13; \\ M(2,F_2)^3 \oplus F \oplus F, & \text{if } \ q \equiv \pm 5 \bmod 13. \end{cases} \qquad \square$$

4. Structure of $U(FD_{17})$.

Theorem 4.1. *Let F be a finite field of characteristic p having $q = p^n$ elements. If $p \nmid |D_{17}|$, then*

$$U(FD_{17}) \cong \begin{cases} GL(2,F)^8 \times C_{q-1}^2, & \text{if } \ q \equiv \pm 1 \bmod 17; \\ GL(2,F_4)^2 \times C_{q-1}^2, & \text{if } \ q \equiv \pm 2, \pm 8 \bmod 17; \\ GL(2,F_8) \times C_{q-1}^2, & \text{if } \ q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \bmod 17; \\ GL(2,F_2)^4 \times C_{q-1}^2, & \text{if } \ q \equiv \pm 4 \bmod 17. \end{cases}$$

Proof. Since, $F(D_{17}/D_{17}') \cong FC_2$,

$$FD_{17} \cong F \oplus F \oplus \Big(\oplus_{i=1}^{k} M(n_i, K_i) \Big),$$

where $n_i \ge 2$ and K_i's are finite fields. Also $\dim_F Z(FD_{17}) = 10$, so $\sum_{i=1}^{k} [K_i : F] = 8$.

By [3], $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^{\pm 1}\}$, $\mathcal{C}_3 = \{r^{\pm 2}\}$, $\mathcal{C}_4 = \{r^{\pm 3}\}$, $\mathcal{C}_5 = \{r^{\pm 4}\}$, $\mathcal{C}_6 = \{r^{\pm 5}\}$, $\mathcal{C}_7 = \{r^{\pm 6}\}$, $\mathcal{C}_8 = \{r^{\pm 7}\}$, $\mathcal{C}_9 = \{r^{\pm 8}\}$ and $\mathcal{C}_{10} = \{s, rs, \ldots, r^{16}s\}$ are all the conjugacy classes of D_{17}.

As in the previous theorem, $x^{q^l} = x$ for all $x \in Z(FD_{17})$ if and only if $17|(q^l - 1)$ or $17|(q^l + 1)$ and $[K_i : F]|l$ for all $i = 1, \dots, k$. Thus the least number t such that $17|(q^t - 1)$ or $17|(q^t + 1)$ is $t = l.c.m.\{[K_i : F] : 1 \leq i \leq k\}$.

Now if,

1. $q \equiv \pm 1 \bmod 17$, then $t = 1$.
2. $q \equiv \pm 2, \pm 8 \bmod 17$, then $t = 4$.
3. $q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \bmod 17$, then $t = 8$.
4. $q \equiv \pm 4 \bmod 17$, then $t = 2$.

Clearly, $m = 34$. Let a be the number of simple components in the Wedderburn decomposition of FD_{17}. Then

1. $q \equiv 1 \bmod 17$.
   $T = \{1\} \bmod 34$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \dots, 10\}$ are the p-regular F-conjugacy classes. Hence $a = 10$.

2. $q \equiv -1 \bmod 17$.
   $T = \{1, -1\} \bmod 34$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \dots, 10\}$ are the p-regular F-conjugacy classes. Hence $a = 10$.

3. $q \equiv \pm 2$ or $\pm 8 \bmod 17$.
   $T = \{1, 9, 13, 15, 19, 21, 25, 33\} \bmod 34$. Since $r^{19} = r^2$, $r^{21} = r^4$, $r^{25} = r^8$, $r^9 = r^{-8}$, $(r^3)^9 = r^{-7}$, $(r^3)^{13} = r^5$, $(r^3)^{15} = r^{-6}$. Therefore the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 2}, r^{\pm 4}, r^{\pm 8}\}$, $\{r^{\pm 3}, r^{\pm 5}, r^{\pm 6}, r^{\pm 7}\}$ and $\{s, rs, \dots, r^{16}s\}$. Hence $a = 4$.

4. $q \equiv \pm 3$ or ± 5 or ± 6 or $\pm 7 \bmod 17$.
   $T = \{1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 23, 25, 27, 29, 31, 33\} \bmod 34$. Since $r^{19} = r^2$, $r^{21} = r^4$, $r^{25} = r^8$, $r^{11} = r^{-6}$, $(r^3)^9 = r^{-7}$, $(r^3)^{13} = r^5$, $(r^3)^{15} = r^{-6}$. Therefore the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, r^{\pm 4}, r^{\pm 5}, r^{\pm 6}, r^{\pm 7}, r^{\pm 8}\}$ and $\{s, rs, \dots, r^{16}s\}$. Hence $a = 3$.

5. $q \equiv \pm 4 \bmod 17$.
   $T = \{1, 13, 21, 33\} \bmod 34$. Since $r^{21} = r^4$, $r^{13} = r^{-4}$, $r^{33} = r^{-1}$, $(r^2)^{13} = r^{-8}$, $(r^6)^{13} = r^{-7}$. Therefore the p-regular F-conjugacy classes are given by $\{1\}$, $\{r^{\pm 1}, r^{\pm 4}\}$, $\{r^{\pm 2}, r^{\pm 8}\}$, $\{r^{\pm 3}, r^{\pm 5}\}$, $\{r^{\pm 6}, r^{\pm 7}\}$ and $\{s, rs, \dots, r^{12}s\}$. Hence $a = 6$.

Now, we have the following possibilities for $[K_i : F]_{i=1}^k$ depending on q.

1. $q \equiv \pm 1 \bmod 17$, then $[K_i : F]_{i=1}^k = (1,1,1,1,1,1,1,1)$.
2. $q \equiv \pm 2, \pm 8 \bmod 17$, then $[K_i : F]_{i=1}^k = (4,4)$.
3. $q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \bmod 17$, then $[K_i : F]_{i=1}^k = (8)$.
4. $q \equiv \pm 4 \bmod 17$, then $[K_i : F]_{i=1}^k = (2,2,2,2)$.

Due to dimension constraints, $n_i > 2$ is impossible for any $1 \le i \le k$. Therefore,

$$FD_{17} \cong \begin{cases} M(2,F)^8 \oplus F \oplus F, & \text{if } q \equiv \pm 1 \bmod 17; \\ M(2,F_4)^2 \oplus F \oplus F, & \text{if } q \equiv \pm 2, \pm 8 \bmod 17; \\ M(2,F_8) \oplus F \oplus F, & \text{if } q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \bmod 17; \\ M(2,F_2)^4 \oplus F \oplus F, & \text{if } q \equiv \pm 4 \bmod 17. \end{cases} \qquad \square$$

5. Structure of $U(FD_{19})$.

Theorem 5.1. *Let F be a finite field of characteristic p having $q = p^n$ elements. If $p \nmid |D_{19}|$, then*

$$U(FD_{19}) \cong \begin{cases} GL(2,F)^9 \times C_{q-1}^2, & \text{if } q \equiv \pm 1 \bmod 19; \\ GL(2,F_9) \times C_{q-1}^2, & \text{if } q \equiv \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9 \bmod 19; \\ GL(2,F_3)^3 \times C_{q-1}^2, & \text{if } q \equiv \pm 7, \pm 8 \bmod 19. \end{cases}$$

Proof. Since $F(D_{19}/D'_{19}) \cong FC_2$, we have

$$FD_{19} \cong F \oplus F \oplus \Big(\oplus_{i=1}^k M(n_i, K_i) \Big),$$

where $n_i \ge 2$ and K_i's are finite fields. As $\dim_F Z(FD_{19}) = 11$, so $\sum_{i=1}^k [K_i : F] = 9$.

By [3], $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^{\pm 1}\}$, $\mathcal{C}_3 = \{r^{\pm 2}\}$, $\mathcal{C}_4 = \{r^{\pm 3}\}$, $\mathcal{C}_5 = \{r^{\pm 4}\}$, $\mathcal{C}_6 = \{r^{\pm 5}\}$, $\mathcal{C}_7 = \{r^{\pm 6}\}$, $\mathcal{C}_8 = \{r^{\pm 7}\}$, $\mathcal{C}_9 = \{r^{\pm 8}\}$, $\mathcal{C}_{10} = \{r^{\pm 9}\}$, $\mathcal{C}_{11} = \{s, rs, \dots, r^{18}s\}$ are all the conjugacy classes of D_{19}.

Now, $x^{q^l} = x$ for all $x \in Z(FD_{19})$ if and only if $19 | (q^l - 1)$ or $19 | (q^l + 1)$ and $[K_i : F] | l$ for all $i = 1, \dots, k$. So the least number t such that $19 | (q^t - 1)$ or $19 | (q^t + 1)$ is $t = l.c.m.\{[K_i : F] : 1 \le i \le k\}$. Thus we have

1. $q \equiv \pm 1 \bmod 19$, then $t = 1$.

2. $q \equiv \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9 \bmod 19$, then $t = 9$.

3. $q \equiv \pm 7, \pm 8 \bmod 19$, then $t = 3$.

Here $m = 38$. Let a be the number of simple components in the Wedderburn decomposition of FD_{19}. Then

1. $q \equiv 1 \bmod 19$.
   $T = \{1\} \bmod 38$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 11\}$ are the p-regular F-conjugacy classes. So $a = 11$.

2. $q \equiv -1 \bmod 19$.
   $T = \{1, -1\} \bmod 38$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 11\}$ are the p-regular F-conjugacy classes. So $a = 11$.

3. $q \equiv 2, 3, -4, -5, -6$ or $-9 \bmod 19$.
   $T = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29, 31, 33, 35, 37\} \bmod 38$. Since $r^{21} = r^2$, $r^{23} = r^4$, $r^{25} = r^6$, $r^{27} = r^8$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, r^{\pm 4}, r^{\pm 5}, r^{\pm 6}, r^{\pm 7}, r^{\pm 8}, r^{\pm 9}\}$ and $\{s, rs, \ldots, r^{18}s\}$. Hence $a = 3$.

4. $q \equiv -2, -3, 4, 5, 6$ or $9 \bmod 19$.
   $T = \{1, 5, 7, 9, 11, 17, 23, 25, 35\} \bmod 38$. Since $r^{17} = r^{-2}$, $r^{23} = r^4$, $r^{25} = r^6$, $r^{35} = r^{-3}$, $r^{11} = r^{-8}$, the p-regular F-conjugacy classes are $\{1\}$, $\{s, rs, \ldots, r^{18}s\}$, $\{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, r^{\pm 4}, r^{\pm 5}, r^{\pm 6}, r^{\pm 7}, r^{\pm 8}, r^{\pm 9}\}$. Hence $a = 3$.

5. $q \equiv 7$ or $-8 \bmod 19$.
   $T = \{1, 7, 11\} \bmod 38$. Since $r^{11} = r^{-8}$, $(r^2)^7 = r^{-5}$, $(r^2)^{11} = r^3$, $(r^4)^7 = r^9$, $(r^4)^{11} = r^6$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 7}, r^{\pm 8}\}$, $\{r^{\pm 2}, r^{\pm 3}, r^{\pm 5}\}$, $\{r^{\pm 4}, r^{\pm 6}, r^{\pm 9}\}$ and $\{s, rs, \ldots, r^{18}s\}$. Hence $a = 5$.

6. $q \equiv -7$ or $8 \bmod 19$.
   $T = \{1, 7, 11, 27, 31, 37\} \bmod 38$. Since $r^{11} = r^{-8}$, $r^{31} = r^{-7}$, $r^{37} = r^{-1}$, $r^{27} = r^8$, $(r^2)^{11} = r^3$, $(r^2)^7 = r^{-5}$, $(r^4)^7 = r^9$, $(r^4)^{11} = r^6$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 7}, r^{\pm 8}\}$, $\{r^{\pm 2}, r^{\pm 3}, r^{\pm 5}\}$, $\{r^{\pm 4}, r^{\pm 6}, r^{\pm 9}\}$ and $\{s, rs, \ldots, r^{18}s\}$. Hence $a = 5$.

We have the following possibilities for $[K_i : F]_{i=1}^k$ depending on q.

1. $q \equiv \pm 1 \bmod 19$, then $[K_i : F]_{i=1}^k = (1, 1, 1, 1, 1, 1, 1, 1, 1)$.

2. $q \equiv \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9 \bmod 19$, then $[K_i : F]_{i=1}^k = (9)$.

3. $q \equiv \pm 7, \pm 8 \bmod 19$, then $[K_i : F]_{i=1}^{k} = (3, 3, 3)$.

Due to dimension constraints, $n_i > 2$ is impossible for any $1 \le i \le k$. Hence

$$FD_{19} \cong \begin{cases} M(2,F)^9 \oplus F \oplus F, & \text{if } q \equiv \pm 1 \bmod 19; \\ M(2,F_9) \oplus F \oplus F, & \text{if } q \equiv \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9 \bmod 19; \\ M(2,F_3)^3 \oplus F \oplus F, & \text{if } q \equiv \pm 7, \pm 8 \bmod 19. \end{cases}$$

□

6. Structure of $U(FD_{23})$.

Theorem 6.1. *Let F be a finite field of characteristic p having $q = p^n$ elements. If $p \nmid |D_{23}|$, then*

$$U(FD_{23}) \cong \begin{cases} GL(2,F)^{11} \times C_{q-1}^2, & \textit{if } p \equiv \pm 1 \bmod 23; \\ GL(2,F_{11}) \times C_{q-1}^2, & \textit{if } p \equiv \pm 2, \pm 3, \dots, \pm 11 \bmod 23. \end{cases}$$

P r o o f. Since $p \nmid |D_{23}|$,

$$FD_{23} \cong M(n_1, K_1) \oplus M(n_2, K_2) \oplus \cdots \oplus M(n_t, K_t),$$

where K_i's are finite fields and at least one $n_k > 1$. Obviously, $F(D_{23}/D'_{23}) \cong F \oplus F$ and $\dim_F Z(FD_{23}) = 13$.

If $p \equiv \pm 1 \bmod 23$, then $p^n \equiv \pm 1 \bmod 23$ for all n. So, for all $1 \le i \le 13$, $\widehat{\mathcal{C}_i}^{p^n} = \widehat{\mathcal{C}_i}$. Thus $x^{p^n} = x$, for all $x \in Z(FD_{23})$ and

$$FD_{23} \cong M(2,F)^{11} \oplus F \oplus F.$$

If $p \equiv \pm 2, \pm 3, \dots, \pm 11 \bmod 23$, then $p^{11n} \equiv \pm 1 \bmod 23$ for all n. So, for all $1 \le i \le 13$, $\widehat{\mathcal{C}_i}^{p^{11n}} = \widehat{\mathcal{C}_i}$. Thus $x^{p^{11n}} = x$, for any $x \in Z(FD_{23})$ and

$$FD_{23} \cong M(2,F_{11}) \oplus F \oplus F.$$

Hence,

$$FD_{23} \cong \begin{cases} M(2,F)^{11} \oplus F \oplus F, & \text{if } p \equiv \pm 1 \bmod 11; \\ M(2,F_{11}) \oplus F \oplus F, & \text{if } p \equiv \pm 2, \pm 3, \dots, \pm 11 \bmod 23. \end{cases}$$

□

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