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THE ALGEBRAIC SOLUTION OF UNILATERAL MATRIX POLYNOMIAL EQUATIONS

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Abstract. In this paper we describe an algebraic method of obtaining all solutions of the unilateral matrix polynomial equation $A_0X^m + A_1X^{m-1} + A_2X^{m-2} + \cdots + A_{m-1}X + A_m = 0$ where the coefficients A_i and X are square matrices of order 2 and m is a positive integer ≥ 2. We also show how an extension of the method can be used to solve the quadratic matrix equation $AX^2 + BX + C = 0$ where A, B, C and X are 3×3 matrices. We give several illustrative examples which show that the algebraic method described in the paper is quite practical and effective in solving such matrix equations.

1. Introduction. This paper is concerned with the unilateral matrix polynomial equation,

$$A_0X^m + A_1X^{m-1} + A_2X^{m-2} + \cdots + A_{m-1}X + A_m = 0, \tag{1}$$

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where m is an integer ≥ 2, the coefficients A_i, $i = 0, 1, \ldots, m$, are square matrices of order k with real entries and X is an unknown $k \times k$ matrix, and $k \leq 3$. Any solution of the matrix equation (1) is called a solvent of the equation. A theoretical method of solving Eq. (1), when all the matrices A_i and the unknown matrix X are square matrices of order k, has been described in [2, pp. 227–231] and [5, pp. 465–467] but, as has been stated in [5, p. 467], this general method "is often of little practical use" in solving a specific equation. Several mathematicians have given numerical methods for obtaining a specific solvent of Eq. (1), especially when $m = 2$ (see, for instance, [1, 3, 4, 6]), but the exact computation of the solvents still poses difficulties.

In Section 2 of this paper we describe an algebraic method of obtaining all the solvents of Eq. (1) when the coefficients A_i and X are square matrices of order 2 and m is a positive integer ≥ 2. When all the entries of the matrices A_i, $i = 0, 1, \ldots, m$, are rational numbers and Eq. (1) also has a solvent whose entries are all rational numbers, the existing methods give only a numerical approximation of the solvent but the method described in this paper yields the precise rational solvent. We illustrate the algebraic method of solving Eq. (1) by giving several numerical examples. In Section 3 we show that a similar approach can be used to solve the unilateral quadratic matrix equation in 3×3 matrices.

2. The unilateral matrix polynomial equation in 2×2 matrices. In Section 2.1 we first prove a lemma that we will use to solve unilateral matrix polynomial equations in 2×2 matrices. In Section 2.2 we will describe our method of obtaining all solvents of Eq. (1) and in Section 2.3 we give several illustrative examples which show that the method is quite practical and effective in solving such matrix equations.

2.1. A preliminary lemma.

Lemma 1. *If p is the trace and q the determinant of an arbitrary 2×2 matrix X, and I is the 2×2 identity matrix, then for any positive integer $n \geq 2$,*

$$X^n = \phi_{n-1}(p, q)X - q\phi_{n-2}(p, q)I, \tag{2}$$

where $\phi_0(p, q) = 1$, $\phi_1(p, q) = p$, and for any positive integer $n \geq 2$,

$$\phi_n(p, q) = p^n - {}^{n-1}C_1 p^{n-2} q + {}^{n-2}C_2 p^{n-4} q^2 - \cdots + {}^{n-j}C_j p^{n-2j} (-q)^j - \cdots, \tag{3}$$

with the last term being $(-q)^{n/2}$ or ${}^{(n+1)/2}C_{(n-1)/2} p q^{(n-1)/2}$ according as n is even or odd.

P r o o f. Since the trace and determinant of the matrix X are p and q respectively, the characteristic equation of X is $\lambda^2 - p\lambda + q = 0$, and it follows from the Cayley-Hamilton theorem that

$$X^2 = pX - qI. \tag{4}$$

On multiplying Eq. (4) by X and replacing X^2 by $pX - q$ on the right-hand side of the resulting equation, we get,

$$X^3 = (p^2 - q)X - pqI. \tag{5}$$

Thus, the relation (2) is true both for $n = 2$ and $n = 3$. We now assume that the relation (2) is true for any arbitrary positive integer $n \geq 3$. On multiplying (2) by X we get,

$$\begin{aligned}
X^{n+1} &= \phi_{n-1}(p,q)X^2 - q\phi_{n-2}(p,q)X,\\
&= \phi_{n-1}(p,q)(pX - qI) - q\phi_{n-2}(p,q)X,\\
&= (p\phi_{n-1}(p,q) - q\phi_{n-2}(p,q))X - q\phi_{n-1}(p,q)I,\\
&= (p^n - ({}^{n-2}C_1 + 1)p^{n-2}q + ({}^{n-3}C_2 + {}^{n-3}C_1)p^{n-4}q^2\\
&\quad - \cdots + ({}^{n-j-1}C_j + {}^{n-j-1}C_{j-1})p^{n-2j}(-q)^j - \cdots)X - q\phi_{n-1}(p,q)I,\\
&= \phi_n(p,q)X - q\phi_{n-1}(p,q)I,
\end{aligned}$$

and hence the lemma is proved by induction. □

2.2. A description of the method of solving matrix equations. We now consider Eq. (1) where the coefficients A_i, $i = 0, 1, \ldots, m$, and the unknown matrix X are all square matrices of order 2.

Let p be the trace and q the determinant of the solvent X so that the characteristic polynomial of X is $\lambda^2 - p\lambda + q$. In view of Lemma 1, for any positive integer $n \geq 2$, we can write the matrix X^n as $f_1(p,q)X + f_2(p,q)$ where $f_1(p,q)$, $f_2(p,q)$ are some functions of p and q. Hence we can reduce Eq. (1) to the type,

$$M_1 X = M_2, \tag{6}$$

where M_1 and M_2 are 2×2 matrices whose entries are functions of p and q.

We first assume that the values of p and q are such that the matrix M_1 is nonsingular. Accordingly we may multiply Eq. (6) by the adjoint of the matrix M_1, denoted by adj M_1, when we get,

$$dX = (\text{adj } M_1) \times M_2, \tag{7}$$

where d is the determinant of the matrix M_1.

We note that the characteristic polynomial of dX is $\lambda^2 - pd\lambda + qd^2$. Since the matrices on the two sides of Eq. (7) are equal, their characteristic polynomials must be necessarily identical. On equating these two characteristic polynomials, we get two equations in p and q. The complete solution of these two equations is readily obtained and is given by a finite number of values of (p, q). For each pair of values of p and q thus obtained, we find the numerical entries of the matrix on the right-hand side of Eq. (7) and in each case, we readily obtain the matrix X by solving Eq. (7).

We now note that even though we have ensured that the characteristic polynomials of the matrices on the two sides of Eq. (7) are identical, this is not sufficient to ensure that the matrices on the two sides of Eq. (7) are equal. Therefore for each solution of Eq. (7) that we have obtained, we need to verify by direct computation whether the matrix X is actually a solvent of Eq. (1). This procedure gives us all solvents X whose trace p and determinant q are such that that the matrix M_1 is nonsingular.

We now consider the case when the values of p and q are such that the M_1 is a singular matrix. It follows from Eq. (6) that in this case the matrix M_2 must also necessarily be a singular matrix. Equating to 0 the determinants of the two matrices M_1 and M_2, we get two simultaneous equations in p and q. We solve these equations for p and q, and thus the matrices M_1 and M_2 are now known and Eq. (6) reduces to a linear equation in X which is readily solved. The trace and determinant of any such matrix X must be p and q respectively for it to be actually a solvent of Eq. (1). As before, we may get several solutions for p and q and each such solution may yield a solvent of Eq. (1).

2.3. Numerical examples. We give below three examples to illustrate the method of solution described in Section 2.2

2.3.1. Example 1. Consider the quadratic equation,

$$AX^2 + BX + C = 0, \tag{8}$$

where A, B, C are the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & -6 \\ 2 & -12 \end{bmatrix}. \tag{9}$$

Here we have $X^2 = pX - qI$, and Eq. (8) reduces to

$$(Ap + B)X - Aq + C = 0, \tag{10}$$

which may be written as Eq. (6) where

$$(11)\qquad M_1 = \begin{bmatrix} p+2 & p+1 \\ 5p-1 & 2p+2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} q-7 & q+6 \\ 5q-2 & 2q+12 \end{bmatrix}.$$

We first assume that the matrix M_1 is nonsingular and hence $(p+1)(3p-5) \neq 0$. Now
adj $M_1 = \begin{bmatrix} 2p+2 & -p-1 \\ -5p+1 & p+2 \end{bmatrix}$ and $\det(M_1) = -(p+1)(3p-5)$,
and hence on multiplying Eq. (11) by the matrix adj M_1, we get

$$(12)\qquad (p+1)(5-3p)X = \begin{bmatrix} -3pq-12p-3q-12 & 0 \\ 33p+11q-11 & -3pq-18p+5q+30 \end{bmatrix}.$$

On equating the characteristic polynomials of the matrices on the two sides of Eq. (12), we get the following two conditions:

$$(13)\qquad \begin{aligned} p(p+1)(3p-5) &= 6pq+30p-2q-18, \\ q(p+1)^2(3p-5)^2 &= 3(3pq+18p-5q-30)(pq+4p+q+4). \end{aligned}$$

Since the first of these two equations is linear in q, Eqs. (13) are readily solved for p and q and we get the following four solutions for (p, q):

$$(0, -9), \quad (5, 6), \quad (2/3, -8/3), \quad (-13/3, 4).$$

With each of these solutions for (p, q), we solve Eq. (12) for X, and verify whether the matrix X so obtained actually satisfies Eq. (8). In fact, in each of the four cases, we find that the matrix X is a solvent of Eq. (8). We thus obtain the following four solvents of the quadratic matrix equation (8):

$$\begin{bmatrix} 3 & 0 \\ -22 & -3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 \\ -11/3 & 2 \end{bmatrix}, \quad \begin{bmatrix} -4/3 & 0 \\ -11/3 & 2 \end{bmatrix}, \quad \begin{bmatrix} -4/3 & 0 \\ 11/6 & -3 \end{bmatrix}.$$

Next we try to determine any solvents X whose trace p and determinant q are such that the matrices M_1 and M_2 are nonsingular. In this case, $\det(M_1)$ and $\det(M_2)$ are both 0, and hence we get $(p+1)(3p-5) = 0$ and $(q+4)(q+6) = 0$. In addition, the condition (6) must be satisfied. Of the four possible solutions for $(p,\ q)$, it is readily seen that there is just one solution namely $(p,\ q) = (5/3,\ -4)$ that yields a solvent X that satisfies the condition (6) and is such that its trace p and determinant q are given by $(p,\ q) = (5/3,\ -4)$. We thus get a fifth solvent

$\begin{bmatrix} -1/3 & -10/11 \\ -11/3 & 2 \end{bmatrix}$ of Eq. (8). There are no other solutions of the matrix equation (8) when the matrices A, B, C are given by (9).

2.3.2. Example 2. As a second example, we consider the quadratic equation,

$$AX^2 + BX + C = 0, \tag{14}$$

where A, B, C are the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}, \tag{15}$$

Proceeding as in the case of the first example, we note that Eq. (14) reduces to Eq. (6) where

$$M_1 = \begin{bmatrix} p+1 & 2p-1 \\ 3p & 5p-5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} q+2 & 2q+1 \\ 3q-2 & 5q-1 \end{bmatrix}. \tag{16}$$

Now adj $M_1 = \begin{bmatrix} 5p-5 & -2p+1 \\ -3p & p+1 \end{bmatrix}$ and $\det(M_1) = -p^2+3p-5$ and hence, assuming that M_1 is nonsingular, Eq. (7) reduces to

$$(-p^2+3p-5)X = \begin{bmatrix} -pq+14p-2q-12 & 7p-5q-6 \\ -8p+3q-2 & -pq-4p+5q-1 \end{bmatrix}. \tag{17}$$

The characteristic polynomials of the matrices on the two sides of Eq. (11) will be identical if the following two equations are satisfied:

$$\begin{aligned} -p(-p^2+3p-5) &= 2pq-10p-3q+13, \\ q(-p^2+3p-5)^2 &= p^2q^2-10p^2q-3pq^2+30pq+5q^2-50q. \end{aligned} \tag{18}$$

We readily obtain six solutions $(p,\ q)$ of (18) given by

$$\begin{gathered} (1,\ 0),\ \ (2,\ 13),\ \ (1+2\sqrt{-3},\ 0),\ \ (1-2\sqrt{-3},\ 0), \\ (2+2\sqrt{-3},\ 1+2\sqrt{-3}),\quad (2-2\sqrt{-3},\ 1-2\sqrt{-3}), \end{gathered}$$

and we thus get the following six solvents of Eq. (14):

$$\begin{bmatrix} -2/3 & -1/3 \\ 10/3 & 5/3 \end{bmatrix}, \quad \begin{bmatrix} 12 & 19 \\ -7 & -10 \end{bmatrix},$$

$$\begin{bmatrix} 2+(8/3)\sqrt{-3} & 1+(4/3)\sqrt{-3} \\ -2-(4/3)\sqrt{-3} & -1-(2/3)\sqrt{-3} \end{bmatrix},$$

$$\begin{bmatrix} 2-(8/3)\sqrt{-3} & 1-(4/3)\sqrt{-3} \\ -2+(4/3)\sqrt{-3} & -1+(2/3)\sqrt{-3} \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} 36/31+(70/31)\sqrt{-3} & 1/31+(14/31)\sqrt{-3} \\ -25/31-(40/31)\sqrt{-3} & 26/31-(8/31)\sqrt{-3} \end{bmatrix},$$

$$\begin{bmatrix} 36/31-(70/31)\sqrt{-3} & 1/31-(14/31)\sqrt{-3} \\ -25/31+(40/31)\sqrt{-3} & 26/31+(8/31)\sqrt{-3} \end{bmatrix}.$$

We also note that in this example there are no solutions for which $\det(M_1) = 0$. Thus the complete solution of Eq. (14) is given by (19).

2.3.3. Example 3. As a final example, we consider the following matrix equation of degree 6 in 2×2 matrices:

$$A_0X^6 + A_1X^5 + A_2X^4 + A_3X^3 + A_4X^2 + A_5X + A_6 = 0, \tag{20}$$

where

$$A_0 = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_5 = \begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} -6910 & -5670 \\ -2580 & -2100 \end{bmatrix}.$$

We first compute the values of X^n, $n = 2, 3, \ldots, 6$, using Lemma 1, and substituting these values in Eq. (20), we reduce it to

$$\begin{aligned} &\{p(p^2-3q)(p^2-q)A_0 + (p^4-3p^2q+q^2)A_1 + p(p^2-2q)A_2 \\ &\quad + (p^2-q)A_3 + pA_4 + A_5\}X - (p^4-3p^2q+q^2)qA_0 \\ &\quad\quad - pq(p^2-2q)A_1 - (p^2-q)qA_2 - pqA_3 - qA_4 + A_6 = 0. \end{aligned} \tag{21}$$

Now on using the values of the matrices A_i given by (2.3), we may write Eq. (21) as Eq. (6) where

$$M_1 = \begin{bmatrix} m_{111} & m_{112} \\ m_{121} & m_{122} \end{bmatrix}, \quad M_2 = \begin{bmatrix} m_{211} & m_{212} \\ m_{221} & m_{222} \end{bmatrix}, \tag{22}$$

with the entries m_{ijk} being given by

$$
\begin{aligned}
m_{111} &= p^5 + 2p^4 - 4p^3q + p^3 - 6p^2q + 3pq^2 - 2pq + 2q^2 + p - 1,\\
m_{112} &= 4p^5 + 3p^4 - 16p^3q + 5p^3 - 9p^2q + 12pq^2 + 2p^2\\
&\qquad - 10pq + 3q^2 + p - 2q + 3,\\
m_{121} &= 2p^5 + p^4 - 8p^3q - 3p^2q + 6pq^2 + p^2 + q^2 + p - q - 2,\\
m_{122} &= p^4 + 3p^3 - 3p^2q + 2p^2 - 6pq + q^2 + 2p - 2q,
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
m_{211} &= p^4q + 2p^3q - 3p^2q^2 + p^2q - 4pq^2 + q^3 - q^2 + q + 6910,\\
m_{212} &= 4p^4q + 3p^3q - 12p^2q^2 + 5p^2q - 6pq^2 + 4q^3\\
&\qquad + 2pq - 5q^2 + q + 5670,\\
m_{221} &= 2p^4q + p^3q - 6p^2q^2 - 2pq^2 + 2q^3 + pq + q + 2580,\\
m_{222} &= p^3q + 3p^2q - 2pq^2 + 2pq - 3q^2 + 2q + 2100.
\end{aligned} \tag{24}
$$

We have now obtained an equation of type (6), and proceeding as in the two numerical examples above, we get two simultaneous equations in p and q. Using MAPLE, we could readily obtain 20 real numerical solutions of these equations. Only 10 of these solutions actually yielded solvents of Eq. (20). These solvents of the matrix equation (20), with entries given up to six decimal places (where applicable), are as follows:

$$
\begin{bmatrix} 1 & 2\\ 3 & 1\end{bmatrix}, \qquad \begin{bmatrix} 633.225155 & 518.263447\\ -772.103109 & -631.925568\end{bmatrix},
$$

$$
\begin{bmatrix} -2.688082 & -1.011306\\ -1.354032 & -2.555052\end{bmatrix}, \qquad \begin{bmatrix} -417.772238 & -339.926101\\ 513.428429, & 417.763067\end{bmatrix},
$$

$$
\begin{bmatrix} 2.448378 & 0.817403\\ 1.234616 & 2.441429\end{bmatrix}, \qquad \begin{bmatrix} -486.557419 & -398.066135\\ 593.006842 & 485.151771\end{bmatrix},
$$

$$
\begin{bmatrix} -63.988867 & 55.063188\\ -73.723790 & 63.644712\end{bmatrix}, \qquad \begin{bmatrix} -614.403218 & 562.367752\\ -666.949067 & 610.440160\end{bmatrix},
$$

$$
\begin{bmatrix} -1.334195 & -2.249765\\ -3.033106 & -1.019130\end{bmatrix}, \qquad \begin{bmatrix} 80.797050 & -75.910545\\ 82.660605 & -77.499605\end{bmatrix}.
$$

It is also possible to compute the complex solvents of Eq. (1) in the same way. As an example, using MAPLE, we quickly obtained the following solvent of

Eq. (20):

$$\begin{bmatrix} -1.726914 - 1.701413i, & -0.226516 - 1.389198i, \\ -0.311837 - 1.851264i, & -1.704104 - 1.511551i \end{bmatrix}.$$

All other complex solvents of Eq. (20) may similarly be obtained.

3. The quadratic matrix equation in 3 × 3 matrices. We will briefly describe in Section 3.1 how to apply the algebraic method to solve the quadratic matrix equation,

$$AX^2 + BX + C = 0, \tag{25}$$

where A, B, C are given 3×3 matrices. In Section 3.2 we give two numerical examples to illustrate the method.

3.1. A brief description of the method. Let the characteristic polynomial of the solvent X be $\lambda^3 - p\lambda^2 + q\lambda - r$. It follows from the Cayley-Hamilton theorem that

$$X^3 - pX^2 + qX - rI = 0, \tag{26}$$

where I is the 3×3 identity matrix.

As in the case of Eq. (1), we will now use the two equations (25) and (26) to construct a linear equation

$$M_1 X = M_2 \tag{27}$$

where the entries of the matrices M_1 and M_2 are functions of p, q and r. As in Section 2, we will multiply Eq. (27) by the matrix adj M_1 to get the equation

$$dX = (\text{adj } M_1) \times M_2, \tag{28}$$

where d is the determinant of the matrix M_1.

The characteristic polynomials of the matrices on the two sides of Eq. (27) must be necessarily identical, and we thus obtain three conditions on the numbers p, q, r. It is significant to note that one of these conditions is linear in r and so the three equations in p, q, r are readily solved. As before, we will get a finite number of solutions for p, q, r and for each solution, we will solve Eq. (28) for X and test whether this actually gives a solvent for Eq. (25). Finally, we check for any solvents such that M_1 and M_2 are both singular. The method yields all solvents of Eq. (25).

We give below two illustrative examples – in the first example, the matrix A is singular while in the second one, A is a nonsingular matrix.

3.2. Numerical examples.

3.2.1. Example 1. Consider the matrix equation

$$AX^2 + BX + C = 0, \tag{29}$$

where A, B, C are given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 7 & -37 \\ 3 & 2 & -7 \\ -10 & -7 & -24 \end{bmatrix}. \tag{30}$$

We note that A is a singular matrix. It is readily verified that Eq. (29) has no solvent of the type $X = pI$ where I is the 3×3 identity matrix and p is a complex number. As already noted, X satisfies Eq. (26) for some values of p, q, r, and so we have,

$$\begin{aligned} & (AX^2 + BX + C)(X - pI) = 0, \\ \text{or,} \quad & AX^3 - (pA - B)X^2 - (pB - C)X - pC = 0, \\ \text{or,} \quad & A(pX^2 - qX + rI) - (pA - B)X^2 - (pB - C)X - pC = 0, \\ \text{or,} \quad & BX^2 - (qA + pB - C)X + rA - pC = 0. \end{aligned}$$

We note that B is a nonsingular matrix, and hence it now follows that

$$X^2 = B^{-1}(qA + pB - C)X - B^{-1}(rA - pC). \tag{31}$$

On substituting the value of X^2 given by (31) in Eq. (29), we get a linear equation of type (27). Using the values of the matrices A, B, C given by (30), we now compute the matrices M_1 and M_2 which are as follows:

$$\begin{aligned} M_1 &= \begin{bmatrix} p + 23q + 228 & 2p - 5q + 173 & 3p - 33q - 171 \\ -p + 7q + 65 & -q + 48 & p - 9q - 44 \\ p + 8q + 89 & p - 2q + 65 & p - 12q - 63 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} 226p + 23r - 7 & 173p - 5r - 7 & -172p - 33r + 37 \\ 62p + 7r - 3 & 47p - r - 2 & -44p - 9r + 7 \\ 82p + 8r + 10 & 63p - 2r + 7 & -64p - 12r + 24 \end{bmatrix}. \end{aligned} \tag{32}$$

As already mentioned, we now multiply Eq. (27) by the matrix adj M_1 and obtain Eq. (28). On equating the characteristic polynomials of the matrices on the two sides of Eq. (28), we get the following three equations in p, q, r:

$$\begin{aligned}(33)\quad & 6p^3 - (60q + 14)p^2 - 24q^2 - 72qr + (36q^2 + 128q + 60r - 3508)p \\ & \qquad + 1494q + 130r + 7712 = 0,\end{aligned}$$

$$\begin{aligned}(34)\quad & (36q - 21312)p^4 + (-720q^2 + 224904q + 1548r - 145152)p^3 \\ & + (4032q^3 - 247272q^2 - 15768qr - 216r^2 - 24284q + 23928r - 5162144)p^2 \\ & + (-4320q^4 + 86304q^3 + 12168q^2r + 2160qr^2 - 10616q^2 \\ & - 125964qr - 2088r^2 + 2023440q + 502548r + 16113632)p \\ & + 4(18q^2 - 65q + 806)(18q^3 - 65q^2 - 24qr - 18r^2 + 198q + 747r - 2056) = 0,\end{aligned}$$

$$\begin{aligned}(35)\quad & (216r + 76032)p^6 + (-6480qr - 1520640q + 72792r + 1343232)p^5 \\ & + (68688q^2r + 8515584q^2 - 1469880qr - 864r^2 - 15459840q \\ & + 1477080r + 45522048)p^4 + (-293760q^3r - 9123840q^3 + 8433504q^2r \\ & + 17280qr^2 + 27574272q^2 - 19228320qr - 16704r^2 - 410657280q \\ & + 44679016r + 315239936)p^3 + (412128q^4r + 2737152q^4 - 10307520q^3r \\ & - 96768q^2r^2 - 4561920q^3 + 54523800q^2r + 204480qr^2 + 212627712q^2 \\ & - 417499320qr - 544992r^2 - 154190080q + 380990384r + 4837906304)p^2 \\ & - 8(18q^2 - 65q + 806)(1620q^3r - 24048q^2r - 720qr^2 + 31680q^2 \\ & + 101775qr + 696r^2 - 103840q - 779602r + 1408352)p \\ & + 8(18q^2 - 65q + 806)^2(18q^2r - 65qr - 12r^2 + 198r + 176) = 0.\end{aligned}$$

Since Eq. (33) is linear in r, we solve it for r and substitute this value in Eqs. (34) and (35) to get two equations in p and q from which we eliminate q and get the following equation in p:

$$\begin{aligned}(36)\quad & (p - 5)(9p^2 - 54p + 197)(27p^3 - 63p - 128)(27p^3 + 81p^2 + 18p + 92) \\ & \times (27p^6 - 324p^5 + 2538p^4 - 11920p^3 + 43651p^2 - 81548p + 133128) \\ & \qquad \times (2052p^3 - 8208p^2 + 214569p - 1115776)^4 = 0.\end{aligned}$$

Eq. (36) has just four real roots and in fact, we get only the following three solutions of Eq. (33), (34) and (35) in which p, q, r have real values:

$$(5,\, 6,\, -4), \quad (-3.134170\ldots,\ 15.913343\ldots,\ -6.856723\ldots),$$

and

$$(2.134170\ldots,\ -0.051893\ldots,\ -0.544361\ldots).$$

With these three sets of values of p, q, r, we can now solve Eq. (28) for X to get three matrices. Of these three matrices, only two (obtained from the first two solutions for p, q, r) are actually solvents of Eq. (29). These two solvents, with entries up to six decimal places, are as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 7.500958 & 5.342903 & 6.563507 \\ -18.743592 & -12.949616 & -16.553419 \\ 5.203588 & 3.604531 & 2.314488 \end{bmatrix}.$$

The complex solvents of Eq. (29) are obtained in the same way. We obtain all the complex solutions of the three equations (33), (34) and (35), and proceeding as before, we obtain six complex solvents, three of which are as follows:

$$\begin{bmatrix} 2.435103 - 1.486059i & 1.975519 - 1.216128i & 2.009199 + 1.637453i \\ -2.964273 + 4.169927i & -2.441717 + 3.477412i & -2.615503 - 5.440214i \\ 2.294813 + 1.441257i & 1.575786 + 0.854872i & 0.939528 + 2.639275i \end{bmatrix},$$

$$\begin{bmatrix} -0.860549 - 2.924762i & -0.195588 - 2.326742i & -1.208716 + 2.353264i \\ 6.852603 + 9.736039i & 3.972296 + 7.652466i & 7.661914 - 6.624169i \\ 2.645221 - 4.808754i & 2.072338 - 3.361189i & -2.178832 - 2.178112i \end{bmatrix},$$

$$\begin{bmatrix} 0.705962 + 0.991727i & 1.258149 - 0.870685i & 1.084798 - 0.286006i \\ 3.241427 - 2.999426i & -0.089908 + 2.633339i & -0.358017 + 0.865010i \\ -2.796573 + 0.121216i & 0.577297 - 0.106421i & 3.518117 - 0.034957i \end{bmatrix}.$$

Replacing i by $-i$, we get three additional solvents of Eq. (29) which are conjugates of the three solvents given above.

No additional solvents of Eq. (29) could be obtained by imposing the condition that M_1 be a singular matrix. Thus, Eq. (29) has a total of eight solvents that have been obtained above.

3.2.2. Example 2. Consider the equation

$$AX^2 + BX + C = 0, \tag{37}$$

where A, B, C are given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 7 & -37 \\ 9 & 7 & -15 \\ -4 & -1 & -18 \end{bmatrix}. \tag{38}$$

Since A is nonsingular, we simply multiply Eq. (37) by A^{-1} and on writing $A^{-1}B = B_1$ and $A^{-1}C = C_1$, we get after suitable transposition,

$$X^2 = -B_1X - C_1. \tag{39}$$

On multiplying Eq. (39) by X and replacing X^2 by $-B_1X - C_1$ in the resulting equation, we get

$$X^3 = (B_1^2 - C_1)X + B_1C_1. \tag{40}$$

Now X satisfies the characteristic equation Eq. (26), and substituting the values of X^2 and X^3 given by Eqs. (39) and (40) in Eq. (26), we get the linear equation (27) where

$$M_1 = B_1^2 + pB_1 - C_1 + qI, \quad M_2 = -B_1C_1 - pC_1 + rI, \tag{41}$$

where I is the 3×3 identity matrix.

Now using the values of the matrices A, B, C given by (38), we get

$$\begin{aligned} M_1 &= \begin{bmatrix} -9 + 3p + q & -60 + 9p & 3 + 2p \\ 16 - 5p & 77 - 12p + q & 3 - 2p \\ -7 + 3p & -26 + 5p & 11 + p + q \end{bmatrix}, \\ M_2 &= \begin{bmatrix} -132 + 21p + r & -71 + 11p & -14 + 13p \\ 159 - 23p & 85 - 12p + r & 15 - 9p \\ -46 + 6p & -25 + 2p & 8 + 14p + r \end{bmatrix}. \end{aligned} \tag{42}$$

As the rest of the computations are now straightforward, we do not give all the details. Following the procedure already described above, we found that there are 8 solvents of Eq. (37) whose entries are all real numbers. This includes the following 2 solvents whose entries are all given by rational numbers:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix},$$

$$\begin{bmatrix} -86585/1583 & -78899/1583 & 17569/1583 \\ 94388/1583 & 86175/1583 & -21088/1583 \\ -44889/1583 & -41371/1583 & 5159/1583 \end{bmatrix}. \tag{43}$$

The remaining 6 solvents, with entries up to 6 decimal places, are as follows:

$$\begin{bmatrix} -3.142438 & -1.779472 & -2.386773 \\ 2.840975 & 1.499434 & 1.532002 \\ -0.100968 & 0.207291 & -4.132044 \end{bmatrix},$$

$$\begin{bmatrix} -190.687705 & -144.878838 & 69.082283 \\ 209.414612 & 159.117683 & -77.188316 \\ -102.890946 & -78.222737 & 35.038780 \end{bmatrix},$$

$$\begin{bmatrix} -1.267111 & -1.898591 & 17.085345 \\ 0.775378 & 1.630639 & -19.915761 \\ 0.926862 & 0.142004 & 6.540251 \end{bmatrix},$$

$$\begin{bmatrix} -3.183177 & 4.672612 & 2.206394 \\ 3.560661 & -0.370179 & -0.450082 \\ -6.719920 & 4.021791 & 4.649578 \end{bmatrix},$$

$$\begin{bmatrix} 0.633603 & 1.321677 & 1.105665 \\ 2.066575 & 0.941549 & -0.019199 \\ -0.847738 & -1.133677 & 2.956088 \end{bmatrix},$$

$$\begin{bmatrix} 5.231065 & -2.714654 & -0.220204 \\ -3.541431 & 5.865087 & 1.598103 \\ 0.113430 & -1.977533 & 2.678895 \end{bmatrix}.$$

We note that no solvents of Eq. (37) could be obtained by imposing the condition that M_1 be a singular matrix. Further, we may obtain all solvents whose entries are complex numbers following the same procedure. As an example, one complex solvent of Eq. (37), with entries written up to six decimal places, is as follows:

$$\begin{bmatrix} 0.426487 - 4.472880i & 1.547688 - 4.169880i & -3.320315 + 1.169993i \\ 2.147651 + 1.750917i & 0.853077 + 1.632307i & 1.713358 - 0.457995i \\ -1.166389 - 6.881601i & -0.785956 - 6.415431i & -3.853357 + 1.800054i \end{bmatrix}.$$

The remaining complex solvents can be similarly computed.

4. Concluding remarks. We thus see that the algebraic method described in this paper is quite effective in obtaining numerical solutions of the quadratic matrix equation in 2×2 as well as 3×3 matrices. It can also be effectively used to solve higher degree unilateral polynomial matrix equations in 2×2 matrices. When the solvent of a matrix equation can be expressed precisely in terms of rational numbers as in the case of the five solvents of Eq. (8) or the first two solvents of Eq. (37) given by (43), the algebraic method yields the exact solvent while the numerical iterative methods yield only an approximation. This is a striking advantage of the algebraic method as compared to the iterative methods of solution.

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