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SOME RESULTS ON UNIQUENESS OF DIFFERENCE POLYNOMIAL

Harina P. Waghamore, Ramya Maligi

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Abstract. In this paper, we consider the uniqueness of the difference polynomial $f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{s_j}$. The results of the paper improve and generalize some recent results due to Y. Liu, J. P. Wang and F. H. Liu [10].

1. Introduction and main results. In this paper a meromorphic function means meromorphic in the complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [6, 7, 8]. By letter E we denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying the relation $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$

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Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $a \in \mathbb{C} \cup \{\infty\}$. We define $\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}$. We say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. And if we do not consider the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM(ignoring multiplicities).

Definition 1 ([5])**.** *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all zeros of $f(z) - a$ where each zero of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a, g)$, we say that $f(z)$, $g(z)$ share the value a with weight k. Obviously, when $k = 0$ (resp. ∞), $f(z)$ and $g(z)$ share the value a IM (resp. a CM).*

Definition 2 ([6])**.** *For $a \in \mathbb{C}\cup\{\infty\}$ and k is a positive integer or infinity. We denote by $\overline{N}_{(k}\left(r, \dfrac{1}{f-a}\right)$ the counting function of the zeros of $f - a$ whose multiplicities are not less than k, where each zero is counted only once. Then*

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Clearly, $N_1\left(r, \dfrac{1}{f-a}\right) = \overline{N}\left(r, \dfrac{1}{f-a}\right)$.

In 2010, Qi et al. [9] studied the uniqueness of the difference monomials and obtained the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be transcendental entire functions with finite order, c be a nonzero complex constant, and $n \geq 6$ be an integer. If $E(1, f^n(z)f(z + c)) = E(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2015, Y. Liu, J. P. Wang and F. H. Liu [10] obtained the following results.

Theorem B. *Let $c \in \mathbb{C} \setminus \{0\}$. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order, and n (≥ 14), k (≥ 3) be two positive integers. If $E_k(1, f^n(z)f(z + c)) = E_k(1, g^n(z)g(z + c))$, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem C. *Let $c \in \mathbb{C}$ and $n \geq 16$ be an integer. Let $f(z)$ and $g(z)$ be* *two transcendental meromorphic functions with finite order. If* $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, *then* $f(z) \equiv t_1 g(z)$ *or* $f(z)g(z) = t_2$, *for some constants* t_1 *and* t_2 *that satisfy* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Theorem D. *Let* $c \in \mathbb{C}$ *and* $n \geq 22$ *be an integer. Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with finite order. If* $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, *then* $f(z) \equiv t_1 g(z)$ *or* $f(z)g(z) = t_2$, *for some constants* t_1 *and* t_2 *that satisfy* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Regarding Theorems B–D, it is natural to ask the following question which is the motive of the present paper.

Question. What would happen if one replaced the difference polynomials $f^n(z)f(z+c)$ by $f^n P(f) \prod_{j=1}^{d} f(z+c_j)^{s_j}$ in Theorem B–D, where n is any positive integer?

In this paper, our main aim is to find the possible answer to the above question. We prove following results which improve and generalizes Theorems B-D. The following theorems are the main results of the paper.

Theorem 1. *Let* $c \in \mathbb{C}$ *and* $n \geq 9 + 2d + 3\lambda + m$, l (≥ 3) *be an integer. Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with finite order. If* $E_l\left(1, f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) = E_l\left(1, g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)$, *then* $f(z) \equiv t_1 g(z)$ *or* $f(z)g(z) = t_2$, *for some constants* t_1 *and* t_2 *that satisfy* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Theorem 2. *Let* $c \in \mathbb{C}$ *and* $n \geq 10 + 3d + 3\lambda + 2m$ *be an integer. Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with finite order. If* $E_2\left(1, f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) = E_2\left(1, g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)$, *then* $f(z) \equiv t_1 g(z)$ *or* $f(z)g(z) = t_2$, *for some constants* t_1 *and* t_2 *that satisfy* $t_1^{n+1} = 1$ *and* $t_2^{n+1} = 1$.

Theorem 3. *Let* $c \in \mathbb{C}$ *and* $n \geq 13 + 6d + 3\lambda + 5m$ *be an integer. Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with finite order. If* $E_1\left(1, f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) = E_1\left(1, g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)$, *then*

$f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Remark. Since Theorems B–D can be obtained from Theorems 1–3, respectively, by putting $m = 0$, $d = 1$ and $\lambda = 1$. Theorems 1–3 improve and generalizes Theorems B–D, respectively.

2. Lemmas.

We need following Lemmas to prove our results.

Lemma 2.1 ([3])**.** *Let f and g be two meromorphic functions and let k be a positive integer. If $E_k(1; f) = E_k(1; g)$, then one of the following cases must occur:*
(i)

$$\begin{aligned} T(r,f)+T(r,g) \le{} & \overline{N}_2(r,f)+\overline{N}_2\left(r,\frac{1}{f}\right)+\overline{N}_2(r,g)+\overline{N}_2\left(r,\frac{1}{g}\right) \\ & +\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)-N_{11}\left(r,\frac{1}{f-1}\right) \\ & +\overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g) \end{aligned} \tag{2.1}$$

(ii)

$$f \equiv \frac{(b+1)g+(a-b-1)}{bg+(a-b)}, \tag{2.2}$$

where a ($\neq 0$), b are two constants.

Lemma 2.2 ([7])**.** *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then $T(r, f(z+ \mid c \mid)) = T(r, f(z)) + S(r, f)$.*

Remark 2.3. It shows in [4], that for $c \in \mathbb{C} \setminus \{0\}$,

$$(1+o(1))T(r, f(r- \mid c \mid), f(z)) \le T(r, f(z+c)) \le (1+o(1))T(r+ \mid c \mid, f(z))$$

hold as $r \to \infty$, for a general meromorphic function. By this and Lemma 2.2, we obtain

$$T(r, f(z+ \mid c \mid)) = T(r, f(z)) + S(r, f).$$

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function of finite order, $n \geq 4+2m+3d+\lambda$ a positive integer, and let $F(z) = f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $F(z) = f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$. If*

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)} \tag{2.3}$$

where $a(\neq 0), b$ are two constants, then $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

P r o o f. Remark 2.3 yields that

$$T(r, F) = T(r, f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}) + S(r, f) \tag{2.4}$$

$$\leq T(r, f^n(z)) + T(r, P(f)) + T(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}) + S(r, f) \tag{2.5}$$

$$= (n+m+\lambda)T(r, f) + S(r, f). \tag{2.6}$$

On the other hand, together the first main Theorem with Remark 2.3, we obtain

$$\begin{aligned}(n+m+\lambda)T(r, f) &= T(r, f^n f^m f^\lambda) + S(r, f)\\ &\leq m\left(r, \frac{F(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)\\ &+ N\left(r, \frac{F(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r, f)\\ &\leq T(r, F(z)) + T\left(r, \frac{f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r, f)\end{aligned} \tag{2.7}$$

$$T(r, F) \geq (n+m-\lambda)T(r, f) + S(r, f) \tag{2.8}$$

Hence, (2.4) and (2.6) yield that

$$S(r, F) = S(r, f). \tag{2.9}$$

Similarly, we obtain

$$T(r,G) \geq (n+m-\lambda)T(r,g) + S(r,g), \tag{2.10}$$

and

$$S(r,G) = S(r,g). \tag{2.11}$$

Set $I_1 = \{r : T(r,g) \geq T(r,f)\} \subseteq (0,\infty)$ and $I_2 = (0,\infty) \setminus I_1$. Then there is at least one I_i $(i = 1,2)$ such that I_i has infinite logarithmic measure. Without loss of generality, we may suppose that I_1 has infinite logarithmic measure. We break the rest of the proof into three cases.

Case 1. $b \neq 0, -1$. If $a-b-1 \neq 0$, then we know from (2.3)

$$\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G + \frac{a-b-1}{b+1}}). \tag{2.12}$$

Together with the first main theorem, the second main theorem with Remark 2.3, (2.8) and (2.12), we obtain

$$\begin{aligned}
(n+m-\lambda)T(r,g) &\leq T(r,G) + S(r,g) \leq \overline{N}(r, \frac{1}{G}) + \overline{N}(r,G) \\
&\leq \overline{N}\left(r, \frac{1}{G - \frac{a-b-1}{b+1}}\right) + S(r,G) + S(r,g) \\
&= \overline{N}(r, \frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r, \frac{1}{F}) + S(r,g) \\
&\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j}}\right) \\
&\quad + \overline{N}\left(r, g^n P(g) \prod_{j=1}^{d} g(z+c_j)^{s_j}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,g) \\
&\leq (1+m+d)T(r,g) + (1+m+d)T(r,g) \\
&\quad + (1+m+d)T(r,f) + S(r,g)
\end{aligned} \tag{2.13}$$

$$(n+m-\lambda)T(r,g) \le (3+3m+3d)T(r,g)+S(r,g), r\in I_1 \tag{2.14}$$

which is impossible, since $n \ge 4+2m+3d+\lambda$. Hence, we obtain $a-b-1=0$, so

$$F(z)=\frac{(b+1)G(z)}{bG(z)+1}.$$

Using the similar method as above, we obtain

$$\begin{aligned}(n+m-\lambda)T(r,g) &\le T(r,G)+S(r,g)\\ &\le \overline{N}(r,\frac{1}{G})+\overline{N}(r,G)+\overline{N}(r,\frac{1}{G+\frac{1}{b}})+S(r,G)\\ &\le (3+3m+3d)T(r,g)+S(r,g), r\in I_1\end{aligned} \tag{2.15}$$

which is a contradiction, since $n \ge 4+2m+3d+\lambda$.

Case 2. $b=-1$, $a\ne -1$. By (2.3) we have

$$F=\frac{a}{a+1-G}. \tag{2.16}$$

Similarly, we get a contradiction. Hence we obtain $a=-1$. So, we get $FG=1$, that is $f^nP(f)\prod_{j=1}^{d} f(z+c_j)^{s_j} g^nP(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}=1$. Set $H(z)=f(z)g(z)$. Suppose that $H(z)$ is not a constant. Then we obtain

$$H^n(z)H(z+c)=1. \tag{2.17}$$

Remark 2.3, the first main Theorem and (2.16) imply that

$$nT(r,H(z))=T(r,H^n(z))=T(r,\frac{1}{H(z+c)})=T(r,H(z))+S(r,H). \tag{2.18}$$

Hence $H(z)$ must be a nonzero constant, since $n\ge 4+2m+3d+\lambda$. Set $H(z)=t_1$ by (2.17), we know $t_1^{n+1}=1$. Thus $f(z)g(z)=t_1$ where $t_1^{n+1}=1$.

Case 3. $b=0$, $a\ne 1$. By (2.3), we obtain $F=\dfrac{G+a-1}{a}$.

Similarly, we get a contradiction, hence we obtain a=1, so we get $F=G$, that is

$$f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}=g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}.$$

Let $H(z)=\dfrac{f(z)}{g(z)}$, using the similar method as above we also obtain that $H(z)$ must be a nonzero constant. Thus we have $f=t_2g$, where $t_2^{n+1}=1$. □

3. Proof of the Theorems.

Proof of Theorem 1. Let $F(z) = f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $G(z) = g^n(z)P(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}$. Since $l \geq 3$, we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)\\
&\quad+\overline{N}_{(k+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{G-1}\right)\\
&\leq \frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\\
&\leq \frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).
\end{aligned}\tag{3.1}$$

(2.1) and (3.1) give that

$$\begin{aligned}
T(r,F)+T(r,G) \leq\ & 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}\\
&+S(r,f)+S(r,g)
\end{aligned}\tag{3.2}$$

Together the definition of F, the first main Theorem with Remark 2.3, we have

$$\begin{aligned}
N_2(r,\frac{1}{F}) &\leq 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{P(f)}\right)+N\left(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)\\
&\leq (2+m+\lambda)T(r,f)+S(r,f).
\end{aligned}\tag{3.3}$$

Similarly,

$$N_2\left(r, \frac{1}{G}\right) \leq (2 + m + \lambda)T(r, g) + S(r, g) \tag{3.4}$$

$$N_2(r, F) \leq (2 + d)T(r, f) + S(r, f). \tag{3.5}$$

Similarly,

$$N_2(r, G) \leq (2 + d)T(r, g) + S(r, g). \tag{3.6}$$

(3.2)–(3.6) yields that

$$\begin{aligned} T(r, F) + T(r, G) \leq 2\{&(2 + m + \lambda)(T(r, f) + T(r, g)) \\ &+ (2 + d)(T(r, f) + T(r, g))\} \\ &+ S(r, f) + S(r, g) \end{aligned} \tag{3.7}$$

Then, by (2.6), (2.8) and (3.7), we obtain

$$(n - m - 3\lambda - 8 - 2d)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g) \tag{3.8}$$

which is a contradiction since $n \geq m + 3\lambda + 2d + 9$. Hence by Lemma 2.1, we have $F = (b + 1)G + \dfrac{(a - b - 1)}{bG + a - b}$, where $a \neq 0$, b are two constants. By Lemma 2.4, we get $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t^{n+1} = 1$ and $t_2^{n+1} = 1$. □

Proof of Theorem 2. Note that

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F - 1}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) - N_{11}\left(r, \frac{1}{F - 1}\right) \\ &\qquad + \frac{1}{2}\overline{N}_{(3}\left(r, \frac{1}{F - 1}\right) + \frac{1}{2}\overline{N}_{(3}\left(r, \frac{1}{G - 1}\right) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{F - 1}\right) + \frac{1}{2}N\left(r, \frac{1}{G - 1}\right) \\ &\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned} \tag{3.9}$$

Then we obtain from and (2.1) and (3.9)

$$T(r,F)+T(r,G)\le 2\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\}$$
$$+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).$$

Obviously combining the first main Theorem and Remark 2.3 we have

$$\begin{aligned}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)&\le\frac{1}{2}N\left(r,\frac{F}{F'}\right)=\frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,f)\\&\le\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\&\le\frac{1}{2}\left[\overline{N}(r,f(z))+\overline{N}(r,P(f))+\overline{N}\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)\right.\\&\quad+\overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)\\&\quad\left.+\overline{N}\left(r,\frac{1}{\prod_{j=1}^{d}f(z+c_j)^{s_j}}\right)\right]+S(r,f)\\&\le(1+m+d)T(r,f)+S(r,f).\end{aligned}\tag{3.10}$$

Similarly, we obtain

$$\overline{N}_{(3}(r,\frac{1}{G-1})\le(1+m+d)T(r,g)+S(r,g).\tag{3.11}$$

Suppose that

$$\begin{aligned}T(r,F)+T(r,G)\le 2&\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\right.\\&\left.+2N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}\\&+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)\\&+S(r,f)+S(r,g).\end{aligned}\tag{3.12}$$

Then we have from (2.8), (2.10), (3.3)–(3.6) and (3.10)–(3.12).

$$(n+m-\lambda)T(r,f)+(n+m-\lambda)T(r,g)\leq T(r,F)+T(r,G)$$
$$\leq (9+3m+2\lambda+3d)(T(r,f)+T(r,g))$$

which is a contradiction, since $n\geq 2m+3\lambda+3d+10$. By Lemma 2.1, we obtain that $F=(b+1)G+\dfrac{(a-b-1)}{bG+a-b}$, where $a\neq 0, b$ are two constants. By Lemma 2.4, we get $f(z)\equiv t_1 g(z)$ or $f(z)g(z)=t_2$, for some constants t_1 and t_2 that satisfy $t^{n+1}=1$ and $t_2^{n+1}=1$. □

P r o o f o f T h e o r e m 3. Since

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)\\
&\leq \frac{1}{2}N\left(r,\frac{1}{F}\right)+\frac{1}{2}N\left(r,\frac{1}{G}\right)\\
&\leq \frac{1}{2}T(r,F)+\frac{1}{2}T(r,F)+S(r,f)+S(r,g).
\end{aligned}\tag{3.13}$$

Then (2.1) becomes

$$\begin{aligned}
T(r,F)+T(r,G)\leq 2\Bigg\{&N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\\
&+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\Bigg\}+S(r,f)+S(r,g)
\end{aligned}\tag{3.14}$$

Combining the first main Theorem and Remark 2.3, we obtain

$$
\begin{aligned}
\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{F}{F'}\right) = N\left(r,\frac{F'}{F}\right) + S(r,f) \\
&\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\
&\le \overline{N}(r,f(z)) + \overline{N}(r,P(f)) \\
&\quad + \overline{N}\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + \overline{N}\left(r,\frac{1}{f(z)}\right) \\
&\quad + \overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\
&\le (2+2m+2d)T(r,f) + S(r,f).
\end{aligned} \tag{3.15}
$$

Similarly, we get

$$
\overline{N}_{(2}(r,\frac{1}{G-1}) \le (2+2m+2d)T(r,g) + S(r,g). \tag{3.16}
$$

Suppose that

$$
\begin{aligned}
T(r,F)+T(r,G) \le 2\Bigg\{ & N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) \\
& + \overline{N}_{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\Bigg\} + S(r,f) + S(r,g)
\end{aligned} \tag{3.17}
$$

Then we obtain from (2.8), (2.10), (3.3)–(3.6) and (3.15)–(3.17)

$$
\begin{aligned}
(n+m-\lambda)T(r,f) &+ (n+m-\lambda)T(r,g) \\
&\le T(r,F) + T(r,G) \\
&\le (12+6m+2\lambda+6d)(T(r,f)+T(r,g)) \\
&\quad + S(r,f) + S(r,g)
\end{aligned} \tag{3.18}
$$

which is impossible, since $n \geq 13 + 6d + 3\lambda + 5m$. By Lemma 2.1, we obtain that $F = (b+1)G + \dfrac{(a-b-1)}{bG + a - b}$, where $a \neq 0, b$ are two constants. By Lemma 2.4, we get $f(z) \equiv t_1 g(z)$ or $f(z)g(z) = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$. □

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Department of Mathematics
Jnanabharathi Campus
Bangalore University
560 056 Bengaluru, India
e-mail: harinapw@gmail.com, harinapw@bub.ernet.in *(Harina P. Waghamore)*
e-mail: ramyamalgi@gmail.com *(Ramya Maligi)*

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