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A NOTE ON DOUBLE FOURIER COEFFICIENTS

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Abstract. We estimate the order of magnitude of double Fourier coefficients of functions of (ϕ,ψ)-(Λ,Γ)-bounded variation in the sense of Vitali and Hardy.

1. Introduction. Let $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ and $\Gamma=\{\gamma_n\}_{n=1}^{\infty}$ be a non-decreasing sequences of positive numbers such that $\lim\limits_{n\to\infty}\Lambda_n=\lim\limits_{n\to\infty}\Gamma_n=\infty$, where $\Lambda_n=\sum\limits_{k=1}^{n}\lambda_k^{-1}$ and $\Gamma_n=\sum\limits_{k=1}^{n}\gamma_k^{-1}$.

A convex function ϕ defined on $[0,\infty)$ such that $\phi(0)=0$, $\dfrac{\phi(x)}{x}\to 0$ as $x\to 0_+$, and $\dfrac{\phi(x)}{x}\to\infty$ as $x\to\infty$, is called an N-function.

We note that an N-function is necessarily continuous and strictly increasing on $[0,\infty)$.

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An N-function ϕ is said to be a Δ_2 function if there is a constant $d \geq 2$ such that $\phi(2x) \leq d\phi(x)$ for all $x \geq 0$.

For $I = [a, b]$ and $J = [c, d]$, define

$$f(I \times J) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

A complex valued measurable function f defined on $\overline{\mathbb{T}}^2$, where $\mathbb{T} = [0, 2\pi)$, is said to be of (ϕ, ψ)-(Λ, Γ)-bounded variation (that is, $f \in (\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$) if

$$V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f, \overline{\mathbb{T}}^2) = \sup_{\mathcal{I},\mathcal{J}} \left(\sum_{\ell} \frac{1}{\gamma_\ell} \psi \left(\sum_{k} \frac{\phi(|f(I_k \times J_\ell)|)}{\lambda_k} \right) \right) < \infty,$$

where Λ and Γ are as defined above; functions ϕ and ψ are convex and strictly increasing on $[0, \infty)$ with $\phi(0) = \psi(0) = 0$; and $\mathcal{I}$ and $\mathcal{J}$ are finite collections of nonoverlapping subintervals $\{I_k\}$ and $\{J_\ell\}$ in $\overline{\mathbb{T}}$ respectively.

Consider a function $f : \overline{\mathbb{T}}^2 \to \mathbb{R}$ defined by $f(x, y) = g(x) + h(y)$, where g and h are any two arbitrary not necessarily bounded functions from $\overline{\mathbb{T}}$ into $\mathbb{R}$. Then $V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f, \overline{\mathbb{T}}^2) = 0$. Thus, a function f with $V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f, \overline{\mathbb{T}}^2) < \infty$ need not be bounded.

If $f \in (\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ is such that the marginal functions $f(0, \cdot) \in \phi\Gamma BV(\overline{\mathbb{T}})$ and $f(\cdot, 0) \in \phi\Lambda BV(\overline{\mathbb{T}})$ (refer [5, p. 2] for the definition of $\phi\Lambda BV(\overline{\mathbb{T}})$) then f is said to be of (ϕ, ψ)-$(\Lambda, \Gamma)^*$-bounded variation (that is, $f \in (\phi, \psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$).

Note that, for $\phi(x) = \psi(x) = x$ and $\Lambda = \Gamma = \{1\}$ classes $(\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ and $(\phi, \psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ reduce to classes $BV_V(\overline{\mathbb{T}}^2)$ (the class of functions of bounded variation in the sense of Vitali (refer [4, p. 279] for the definition of $BV_V(\overline{\mathbb{T}}^2)$)) and $BV_H(\overline{\mathbb{T}}^2)$ (the class of functions of bounded variation in the sense of Hardy (refer [4, p. 280] for the definition of $BV_H(\overline{\mathbb{T}}^2)$)) respectively; for $\psi(x) = x$ classes $(\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ and $(\phi, \psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ reduce to classes $\phi(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ [6, Definition 1] and $\phi(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ respectively; for $\psi(x) = x$ and $\phi(x) = x^p$ $(p \geq 1)$ classes $(\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ and $(\phi, \psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ reduce to classes $(\Lambda, \Gamma)BV^{(p)}(\overline{\mathbb{T}}^2)$ [7, Definition 2.1] and $(\Lambda, \Gamma)^* BV^{(p)}(\overline{\mathbb{T}}^2)$ respectively; for $\phi(x) = \psi(x) = x$ classes $(\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ and $(\phi, \psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ reduce to classes $(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ [2, Definition 2] and $(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ respectively.

For a function of two variables several definitions of bounded variation are given and various properties are studied (see, for example, [1]). In 2004, V. Fülöp and F. Móricz [3] estimated the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy. Recently in [6], we have estimated the order magnitude of multiple Fourier coefficients of functions of ϕ-$(\Lambda^1, \dots, \Lambda^N)$-bounded variation in the sense of Vitali and Hardy. Here, we estimate the order of magnitude of double Fourier coefficients of functions of (ϕ, ψ)-(Λ, Γ)-bounded variation in the sense of Vitali and Hardy. Our new results with $\psi(x) = x$ gives our earlier results [6] for functions of two variables.

2. New results. Given a complex-valued function $f \in L^1(\overline{\mathbb{T}}^2)$, where f is 2π-periodic in each variable, its double Fourier series is defined as

$$f(x, y) \sim \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \hat{f}(m, n)\ e^{i(mx+ny)},$$

where the Fourier coefficients $\hat{f}(m, n)$ are defined by

$$\hat{f}(m, n) = \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} f(x, y)\ e^{-i(mx+ny)} dx\ dy.$$

We prove the following results.

Theorem 2.1. *If $f \in (\phi, \psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ and $(m, n) \in \mathbb{Z}^2$ is such that $m \cdot n \neq 0$, then*

$$\hat{f}(m, n) = O\left(\phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\psi^{-1}\left(\frac{1}{\Gamma_{|n|}}\right)\right)\right). \tag{2.1}$$

P r o o f. Since

$$\hat{f}(m, n) = \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} f(x, y)\ e^{-imx}\ e^{-iny}\ dx\ dy,$$

we have

$$4\ |\hat{f}(m, n)| = \frac{1}{4\pi^2} \left| \iint_{\overline{\mathbb{T}}^2} \left(f\left(\left[x, x + \frac{\pi}{m}\right] \times \left[y, y + \frac{\pi}{n}\right]\right)\right) e^{-imx}\ e^{-iny}\ dx\ dy \right|.$$

Because of the periodicity of f in each variable, we get

$$\iint_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x, y)|\ dx\ dy = \iint_{\overline{\mathbb{T}}^2} \left| f\left(\left[x, x + \frac{\pi}{m}\right] \times \left[y, y + \frac{\pi}{n}\right]\right)\right|\ dx\ dy,$$

where

$$\Delta f_{jk}(x, y) = f\left(\left[x + \frac{(j-1)\pi}{m}, x + \frac{j\pi}{m}\right] \times \left[y + \frac{(k-1)\pi}{n}, y + \frac{k\pi}{n}\right]\right),$$

for any $j, k \in \mathbb{Z}$.
Therefore

$$|\hat{f}(m,n)| \leq \frac{1}{16\pi^2} \iint_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy$$
$$\leq \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy.$$

For $c > 0$, using Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(m,n)|) \leq \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} \phi(c|\Delta f_{jk}(x,y)|)\ dx\ dy.$$

Multiplying both the sides of above inequality by λ_j^{-1}, summing over $j = 1$ to $|m|$, and letting $\Lambda_{|m|} = \sum_{j=1}^{|m|} \lambda_j^{-1}$, we get

$$\phi(c|\hat{f}(m,n)|)\left(\Lambda_{|m|}\right) \leq \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} \left(\sum_{j=1}^{|m|} \frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j}\right)\ dx\ dy.$$

Again, using Jensen's inequality for integrals, we have

$$\psi\left(\phi(c|\hat{f}(m,n)|)\left(\Lambda_{|m|}\right)\right) \leq \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} \psi\left(\sum_{j=1}^{|m|} \frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j}\right)\ dx\ dy.$$

Multiplying both the sides of above inequality by γ_k^{-1}, summing over $k = 1$ to $|n|$, and letting $\Gamma_{|n|} = \sum_{k=1}^{|n|} \gamma_k^{-1}$, we get

$$\psi\left(\phi(c|\hat{f}(m,n)|)\left(\Lambda_{|m|}\right)\right)\left(\Gamma_{|n|}\right) \leq \frac{1}{4\pi^2} \iint_{\overline{\mathbb{T}}^2} \sum_{k=1}^{|n|} \frac{1}{\gamma_k} \psi\left(\sum_{j=1}^{|m|} \frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j}\right) dx\, dy$$

$$\leq V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(cf, \overline{\mathbb{T}}^2), \tag{2.2}$$

where $cf \in (\phi,\psi)(\Lambda,\Gamma)BV(\overline{\mathbb{T}}^2)$ for $c \in (0,1]$.

Since ϕ and ψ are convex and $\phi(0) = \psi(0) = 0$, for $c \in (0,1]$ we have $\phi(cx) \le c\phi(x)$ and $\psi(cx) \le c\psi(x)$, and hence we can choose sufficiently small $c \in (0,1]$ such that $V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(cf, \overline{\mathbb{T}}^2) \le 1$. Thus, in view of equation (2.2), we get

$$|\hat{f}(m,n)| \le \frac{1}{c}\left(\phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\psi^{-1}\left(\frac{1}{\Gamma_{|n|}}\right)\right)\right).$$

This completes the proof of theorem. □

Lemma 2.2. *If $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$, then f is bounded on $\overline{\mathbb{T}}^2$.*

Proof. For any $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$,

$$\begin{aligned} |f(x,y)| &\le |f([0,x]\times[0,y])| + |f(0,y) - f(0,0)| + |f(x,0) - f(0,0)| + |f(0,0)| \\ &\le \phi^{-1}(\lambda_1 \psi^{-1}(\gamma_1 V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f, \overline{\mathbb{T}}^2))) + \phi^{-1}(\gamma_1 V_{\Gamma_\psi}(f(0,\cdot), \overline{\mathbb{T}})) \\ &\quad + \phi^{-1}(\lambda_1 V_{\Lambda_\phi}(f(\cdot,0), \overline{\mathbb{T}})) + |f(0,0)| \end{aligned}$$

implies f is bounded on $\overline{\mathbb{T}}^2$. □

Corollary 2.3. *If $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$ and $(m,n) \in \mathbb{Z}^2$ is such that $m \cdot n \ne 0$, then* (2.1) *holds true.*

Proof. In view of Lemma 2.2, $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$ is bounded on $\overline{\mathbb{T}}^2$. Since $(\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2) \subset (\phi,\psi)(\Lambda,\Gamma) BV(\overline{\mathbb{T}}^2)$, the corollary follows from Theorem 2.1. □

Lemma 2.4. *If ϕ and ψ are Δ_2 and $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$, then*

$$\|V_{\Lambda_\phi}(f(.,y), \overline{\mathbb{T}})\|_\infty \le d\left(\psi^{-1}(\gamma_1\ V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f, \overline{\mathbb{T}}^2)) + V_{\Lambda_\phi}(f(\cdot,0), \overline{\mathbb{T}})\right),$$

where

$$\|V_{\Lambda_\phi}(f(.,y), \overline{\mathbb{T}})\|_\infty = \sup_{y \in \overline{\mathbb{T}}} V_{\Lambda_\phi}(f(.,y), \overline{\mathbb{T}}).$$

Proof. Since ϕ is increasing and Δ_2, for any $x, y \ge 0$, we have

$$\phi(x+y) \le \phi(2\max\{x,y\}) \le d\phi(\max\{x,y\}) \le d(\phi(x) + \phi(y)).$$

For any $y \in \overline{\mathbb{T}}$ and for any finite collection of non-overlapping subintervals $\{[x_j, x_{j+1}]\}$ in $\overline{\mathbb{T}}$, we have

$$\sum_j \frac{\phi(|f(x_{j+1},y) - f(x_j,y)|)}{\lambda_j}$$

$$\begin{aligned}
&= \sum_j \frac{\phi(|f([x_j,x_{j+1}]\times[0,y]) + f(x_{j+1},0) - f(x_j,0)|)}{\lambda_j} \\
&\leq \sum_j \frac{\phi(|f([x_j,x_{j+1}]\times[0,y])| + |f(x_{j+1},0) - f(x_j,0)|)}{\lambda_j} \\
&\leq \sum_j \frac{d(\phi(|f([x_j,x_{j+1}]\times[0,y])|) + \phi(|f(x_{j+1},0) - f(x_j,0)|))}{\lambda_j} \\
&= d\left(\sum_j \frac{\phi(|f([x_j,x_{j+1}]\times[0,y])|)}{\lambda_j} + \sum_j \frac{\phi(|f(x_{j+1},0) - f(x_j,0)|)}{\lambda_j}\right).
\end{aligned}$$

Thus,

$$V_{\Lambda_\phi}(f(.,y),\overline{\mathbb{T}}) \leq d\left(\psi^{-1}(\gamma_1\ V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(f,\overline{\mathbb{T}}^2)) + V_{\Lambda_\phi}(f(\cdot,0),\overline{\mathbb{T}})\right),\ \textit{for all } y \in \overline{\mathbb{T}}.$$

Hence, the lemma follows. □

Corollary 2.5. *If* ϕ *and* ψ *are* Δ_2, $f \in (\phi,\psi)(\Lambda,\Gamma)^* BV(\overline{\mathbb{T}}^2)$ *and* $(m,0) \in \mathbb{Z}^2$ *is such that* $m \neq 0$, *then*

$$\hat{f}(m,0) = O\left(\phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\right)\right).$$

P r o o f. Since

$$\hat{f}(m,0) = \frac{1}{4\pi^2}\iint_{\overline{\mathbb{T}}^2} f(x,y)\ e^{-imx}\ dx\ dy,$$

we have

$$2|\hat{f}(m,0)| = \frac{1}{4\pi^2}\left|\iint_{\overline{\mathbb{T}}^2}\left(f\left(x+\frac{\pi}{m},y\right) - f(x,y)\right)\ e^{-imx}\ dx\ dy\right|.$$

Because of the periodicity of f in each variable, we get

$$\iint_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)|\ dx\ dy = \iint_{\overline{\mathbb{T}}^2}\left|f\left(x+\frac{\pi}{m},y\right) - f(x,y)\right|\ dx\ dy,$$

where

$$\Delta f_j(x,y) = f\left(x+\frac{j\pi}{m},y\right) - f\left(x+\frac{(j-1)\pi}{m},y\right),\ \text{for any } j \in \mathbb{Z}.$$

Therefore

$$\begin{aligned}|\hat{f}(m,0)| &\leq \frac{1}{8\pi^2}\iint_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)|\ dx\ dy \\ &\leq \frac{1}{4\pi^2}\iint_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)|\ dx\ dy.\end{aligned}$$

For $c > 0$, using Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(m,0)|) \leq \frac{1}{4\pi^2}\iint_{\overline{\mathbb{T}}^2} \phi(c|\Delta f_j(x,y)|)\ dx\ dy.$$

Multiplying both the sides of above inequality by λ_j^{-1}, summing over $j = 1$ to $|m|$, and letting $\Lambda_{|m|} = \sum_{j=1}^{|m|} \lambda_j^{-1}$, we get

$$\begin{aligned}\phi(c|\hat{f}(m,0)|)(\Lambda_{|m|}) &\leq \frac{1}{4\pi^2}\iint_{\overline{\mathbb{T}}^2} \left(\sum_{j=1}^{|m|} \frac{\phi(c|\Delta f_j(x,y)|)}{\lambda_j}\right)\ dx\ dy \\ &\leq V_{\Lambda_\phi}(cf(.,y),\overline{\mathbb{T}}).\end{aligned}$$

Thus, in view of Lemma 2.4, we get

$$\phi(c|\hat{f}(m,0)|)(\Lambda_{|m|}) \leq d[\psi^{-1}(\gamma_1 V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(cf,\overline{\mathbb{T}}^2)) + V_{\Lambda_\phi}(cf(.,0),\overline{\mathbb{T}})]. \tag{2.3}$$

Since ϕ and ψ are convex and $\phi(0) = \psi(0) = 0$, so we can choose sufficiently small $c \in (0,1]$ such that $V_{(\Lambda,\Gamma)_{(\phi,\psi)}}(cf,\overline{\mathbb{T}}^2) \leq \frac{1}{\gamma_1}\psi\left(\frac{1}{2d}\right)$ and $V_{\Lambda_\phi}(cf(.,0),\overline{\mathbb{T}}) \leq \frac{1}{2d}$. Hence, from equation (2.3), we have

$$|\hat{f}(m,0)| \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\right).$$

This completes the proof of corollary. □

Similarly, one gets the analogue of the above Corollary, which is stated below.

Corollary 2.6. *If ϕ and ψ are Δ_2, $f \in (\phi,\psi)(\Lambda,\Gamma)^*BV(\overline{\mathbb{T}}^2)$ and $(0,n) \in \mathbb{Z}^2$ is such that $n \neq 0$, then*

$$\hat{f}(0,n) = O\left(\phi^{-1}\left(\frac{1}{\Gamma_{|n|}}\right)\right).$$

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