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SUPPLEMENTED PROPERTY IN THE LATTICES

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Abstract. Let L be a lattice with the greatest element 1. In a manner analogous to a module over a ring, we introduce (amply) supplemented filters of L. The basic properties and possible structures of such filters are investigated.

1. Introduction. The notion of an order plays an important role not only throughout mathematics but also in adjacent such as logic, computer science and engineering. They are also useful in other disciplines of mathematics such as combinatorics, number theory and group theory and, hence, ought to be in the literature. Moreover, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 3, 4, 5, 6, 7, 8, 9]). Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular. Wisbauer calls a module M supplemented if, for every submodule N of M, there is a submodule K of M such that $M = N + K$ and $N \cap K$ is a small submodule of K. In [10], the basic properties of supplemented modules are given.

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Let L be a distributive lattice with 1. In the present paper, we are interested in investigating (amply) supplemented filters to use other notions of supplemented, and associate which exist in the literature as laid forth in [10]. Here is a brief outline of the article. If H is a subset of a lattice L, then the filter generated by H, denoted by $T(H)$, is the intersection of all filters that is containing H. Among many results in this paper, in Section 2, it is defined (Definition 2.3) that a subfilter G of a filter F of L is called small in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$. Properties of small filters are given in the Lemma 2.4. It is shown (Theorem 2.6) that if F is a filter of L, then $\text{rad}(F) = T(\cup_{G \ll F} G)$. Some basic properties of supplemented filters are given in the Theorem 2.9. It is proved (Theorem 2.11) that if $F = T(F_1 \cup F_2)$ with F_1 and F_2 supplemented filters, then F is also supplemented. Moreover, Let F be a supplemented filter of L. Then there exist a semisimple subfilter K and a subfilter G with essential radical such that $F = G \oplus K$ (Theorem 2.17). In Section 3, we investigate some properties of amply supplemented filters of L. It is shown (Theorem 3.7) that for a filter F of L the following assertions are equivalent:

(1) F is amply supplemented and every supplement subfilter of F is a direct summand;

(2) If G is a subfilter of F, then there are subfilters X, X' of F such that $F = X \oplus X'$ with $X \subseteq G$ and $X' \cap G \ll X'$;

(3) (i) Every non-small subfilter of F contains a direct summand $U \neq \{1\}$, and

(ii) every subfilter of F contains a maximal direct summand of F.

Let us recall some notions and notations of lattice theory from [5]. By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be maximal if E is a filter in L with $P \subsetneqq E$, then $E = L$. If F is a filter of a lattice L, then the radical of F, denoted by $\text{rad}(F)$, is the intersection of all maximal subfilters of F. Let H be subset of a lattice L. Then the filter generated by H, denoted by $T(H)$, is the intersection of all filters that is containing H. A filter F is called finitely generated if there is a finite subset H of F such that $F = T(H)$.

Lemma 1.1. *A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$ [8, 9].*

2. Some basic properties of supplemented filters. Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. Our starting point is the following proposition:

Proposition 2.1. (1) *Let H be an arbitrary non-empty subset of L. Then $T(H) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in H$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and H is a subset of L with $H \subseteq F$, then $T(H) \subseteq F$, $T(F) = F$ and $T(T(H)) = T(H)$*

(2) *If F_1, F_2 and G are subfilters of a filter F of L, then $T(T(F_1 \cup F_2) \cup G) \subseteq T(F_1 \cup T(F_2 \cup G))$. In particular, if $F = T(T(F_1 \cup F_2) \cup G)$, then $F = T(T(G \cup F_2) \cup F_1) = T(T(F_1 \cup G) \cup F_2)$.*

(3) *(Modular law) If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.*

P r o o f. (1) It is straightforward.

(2) Let $z \in T(T(F_1 \cup F_2) \cup G)$. Then $a \wedge b \leq z$, where $a \in T(F_1 \cup F_2)$ and $b \in G$. Therefore $d \wedge e \leq a$ for some $d \in F_1$ and $e \in F_2$. Then $d \wedge e \wedge b \leq a \wedge b \leq z$ gives $z \in T(F_1 \cup T(F_2 \cup G))$. The in particular statement is clear.

(3) Since $F_2 \cup (F_1 \cap F_3) \subseteq T(F_2 \cup F_3)$, $T(F_2 \cup (F_1 \cap F_3)) \subseteq T(F_2 \cup F_3)$ and $T(F_2 \cup (F_1 \cap F_3)) \subseteq F_1$ gives $T(F_2 \cup (F_1 \cap F_3)) \subseteq F_1 \cap T(F_2 \cup F_3)$. For the reverse inclusion, assume that $x \in F_1 \cap T(F_2 \cup F_3)$. Then $x \in F_1$ and $x = (f_2 \wedge f_3) \vee x = (x \vee f_2) \wedge (x \vee f_3) \leq x$ for some $f_2 \in F_2$ and $f_3 \in F_3$; so $x \in T(F_2 \cup (F_1 \cap F_3))$ since $f_2 \vee x \in F_2$ and $x \vee f_3 \in F_1 \cap F_3$. This completes the proof. □

Proposition 2.2. *Any finitely generated filter F contains a maximal subfilter.*

P r o o f. If $\{x_1, \cdots, x_n\}$ with $x_i \in F$ is any minimal generating set of F, with n as small as possible, then the subfilter $G = T(\{x_1, \cdots, x_{n-1}\})$ is a proper. By Zorn's Lemma there exists a subfilter H of F maximal with respect to the two properties that (1) $G \subseteq H$, but (2) $x_n \notin H$. Suppose that U is a subfilter of

F containing H properly. By the maximality of H with respect to (2), $x_n \in U$. But then $F \subseteq U$, and $U = F$. Hence H is a maximal subfilter of F. $\Box$

Definition 2.3. *A subfilter G of a filter F of L is called small in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$*

Clearly, if F is a filter of L, then $\{1\} \ll F$.

Lemma 2.4. *Let F be a filter of L. Then the following hold:*

(1) *If $A \ll F$ and $C \subseteq A$, then $C \ll F$.*

(2) *If A, B are subfilters of F with $A \ll B$, then $A \ll F$.*

(3) *If $F_1, F_2, \cdots, F_n$ are small subfilters of F, then $T(F_1 \cup F_2 \cup \cdots \cup F_n)$ is also small in F.*

(4) *If A, B, C and D are subfilters of F with $A \ll B$ and $C \ll D$, then $T(A \cup C) \ll T(B \cup D)$.*

Proof. (1) Let $T(C \cup H) = F$ for subfilter H of F. Then $F = T(C \cup H) \subseteq T(A \cup H) \subseteq F$ gives $T(A \cup H) = F$; so $H = F$. Thus $C \ll F$.

(2) Let $T(A \cup G) = F$ for subfilter G of F. Since $A \subseteq B$, $B = B \cap F = B \cap (T(A \cup G)) = T(A \cup (B \cap G))$ by Proposition 2.1. Now $A \ll B$ gives $B = B \cap G$; so $A \subseteq B \subseteq G$. Hence $F = T(A \cup G) = T(G) = G$. Thus $A \ll F$.

(3) It is enough to show that $T(F_1 \cup F_2) \ll F$. Let G be a subfilter of F such that $T(T(F_1 \cup F_2) \cup G) = F$. By Proposition 2.1, $F = T(T(F_1 \cup F_2) \cup G) \subseteq T(F_1 \cup T(F_2 \cup G)) \subseteq F$; so $F = T(F_1 \cup T(F_2 \cup G))$. Now $F_1 \ll F$ gives $F = T(F_2 \cup G)$; hence $G = F$ since $F_2 \ll F$. Thus $T(F_1 \cup F_2) \ll F$.

(4) By (2), $A \subseteq B \subseteq T(B \cup D)$ gives $A \ll T(B \cup D)$. Similarly, $C \ll T(B \cup D)$. Now the assertion follows from (3). $\Box$

Lemma 2.5. *Let G be a proper subfilter of a filter $F \neq \{1\}$ of L, and let $x \in F \setminus G$. If $F = T(G \cup T(\{x\})$, then F has a maximal subfilter K with $G \subseteq K$ and $x \notin K$.*

Proof. At first we show that F has a subfilter K maximal with respect to $G \subseteq K$ and $x \notin K$. Consider the set Ω of all subfilters H of F such that $G \subseteq H$ and $x \notin H$. This set is not empty since $G \in \Omega$. Clearly, Ω is closed under taking unions of chains and so the result follows by Zorn's Lemma. Let K be the maximal element of Ω. Let U be a subfilter of F such that $K \subsetneqq U \subseteq F$. Then $x \in U$ by maximality of K, and so $F = T(G \cup T(\{x\}) \subseteq U$; hence $F = U$. Thus K is maximal subfilter of F with $G \subseteq K$ and $x \notin K$. $\Box$

Theorem 2.6. *Let F be a filter of L. Then the following hold:*

(1) $\mathrm{rad}(F) = T(\cup_{G \ll F} G)$.

(2) *Every finitely generated subfilter of* $\mathrm{rad}(F)$ *is small in* $\mathrm{rad}(F)$.

(3) $\mathrm{rad}(F) = F$ *holds if and only if all finitely generated subfilters of F are small in F.*

Proof. (1) Let $G \ll F$. If K is a maximal subfilter of F and $G \nsubseteq K$, then $T(G \cup K) = F$; but since $G \ll F$, we have $K = F$, a contradiction. We infer that every small subfilter of F is contained in $\mathrm{rad}(F)$; hence $T(\cup_{G \ll F} G) \subseteq \mathrm{rad}(F)$. On the other hand, let $x \in F$. If $H \subseteq F$ with $T(H \cup T(\{x\})) = F$, then by Lemma 2.5, either $H = F$ or there is a maximal subfilter K of F with $H \subseteq K$ and $x \notin K$. If $x \in \mathrm{rad}(F)$, then the latter cannot occur; thus $x \in \mathrm{rad}(F)$ forces $T(\{x\}) \ll F$, and we have equality.

(2) Let G be a finitely generated subfilter of $\mathrm{rad}(F)$. Then $G = T(A)$, where $A = \{a_1, \cdots, a_n\} \subseteq G$. As $a_1 \in \mathrm{rad}(F)$, $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \leq a_1$, where $x_{i_1} \in F_{i_1} \ll F, \cdots, x_{i_k} \in F_{i_k} \ll F$. By Lemma 2.4 (3), $T(\{a_1\}) \subseteq T(F_{i_1} \cup \cdots \cup F_{i_k}) \ll F$; so $T(\{a_1\}) \ll \mathrm{rad}(F)$ by Lemma 2.4 (1). Similarly, $T(\{a_i\}) \ll \mathrm{rad}(F)$ for $2 \leq i \leq n$ which implies that $G \ll \mathrm{rad}(F)$ by Lemma 2.4 (3).

(3) If $F = \mathrm{rad}(F)$, then by (2), all finitely generated subfilters of F are small in F. Conversely, assume that $x \in F$. Then $T(\{x\}) \ll F$ gives $x \in T(\{x\}) \subseteq \mathrm{rad}(F)$ by (1); so $F \subseteq \mathrm{rad}(F)$, as required. □

Definition 2.7. *Let G be a subfilter of a filter F of L. A subfilter $H \subseteq F$ is called a supplement of G in F if H is a minimal element in the set of subfilters $U \subseteq F$ with $T(G \cup U) = F$. A filter F of L is called supplemented if every subfilter of F has a supplemented in F.*

Lemma 2.8. *Let G be a subfilter of a filter F of L. H is a supplement of G in F if and only if $T(G \cup H) = F$ and $G \cap H \ll H$.*

Proof. Let H be a supplement of G in F (so $T(G \cup H) = F$). Let $X \subseteq H$ with $T(X \cup (G \cap H)) = H$. Then by Proposition 2.1, we have

$$F = T(H \cup G) = T(T((G \cap H) \cup X) \cup G) = T(T(G \cup (G \cap H)) \cup X) =$$

$T(T(G) \cup X) = T(G \cup X)$ which implies that $H = X$ by minimality of H. Thus $G \cap H \ll H$. Conversely, assume that $T(G \cup H) = F$ and $G \cap H \ll H$. For $Y \subseteq H$ with $T(Y \cup G) = F$, we have $H = H \cap F = H \cap T(Y \cup G) = T(Y \cup (H \cap G))$ by Proposition 2.1. Now $G \cap H \ll H$ gives $Y = H$, as needed. □

Theorem 2.9. *Let G, H be subfilters of the filter F of L. Assume H to be a supplement of G. Then the following hold:*

(1) *If $T(U \cup H) = F$ for some $U \subseteq G$, then H is a supplement of U.*

(2) *If $K \ll F$, then H is a supplement of $T(G \cup K)$.*

(3) $\mathrm{rad}(H) = H \cap \mathrm{rad}(F)$.

(4) *If G is a maximal subfilter of F, then H is cyclic, and $G \cap H = \mathrm{rad}(H)$ is a (the unique) maximal subfilter of H.*

(5) *If F is finitely generated, then H also is finitely generated.*

Proof. (1) By Lemma 2.8, it is enough to show that $U \cap H \ll H$. Assume that $X \subseteq H$ such that $T(X \cup (U \cap H)) = H$. Now $H = T(X \cup (U \cap H)) \subseteq T(X \cup (G \cap H)) \subseteq H$ gives $H = T(X \cup (G \cap H))$; hence $X = H$ since $G \cap H \ll H$. Thus H is a supplement of U.

(2) By Proposition 2.1, $F = T(G \cup H) \subseteq T(T(G \cup K) \cup H) =$

$$T(T(G \cup H) \cup K) = T(F \cup K) = T(F) = F;$$

so $T(T(G \cup K) \cup H) = F$. Assume that Y is a subfilter of H such that $T(T(G \cup K) \cup Y) = F$; we show that $Y = H$. By Proposition 2.1, $F = T(T(G \cup K) \cup Y) \subseteq T(T(G \cup Y) \cup K) \subseteq F$ which implies that $F = T(T(G \cup Y) \cup K)$. Now $K \ll F$ gives $T(G \cup Y) = F$; hence $Y = H$ by minimality of H. Thus H is a supplement of $T(G \cup K)$.

(3) At first we show that if $K \ll F$, then $K \cap H \ll H$. Assume that $K \ll F$ and let X be a subfilter of H such that $T(X \cup (K \cap H)) = H$. Then by Proposition 2.1, $F = T(G \cup H) = T(G \cup T(X \cup (K \cap H))) \subseteq$

$$T(G \cup T(X \cup K)) \subseteq T(K \cup T(G \cup X)) \subseteq F;$$

so $T(K \cup T(G \cup X)) = F$. Now $K \ll F$ gives $T(G \cup X) = F$; hence $X = H$ by minimality of H. Clearly, $\mathrm{rad}(H) \subseteq H \cap \mathrm{rad}(F)$. For the reverse inclusion, assume that $x \in H \cap \mathrm{rad}(F)$. Then by Theorem 2.6, $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \leq x$, where $x_{i_1} \in F_{i_1} \ll F, \cdots, x_{i_k} \in F_{i_k} \ll F$. By Lemma 2.4 (3), $T(F_{i_1} \cup \cdots \cup F_{i_k}) \ll F$; so $T(F_{i_1} \cup \cdots \cup F_{i_k}) \cap H \ll H$. Now $x \in T(F_{i_1} \cup \cdots \cup F_{i_k}) \cap H \subseteq \mathrm{rad}(H)$, and so we have equality.

(4) There is an element $a \in F$ such that $a \notin G$, so $G \subsetneqq T(G \cup T(\{a\})) \subseteq F$ gives $T(G \cup T(\{a\})) = F$ which implies that $H = T(\{a\})$ since H is a supplement of G. Let K be a subfilter of H such that $G \cap H \subsetneqq K \subseteq H$. There exists $x \in K \subseteq H$ such that $x \notin G$. Now G is maximal gives $T(G \cup T(\{x\})) = F$; hence $F = T(G \cup K)$ which implies that $K = H$. Thus $G \cap H$ is a maximal subfilter of H, so $\mathrm{rad}(H) \subseteq G \cap H$. As $G \cap H \ll H$, $G \cap H \subseteq \mathrm{rad}(H)$. Thus $G \cap H = \mathrm{rad}(H)$.

(5) There is a finite subset $A = \{x_1, x_2, \cdots, x_n\}$ of F such that $F = T(A) = T(G \cup H)$. Without loss of generality, we can assume that $B = \{x_1, \cdots, x_t\}$

$\subseteq H$ and $C = \{x_{t+1}, \cdots, x_n\} \subseteq G$. Set $H' = T(B)$ and $G' = T(C)$. Let $x \in F$. Then $x = (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_s}) \vee x$ for some $x_{i_1}, \cdots, x_{i_s} \in A$. So $x \in T(B \cup C) \subseteq T(T(B) \cup T(C)) \subseteq T(H' \cup G)$; hence $F = T(G \cup H')$. Now H is a supplement of G gives $H = H' = T(B)$. This completes the proof. □

Proposition 2.10. *Let F_1 and G be subfilters of a filter F of L with F_1 supplemented. If there is a supplement for $T(F_1 \cup G)$ in F, then the same is true for G.*

Proof. Let X be a supplement of $T(F_1 \cup G)$ in F; so $T(X \cup T(F_1 \cup G)) = F$ and $X \cap T(F_1 \cup G) \ll X$. Since F_1 is supplemented, $B = T(X \cup G) \cap F_1 \subseteq T(X \cup G)$ has a supplement in F_1, say Y(so $T(Y \cup B) = F_1$). Now we show that $T(X \cup Y)$ is a supplement of G in F. By Proposition 2.1, we have $F = T(X \cup T(F_1 \cup G)) \subseteq T(F_1 \cup T(X \cup G)) =$

$$T(T(B \cup Y) \cup T(X \cup G)) \subseteq T(Y \cup T(B \cup T(X \cup G))) =$$

$T(Y \cup T(X \cup G)) \subseteq T(G \cup T(X \cup Y)) \subseteq F$; hence $F = T(G \cup T(X \cup Y))$. It is enough to show that $T(X \cup Y) \cap G \ll T(X \cup Y)$. As Y is a supplement of $T(X \cup G) \cap F_1$ in F_1, $Y \cap T(X \cup G) = Y \cap (T(X \cup G) \cap F_1) \ll Y$. Since $T(G \cup Y) \subseteq T(F_1 \cup G)$ and $F = T(G \cup T(X \cup Y)) = T(X \cup T(G \cup Y))$, Theorem 2.9 (1) gives X also is a supplement of $T(G \cup Y)$ in F which implies that $T(G \cup Y) \cap X \ll X$. Now by Lemma 2.4 (4), $T(X \cup Y) \cap G \subseteq T((X \cap T(G \cup Y) \cup (Y \cap T(X \cup G)) \ll T(X \cup Y)$; hence $T(X \cup Y) \cap G \ll T(X \cup Y)$ by Lemma 2.4 (1). □

Theorem 2.11. *Let $F = T(F_1 \cup F_2)$. If F_1 and F_2 are supplemented filters, then F is a supplemented filter.*

Proof. If G is any subfilter of F, then $T(F_2 \cup G \cup F_1) = F$. Let H be a supplement of $D = T(F_2 \cup G) \cap F_1 \subseteq T(F_2 \cup G)$ in F_1; so $T(H \cup D) = F_1$ and $D \cap H \ll H$. Moreover, $D, F_2 \cup G \subseteq T(F_2 \cup G)$ gives $T(D \cup (F_2 \cup G)) \subseteq T(F_2 \cup G)$. Now by Proposition 2.1, we have $F = T(F_2 \cup G \cup F_1) =$

$$T(F_2 \cup G \cup T(H \cup D)) \subseteq T(H \cup T((F_2 \cup G) \cup D)) \subseteq$$

$T(H \cup T(F_2 \cup G)) \subseteq F$; hence $F = T(H \cup T(F_2 \cup G))$ which implies that H is a supplement of $T(F_2 \cup G)$ in F since $H \cap T(F_2 \cup G) = H \cap T(F_2 \cup G) \cap F_1 \ll H$. Now the assertion follows from Proposition 3.10. □

Corollary 2.12. *If $F_1, \cdots, F_n$ are supplemented filters of L, then $T(U_{i=1}^n F_i)$ is a supplemented filter.*

A lattice L is called semisimple, if for each proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$). In this case, we say that F is a direct summand of L, and we write $L = F \oplus G$. A filter F of L is called a semisimple filter, if every subfilter of F is a direct summand. A simple filter is a filter that has no filters besides the $\{1\}$ and itself

Proposition 2.13. *Every direct summand of a supplemented filter is a supplemented filter.*

Proof. Let X be a direct summand of a supplemented filter F. Then there is a subfilter X' of F such that $F = T(X \cup X')$ and $X \cap X' = \{1\}$. Let $G \subseteq X$. Then there is a subfilter H of F with $F = T(G \cup H)$ and $G \cap H \ll H$. By Proposition 2.1, $X = X \cap T(G \cup H) = T(G \cup (X \cap H))$. Let $K \subseteq X \cap H$ be a subfilter of X such that $X = T(G \cup K)$ which implies that $F = T(T(G \cup K) \cup X') \subseteq T(G \cup T(K \cup X')) \subseteq F$; so $F = T(G \cup T(K \cup X'))$. Now H is a supplement of G in F gives $H = T(K \cup X')$. Hence, $H \cap X = X \cap T(K \cup X') = T(K \cup (X \cap X')) = K$. This completes the proof. □

Proposition 2.14. *Let F be a supplemented filter of L. If H is a subfilter of F with $H \cap \mathrm{rad}(F) = \{1\}$, then H is semisimple. In particular, if $\mathrm{rad}(F) = \{1\}$, then F is semisimple.*

Proof. Let H' be any subfilter of H. By assumption, there is a subfilter K of F with $F = T(H' \cup K)$ and $H' \cap K \ll K$ (so $H' \cap K \subseteq \mathrm{rad}(K)$). By Proposition 2.1, $H = H \cap T(H' \cup K) = T(H' \cup (H \cap K))$. As $(H \cap K) \cap H' = K \cap H' \subseteq H \cap \mathrm{rad}(K) \subseteq H \cap \mathrm{rad}(F) = \{1\}$, we get $(H \cap K) \cap H' = \{1\}$ and $H = T(H' \cup (H \cap K))$. Thus H is semisimple. The in particular statement is clear. □

A subfilter G of F is called essential in F (notation $G \leq_e F$) if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. G is called essential radical if $\mathrm{rad}(G) \leq_e G$. Let G be a subfilter of a filter F of L. If subfilter H of F is maximal with respect to $G \cap H = \{1\}$, then we say that H is a F-complement of G. Using the maximal principle we readily see that if G is a subfilter of F, then the set of those subfilters of F whose intersection with G is $\{1\}$ contains a maximal element H. Thus every subfilter G of F has a F-complement.

Lemma 2.15. *Let F be a filter of L. Then the following hold:*

(1) *A subfilter G of F is essential if and only if for each $1 \neq x \in F$ there exists an element $a \in L$ such that $1 \neq a \vee x \in G$.*

(2) *Assume that G_1, F_1, G_2 and F_2 are subfilters of F and let $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F = T(F_1 \cup F_2)$ with $F_1 \cap F_2 = \{1\}$. Then $T(G_1 \cup G_2) \leq_e T(F_1 \cup F_2)$ if and only if $G_1 \leq_e F_1$ and $G_2 \leq_e F_2$.*

(3) *If H is a F-complement of G, then $T(G \cup H) \leq_e F$.*

P r o o f. (1) If $G \leq_e F$ and $1 \neq x \in F$, then $T(\{x\}) \cap G \neq \{1\}$; so there is an element $1 \neq y \in G$ with $y = y \vee x \in G$. Conversely, if the condition holds and $1 \neq x \in H \subseteq F$, there is an $a \in L$ such that $1 \neq a \vee x \in G \cap H$.

(2) Suppose, say G_1 is not essential in F_1; so $G_1 \cap H_1 = \{1\}$ for some subfilter $H_1 \neq \{1\}$ of F_1. Let $y \in T(G_1 \cup G_2) \cap H_1$. Then $y \in H_1$ and $y = (g_1 \wedge g_2) \vee y = (y \vee g_1) \wedge (y \vee g_2)$ for some $g_1 \in G_1$ and $g_2 \in G_2$. Since H_1 and G_1 are filters, $y \vee g_1 \in H_1 \cap G_1 = \{1\}$; hence $g_2 \leq y$ which implies that $y \in G_2$. Therefore $y \in F_1 \cap F_2 = \{1\}$. Thus $T(G_1 \cup G_2) \cap H_1 = \{1\}$ which is impossible. Thus $G_1 \leq_e F_1$ and $G_2 \leq_e F_2$. Conversely, assume that $1 \neq x = (f_1 \wedge f_2) \vee x \in T(F_1 \cup F_2)$ for some $f_i \in F_i$. Then by (1), there is an $a_1 \in L$ such that $1 \neq a_1 \vee f_1 \in G_1$. If $a_1 \vee f_2 \in G_2$, then $1 \neq a_1 \vee x = a_1 \vee (f_1 \wedge f_2) \vee x = x \vee ((a_1 \vee f_1) \wedge (a_1 \vee f_2)) \in T(G_1 \cup G_2)$. If $a_1 \vee f_2 \notin G_2$, then again by (1), there is $a_2 \in L$ with $1 \neq a_2 \vee a_1 \vee f_2 \in G_2$, and we have $1 \neq a_1 \vee a_2 \vee x \in T(G \cup G_2)$. Thus $T(G_1 \cup G_2) \leq_e T(F_1 \cup F_2)$.

(3) If $\{1\} \neq K \subseteq F$ and $T(G \cup H) \cap K = \{1\}$, then we show that $G \cap T(H \cup K) = \{1\}$. Let $x \in G \cap T(H \cup K)$. Then $x \in G$ and $x = (a \wedge b) \vee x = (x \vee a) \wedge (x \vee b)$ for some $a \in H$ and $b \in K$. As $a \vee x \in G \cap H = \{1\}$, we get $b \leq x$; hence $x \in K$. Thus $x \in K \cap T(G \cup H) = \{1\}$, contrary to the maximality of H. □

Proposition 2.16. *Let F be a filter of L. Then the following hold:*
(1) *If $F = G \oplus H$, then* $\mathrm{rad}(F) = \mathrm{rad}(G) \oplus \mathrm{rad}(H))$.
(2) *If F is semisimple, then* $\mathrm{rad}(F) = \{1\}$.

P r o o f. (1) By assumption, $\mathrm{rad}(G) \cap \mathrm{rad}(H) \subseteq G \cap H = \{1\}$ and G, H are mutual supplements since $G \cap H = \{1\} \ll H, G$ which implies that $\mathrm{rad}(G) = G \cap \mathrm{rad}(F)$ and $\mathrm{rad}(H) = H \cap \mathrm{rad}(F)$ by Theorem 2.3 (3). Since the inclusion $T((\mathrm{rad}(F) \cap G) \cup (\mathrm{rad}(F) \cap H)) \subseteq \mathrm{rad}(F)$ is clear, we will prove the reverse inclusion. Let $x \in \mathrm{rad}(F)$. By Theorem 2.6, $x = (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}) \vee x$, where $x_{i_1} \in F_{i_1} \subseteq \mathrm{rad}(F), \cdots, x_{i_k} \in F_{i_k} \subseteq \mathrm{rad}(F)$. Then $F = T(H \cup G)$ gives for each $1 \leq j \leq k$, either $x_{i_j} \in \mathrm{rad}(F) \cap G$ or $x_{i_j} \in \mathrm{rad}(F) \cap H$; hence $x \in T((\mathrm{rad}(F) \cap G) \cup (\mathrm{rad}(F) \cap H))$, and so we have equality.

(2) Since every proper subfilter of F is a direct summand, the only proper small subfilter of F can only be $\{1\}$. Thus $\mathrm{rad}(F) = \{1\}$. □

Theorem 2.17. *Let F be a supplemented filter of L. Then there exist a semisimple subfilter K and a subfilter G with essential radical such that $F = G \oplus K$.*

P r o o f. By Lemma 2.15 (3), for $\mathrm{rad}(F)$, there exists subfilter K of F such that $K \cap \mathrm{rad}(F) = \{1\}$ and $T(K \cup \mathrm{rad}(F)) \leq_e F$. Since F is supplemented, there is a subfilter G of F such that $F = T(K \cup G)$ and $K \cap G \ll G$ (so $K \cap G \subseteq \mathrm{rad}(G)$). Since $K \cap G = K \cap (K \cap G) \subseteq K \cap \mathrm{rad}(G) \subseteq K \cap \mathrm{rad}(F) = \{1\}$; hence $F = T(G \cup K)$ with $K \cap G = \{1|$. By Proposition 2.14, K is semisimple. By Proposition 2.16, $\mathrm{rad}(F) = T(\mathrm{rad}(G) \cup \mathrm{rad}(K)) = T(\mathrm{rad}(G) \cup \{1\}) = \mathrm{rad}(G)$. Since $T(K \cup \mathrm{rad}(G)) \leq_e F = T(K \cup G)$, $\mathrm{rad}(G) \leq_e G$ by Lemma 2.15 (2), as required. □

Definition 2.18. *Let F be a filter of a lattice L.*

(1) *F is called hollow if $F \neq \{1\}$ and every proper subfilter G of F is small in F.*

(2) *F is called indecomposable if $F \neq \{1\}$ and $F = T(G \cup H)$ with $H \cap H = \{1\}$, then either $G = \{1\}$ or $H = \{1\}$.*

(3) *F is called local if it has exactly one maximal subfilter that contains all proper subfilters.*

Remark 2.19. (1) Assume that F is a hollow filter and let $F = T(G \cup H)$ with $H \cap G = \{1\}$ for some subfilters H, G of F. If $F \neq G$, then F is a hollow filter gives $H = F$; hence $G = \{1\}$. Thus every hollow filter is indecomposable.

(2) Assume that F is a local filter with unique maximal subfilter of P and let G be a proper subfilter of F with $T(G \cup H) = F$ for some subfilter H of F. If $H \neq F$, then $F \subseteq T(P \cup G) = T(P) = P$, a contradiction. Thus $F = H$. So every local filter is hollow.

(3) A filter F is uniserial if the set of all of its subfilters is linearly ordered by set inclusion. Not that any proper subfilter $G \subsetneq F$ of any uniserial filter F is small in F. Thus every uniserial filter is hollow, and either $\mathrm{rad}(F) \neq F$ is the unique maximal subfilter or F contains no maximal subfilters. Moreover, simple filters are hollow.

(4) Assume that F is a hollow filter and let G be a proper subfilter of F. Then $T(G \cup F) = F$ and $G \cap F = G \ll F$. So hollow (resp. local) filters are supplemented.

(5) By Theorem 2.9 (4), the supplement of a maximal subfilter in a filter of L is a local filter.

(6) A filter F of L is called f-supplemented if every finitely generated subfilter of F has a supplemented in F. By Theorem 2.6 (3), $\mathrm{rad}(F) = F$ holds if and only if all finitely generated subfilters of F are small in F. This implies F to be f-supplemented but need not imply F to be hollow.

Proposition 2.20. *Assume that F is a filter of L and let Ω be family of all hollow subfilters of F. If every proper subfilter of F is contained in a maximal one, and every maximal subfilter has a supplement in F, then $F = T(\cup_{G \in \Omega} G)$.*

P r o o f. Let $H = T(\cup_{G\in\Omega} G)$ and assume $H \neq F$. Then by assumption, there exist a maximal subfilter H' of F with $H \subseteq H'$ and a supplement K of H' in F. By Remark 2.19 (5), K is local; hence K is hollow by Remark 2.19 (2) which implies that $K \subseteq H \subseteq H'$, a contradiction. Thus $H = F$. □

Theorem 2.21. *Let $F \neq \{1\}$ be a filter of a lattice L. Then the following assertions are equivalent:*

(1) *F is hollow and* $\mathrm{rad}(F) \neq F$.

(2) *F is hollow and $F = T(\{a\})$ for some $a \in F$.*

(3) *F is local.*

P r o o f. (1) $\Rightarrow$ (2) Since $\mathrm{rad}(F) \neq F$ and F is hollow, $\mathrm{rad}(F) \ll F$. We claim that $\mathrm{rad}(F)$ is a maximal subfilter of F. Otherwise, there exists a subfilter H of F such that $\mathrm{rad}(F) \subsetneqq H \subsetneqq F$; hence there is a maximal subfilter M of F such that $H \nsubseteq M$ which implies that there is an element $x \in H$ such that $x \notin M$. Then $M \subsetneqq T(M\cup\{x\}) \subseteq F$ gives $T(M\cup\{x\}) = F$; so $T(H\cup M) = F$. Now F is a hollow filter gives $H = F$, a contradiction. Thus $\mathrm{rad}(F)$ is a maximal subfilter of F. There is an element $a \in F$ such that $a \notin \mathrm{rad}(F)$; so $T(\mathrm{rad}(F) \cup T(\{a\})) = F$ which implies that $F = T(\{a\})$ since $\mathrm{rad}(F) \ll F$, as required.

(2) $\Rightarrow$ (3) By Proposition 2.2, F has a maximal subfilter H. Let H' be a maximal subfilter of F with $H \neq H'$. Now $T(H \cup H') = F$ gives $H = F$ which is a contradiction. Thus F is local.

(3) $\Rightarrow$ (1) By Remark 2.19 (2), F is hollow. Moreover, $\mathrm{rad}(F) \neq F$ since F is local. □

3. Amply supplemented filters. In this section, we define the concept of an amply supplemented filters of a lattice and we prove some basic properties concerning such filters. We begin with the key definition of this section.

Definition 3.1. *A subfilter G of a filter F of L has ample supplements in F if, for every subfilter H of F with $F = T(H \cup G)$, there is a supplement H' of G with $H' \subseteq H$. If every subfilter of a filter F has ample supplements in F, then we call F amply supplemented.*

Theorem 3.2. *Let F be an amply supplemented filter of L. Then the following hold:*

(1) *Every supplement of a subfilter of F is an amply supplemented filter.*

(2) *Every direct summand of F is amply supplemented.*

(3) *Every subfilter G of F is of the form $G = T(X \cup Y)$ with X supplemented and $Y \ll F$.*

(4) *If H is a hollow subfilter of F that is not small in F, then H is amply supplemented.*

Proof. (1) Let H be a supplement $G \subseteq F$ in F, so $F = T(G \cup H)$ and $G \cap H \ll H$. Now we show that H is amply supplemented. Let X be a subfilter of H such that $H = T(X \cup Y)$. By Proposition 2.1,

$$F = T(G \cup H) = T(G \cup T(X \cup Y)) \subseteq T(Y \cup T(G \cup X)) \subseteq F;$$

so $F = T(Y \cup T(G \cup X))$. By assumption, there is a supplement Y' of $T(G \cup X)$ in F such that $Y' \subseteq Y$. Hence $F = T(Y' \cup T(G \cup X))$ and $X \cap Y' \subseteq T(G \cup X) \cap Y' \ll Y'$ which implies that $X \cap Y' \ll Y'$ by Lemma 2.4 (1). As $F = T(G \cup T(X \cup Y'))$ (since $F = T(Y' \cup T(G \cup X))$), we get $T(X \cup Y') = H$ since H is a supplement of G. Thus Y' is a supplement of X in H.

(2) Let G be a direct summand of F, so there exists a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H = \{1\} \ll G$. So G is a supplement of H in F. Now the assertion follows from (1).

(3) Let H be a supplement of G, so $T(G \cup H) = F$ and $G \cap H \ll H$; so $G \cap H \ll F$ by Lemma 2.4 (2). Set $G \cap H = Y$. Since F is amply supplemented and $T(G \cup H) = F$, there is a supplement X of H in F with $X \subseteq G$ (so X is supplemented by (1)); hence $T(X \cup H) = F$. By Proposition 2.1, $G = G \cap F = G \cap T(X \cup H) = T(X \cup (G \cap H)) = T(X \cup Y)$, as needed.

(4) By assumption, there exists a proper subfilter G of F with $T(G \cup H) = F$. So H is hollow gives $H \cap G \ll H$. Thus H is a supplement of G in F. Now the assertion follows from (1). □

Theorem 3.3. *Let F be a filter of L and $F = T(F_1 \cup F_2)$. If the subfilters F_1, F_2 have ample supplements in F, then $F_1 \cap F_2$ also has ample supplements in F.*

Proof. Let H be a subfilter of F such that $F = T(H \cup (F_1 \cap F_2))$. By Proposition 2.1, $F_1 \cap F_2 \subseteq F_1$ gives

$$F_1 = F_1 \cap T(H \cup (F_1 \cap F_2)) = T((F_1 \cap F_2) \cup (F_1 \cap H))$$

which implies that $F = T(F_1 \cup F_2) = T(T((F_1 \cap F_2) \cup (F_1 \cap H)) \cup F_2) \subseteq$

$$T((F_1 \cap H) \cup T((F_1 \cap F_2) \cup F_2)) = T(F_2 \cup (F_1 \cap H)) \subseteq F;$$

so $F = T(F_2 \cup (F_1 \cap H))$. Similarly, $F = T(F_1 \cup (F_2 \cap H))$. Therefore there is a supplement H'_2 of F_1 in F with $H'_2 \subseteq F_2 \cap H$ and a supplement H'_1 of F_2 in F with $H'_1 \subseteq F_1 \cap H$ which implies that $T(H'_1 \cup H'_2) \subseteq T(H \cap (F_1 \cup F_2)) \subseteq H$.

So $T(H_2' \cup F_1) = F$, $H_2' \cap F_1 \ll H_2'$, $T(H_1' \cup F_2) = F$ and $H_1' \cap F_2 \ll H_1'$. By Proposition 2.1, $F_1 = F_1 \cap T(H_1' \cup F_2) = T(H_1' \cup (F_1 \cap F_2))$; hence $F = T(F_1 \cup H_2') = T(H_2' \cup T(H_1' \cup (F_1 \cap F_2)) \subseteq T(T(H_1' \cup H_2') \cup (F_1 \cap F_2)) \subseteq F$ which implies that $F = T(T(H_1' \cup H_2') \cup (F_1 \cap F_2))$. By Proposition 2.1 and Lemma 2.4 (4), $T(H_1' \cup H_2') \cap (F_1 \cap F_2) = T(H_1' \cup (H_2' \cap F_1)) \cap F_2 = T((H_2' \cap F_1) \cup (F_2 \cap H_1')) \ll T(H_1' \cup H_2')$. This completes the proof. □

Proposition 3.4. *Let G be a subfilter of a filter F of L. Then the following are equivalent:*

(1) *There is a decomposition $F = X \oplus X'$ with $X \subseteq G$ and $X' \cap G \ll X'$;*

(2) *There is a direct summand X of F with $X \subseteq G$, $G = T(X \cup Y)$ and $Y \ll F$;*

(3) *G has a supplement H in F such that $G \cap H$ is a direct summand in G.*

Proof. (1) ⇒ (2) Let $F = T(X \cup X')$, $X \cap X' = \{1\}$, $X \subseteq G$ and $X' \cap G \ll X'$. By Proposition 2.1, $G = G \cap T(X \cup X') = T(X \cup (X' \cap G))$. Set $G \cap X' = Y$. Then $G = T(X \cup Y)$ with $X \cap Y = \{1\}$. Moreover, $Y \ll X'$ gives $Y \ll F$ by Lemma 2.4 (2). This completes the proof.

(2) ⇒ (1) Let there is a direct summand X of F with $X \subseteq G$, $G = T(X \cup Y)$ and $Y \ll F$; so $F = T(X \cup X')$ with $X \cap X' = \{1\}$ for some subfilter X' of F. Then X' is a supplement of X in F. By Theorem 2.9 (2), $y \ll F$ gives X' is a supplement of $T(X \cup Y)$; hence $T(X \cup Y) \cap X' = G \cap X' \ll X'$, as required.

(1) ⇒ (3) Let $F = T(X \cup X')$, $X \cap X' = \{1\}$, $X \subseteq G$ and $X' \cap G \ll X'$. By Lemma 2.4 (2), $X' \cap G \ll X'$ gives $X' \cap G \ll F$; hence X' is a supplement of $T(X \cup (X' \cap G)) = G \cap T(X \cup X')) = G$ by Proposition 2.1 and Theorem 2.9 (2). Now since $G = T(X \cup (X' \cap G))$ and $X' \cap (X' \cap G) = \{1\}$, we get $G \cap X'$ is a direct summand in G.

(3) ⇒ (1) Let X' be a supplement of G in F with $G = T(X \cup (X' \cap G))$ and $X \cap (X' \cap G) = \{1\}$ for some subfilter X of G. So $T(G \cup X') = F$ and $X' \cap G \ll X'$. Then by Proposition 2.1, $F = T(G \cup X') =$

$$T(G \cup T(X \cup (X' \cap G))) \subseteq T(X \cup T(X' \cup (G \cap X'))) = T(X \cup X') \subseteq F;$$

so $F = T(X \cup X')$. Also, $X \cap X' = X \cap G \cap X' = \{1\}$. This completes the proof. □

Definition 3.5. *Let F be a filter of L.*

(1) *For subfilters $G \subseteq H$ of F, we say H lies above G in F if $T(G \cup K) = F$ holds for all $K \subseteq F$ with $T(K \cup H) = F$.*

(2) *A subfilter G of F is called coclosed in F if G has no proper subfilter K such that G lies above K.*

Proposition 3.6. *Let H be a subfilter of a filter F of L. If H is a supplement in F, then for all $K \subseteq H$, $K \ll F$ implies $K \ll H$.*

Proof. Assume that H is a supplement of $G \subseteq F$ (so $T(H \cup G) = F$). For all subfilters $K \subseteq H$ such that H lies above K, we have that $T(H \cup G) = F$ implies $T(G \cup K) = F$. By the minimality of H with respect to this property we get $K = H$. Hence H is coclosed. Now let $K \ll F$ and $K \subseteq H$. Assume $H = T(K \cup X)$ for $X \subseteq H$; then for every $Y \subseteq F$ with $F = T(H \cup Y) = T(Y \cup T(K \cup X)) \subseteq T(K \cup T(X \cup Y)) \subseteq F$ we get $F = T(X \cup Y)$ since $K \ll F$; hence H lies above X. As H is coclosed, we get $H = X$ and thus $K \ll H$. □

Theorem 3.7. *For a filter F of L the following assertions are equivalent:*

(1) *F is amply supplemented and every supplement subfilter of F is a direct summand;*

(2) *If G is a subfilter of F, then there are subfilters X, X' of F such that $F = X \oplus X'$ with $X \subseteq G$ and $X' \cap G \ll X'$;*

(3) *(i) Every non-small subfilter of F contains a direct summand $U \neq \{1\}$, and*

(ii) every subfilter of F contains a maximal direct summand of F.

Proof. (1) $\Rightarrow$ (2) Let H be a supplement of G in F, so $F = T(G \cup H)$ and $G \cap H \ll H$ (so $G \cap H \ll F$ by Lemma 2.4 (2)). By assumption, there is a supplement X of H with $X \subseteq G$. By (1), there is a subfilter X' of F such that $T(X \cup X') = F$ and $X \cap X' = \{1\}$. Since X' is a supplement of X and $G \cap H \ll F$, X' is a supplement of $T(X \cup (G \cap H)) = G \cap T(X \cap H)) = G$ by Theorem 2.9 (2) and Proposition 2.1; hence $G \cap X' \ll X'$, as needed.

(2) $\Rightarrow$ (1) If (2) holds, then F is supplemented by Proposition 3.4, and every subfilter G of F is of the form $G = T(X \cup Y)$, with X a direct summand F and $Y \ll F$. Since X is again supplemented (see Proposition 2.13) it follows, from Proposition 3.4, that F is amply supplemented. Now we see, from the proof of (3) $\Rightarrow$ (1) Proposition 3.4, that supplements are direct summands.

(2) $\Rightarrow$ (3) Assume that G is a subfilter of F and let $F = T(X \cup X')$ and $X \cap X' = \{1\}$, with $X \subseteq G$ and $X' \cap G \ll X'$. If G is not small in F, then $X' \neq F$; hence $X \neq \{1\}$. Now we show that X is a maximal direct summand of F such that $X \subseteq G$. Let X_1 be a direct summand of F with $X \subseteq X_1 \subseteq G$; so $F = T(X_1 \cup H)$ and $X_1 \cap H = \{1\}$ for some subfilter H of F. By Lemma 2.4. $X_1 \cap X' \subseteq X_1 \cap G \ll X'$ gives $X_1 \cap X' \ll F$. By Proposition 2.1, $X_1 = X_1 \cap T(X \cup X') = T(X \cup (X_1 \cap X')$. Then $F = T(X_1 \cup H) = T(T(X \cup (X_1 \cap X')) \cup H) \subseteq$

$T((X_1 \cap X') \cup T(X \cup H)) \subseteq F$; so $F = T((X_1 \cap X') \cup T(X \cup H))$. Now $X_1 \cap X' \ll F$ gives $F = T(X \cup H)$. Hence $X_1 = X_1 \cap T(X \cup H) = T(X \cup (H \cap X_1)) = T(X) = X$, as needed.

(3) $\Rightarrow$ (2) Let G be a subfilter of F and assume X to be a maximal direct summand of F with $X \subseteq G$, $F = T(X \cup X')$ and $X \cap X' = \{1\}$. If $G \cap X'$ is not small in X' (so $G \cap X'$ is not small in F by Proposition 3.6), then by (3) (i), there is a direct summand H of F with $H \neq \{1\}$ and $H \subseteq G \cap X'$. Since $H \cap X \subseteq G \cap X \cap X' = \{1\}$, we get $H \cap X = \{1\}$. There is a subfilter H' of F with $F = T(H \cup H')$ and $H \cap H' = \{1\}$. By Proposition 2.1,

$$F \subseteq T(T(X \cup H') \cup H) \subseteq T(X \cup T(H \cup H')) = T(X \cup F) = F;$$

so $T(T(X \cup H') \cup H) = F$. Let $y \in T(X \cup H') \cap H$. Then $y = (x \wedge h') \vee y = (x \vee y) \wedge (y \vee h') \in H$ for some $x \in X$ and $h' \in H'$. Since H, H' are filters, $y \vee h' \in H \cap H' = \{1\}$ which implies that $x \vee y = y$; hence $x \leq y$. Now X is a filter gives $y \in H \cap X = \{1\}$; so $T(X \cup H') \cap H = \{1\}$. Then $T(X \cup H')$ is a direct summand of F, contradicting to choice of X. Thus we have $G \cap X' \ll X'$. $\square$

Proposition 3.8. *Let F be a filter of L. If every subfilter of F is a supplemented filter, then F is an amply supplemented filter.*

P r o o f. Let G and H be subfilters of F such that $F = T(G \cup H)$. By assumption, There exists a subfilter H' of H such that $H = T(H' \cup (H \cap G))$ and $(G \cap H) \cap H' = H' \cap G \ll H'$. Then $H = T(H' \cup (H \cap G)) \subseteq T(H' \cup G)$ gives $F = T(G \cup H) \subseteq T(G \cup T(H' \cup G)) = T(H' \cup G) \subseteq F$; hence $F = T(H' \cup G)$, as required. $\square$

Corollary 3.9. *The following statements are equivalent for a lattice L.*
(1) *Every filter is amply supplemented.*
(2) *Every filter is supplemented.*

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