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THE CENTER OF THE TOTAL RING OF FRACTIONS

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Abstract. Let A be a right Ore domain, $Z(A)$ be the center of A and $Q_r(A)$ be the right total ring of fractions of A. If K is a field and A is a K-algebra, in this short paper we prove that if A is finitely generated and $\mathrm{GKdim}(A) < \mathrm{GKdim}(Z(A)) + 1$, then $Z(Q_r(A)) \cong Q(Z(A))$. Many examples that illustrate the theorem are included, most of them within the skew PBW extensions.

1. Introduction. Given an Ore domain A, it is interesting to know when $Z(Q(A)) \cong Q(Z(A))$, where $Z(A)$ is the center of A and $Q(A)$ is the total ring of fractions of A. This question became important after the formulation of the Gelfand-Kirillov conjecture in [4]: *Let $\mathcal{G}$ be an algebraic Lie algebra of finite dimension over a field K, with* $\mathrm{char}(K) = 0$. *Then, there exist integers $n, k \geq 1$ such that*

$$Q(\mathcal{U}(\mathcal{G})) \cong Q(A_n(K[s_1, \ldots, s_k])), \quad (1.1)$$

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where $\mathcal{U}(\mathcal{G})$ is the enveloping algebra of $\mathcal{G}$ and $A_n(K[s_1, \dots, s_k])$ is the general Weyl algebra over K. In the investigation of this famous conjecture the isomorphism between the center of the total ring of fractions and the total ring of fractions of the center occupies a special key role. There are remarkable examples of algebras for which the conjecture holds and they satisfy the isomorphism. For example, if $\mathcal{G}$ is a finite dimensional nilpotent Lie algebra over a field K, with $\mathrm{char}(K) = 0$, then the conjecture holds and $Z(Q(\mathcal{U}(\mathcal{G}))) \cong Q(Z(\mathcal{U}(\mathcal{G})))$ ([4], Lemma 8). More recently, the quantum version of the Gelfand-Kirillov conjecture has occupied the attention of many researchers. One example of this is the following (see [1], Theorem 2.15): Let $U_q^+(sl_m)$ be the quantum enveloping algebra of the Lie algebra of strictly superior triangular matrices of size $m \times m$, $m \geq 3$, over a field K. If $m = 2n + 1$, then $Q(U_q^+(sl_m)) \cong Q(\mathrm{K_q}[x_1, \dots, x_{2n^2}])$, where $\mathrm{K_q}[x_1, \dots, x_{2n^2}]$ is the quantum ring of polynomials, $\mathrm{K} := Q(Z(U_q^+(sl_m)))$ and $\mathrm{q} := [q_{ij}] \in M_{2n^2}(K)$, with $q_{ii} = 1 = q_{ij}q_{ji}$, and q_{ij} is a power of q for every $1 \leq i, j \leq 2n^2$. If $m = 2n$, then $Q(U_q^+(sl_m)) \cong Q(\mathrm{K_q}[x_1, \dots, x_{2n(n-1)}])$, where $\mathrm{q} := [q_{ij}] \in M_{2n(n-1)}(K)$, with $q_{ii} = 1 = q_{ij}q_{ji}$, and q_{ij} is a power of q for every $1 \leq i, j \leq 2n(n-1)$. Moreover, in both cases $Z(Q(U_q^+(sl_m))) \cong Q(Z(U_q^+(sl_m)))$.

Let A be a right Ore domain. In this paper we study the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$, where $Z(A)$ is the center of A and $Q_r(A)$ is the right total ring of fractions of A. The main tool that we will use is the Gelfand-Kirillov dimension, so we will assume that A is a K-algebra, where K is an arbitrary field. The principal result is Theorem 2.3 proved in Section 2. The result can also be interpreted as a way of computing the center of $Q_r(A)$. In Section 3 we include many examples that illustrate the theorem, most of them within the skew PBW extensions.

We start with the following known facts about skew PBW extensions that will be used in the examples.

Definition 1.1 ([3]). *Let R and A be rings. We say that A is a skew PBW extension of R (also called a $\sigma - PBW$ extension of R) if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finitely many elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis*

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \dots\}.$$

The set Mon(A) *is called the set of standard monomials of A.*

(iii) *For every* $1 \leq i \leq n$ *and* $r \in R - \{0\}$ *there exists* $c_{i,r} \in R - \{0\}$ *such that*

$$x_i r - c_{i,r} x_i \in R. \tag{1.2}$$

(iv) *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.3}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ there are n injective endomorphisms $\sigma_1, \ldots, \sigma_n$ of R and σ_i-derivations, as the following proposition shows.

Proposition 1.2 ([3], Proposition 3)**.** *Let* A *be a skew* PBW *extension of* R. *Then, for every* $1 \leq i \leq n$, *there exist an injective ring endomorphism* $\sigma_i : R \to R$ *and a* σ_i*-derivation* $\delta_i : R \to R$ *such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

A particular case of skew PBW extension is when all σ_i are bijective and the constants c_{ij} are invertible.

Definition 1.3 ([3])**.** *Let* A *be a skew* PBW *extension.* A *is bijective if* σ_i *is bijective for every* $1 \leq i \leq n$ *and* $c_{i,j}$ *is invertible for any* $1 \leq i < j \leq n$.

Proposition 1.4 ([7], Theorem 3.6)**.** *Let* $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ *be a bijective skew* PBW *extension of a right Ore domain* R. *Then* A *is also a right Ore domain.*

Proposition 1.5 ([11], Theorem 14)**.** *Let* R *be a* K*-algebra with a finite dimensional generating subspace* V *and let* $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ *be a bijective skew* PBW *extension of* R. *If* σ_i, δ_i *are* K*-linear and* $\sigma_i(V) \subseteq V$, *for* $1 \leq i \leq n$, *then*

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + n.$$

Proposition 1.6. *Let* R *be a commutative domain,* σ *an automorphism of* R *and*

$$R^\sigma := \{r \in R | \sigma(r) = r\}.$$

If σ *has infinite order, then* $Z(R[x;\sigma]) = R^\sigma$. *If* σ *has finite order* v, *then* $Z(R[x;\sigma]) = R^\sigma[x^v]$.

Proof. The proof when R is a field can be found in [12], Proposition 1.6.25. For completeness we include the proof in the general case. Firstly observe that R^σ is a subring of $R[x;\sigma]$, whence $R^\sigma[x^v]$ is the subring of $R[x;\sigma]$ generated by R^σ and x^v, this implies that the elements of $R^\sigma[x^v]$ are polynomial in x^v with coefficients in R^σ.

Let $p(x) := p_0 + p_1x + \cdots + p_nx^n \in Z(R[x;\sigma])$, then for every $r \in R$, $rp(x) = p(x)r$, so for every $0 \le i \le n$, we get $rp_i = p_i\sigma^i(r) = \sigma^i(r)p_i$. If the order of σ is infinite, then $\sigma^i \neq i_R$ for every $i \ge 1$, hence $p_i = 0$ for $i \ge 1$. Thus, in this case $p(x) = p_0$; moreover, $p(x)$ commutes with x, so $p_0x = xp_0$, whence $p_0 = \sigma(p_0)$, i.e., $p_0 \in R^\sigma$. Therefore, if σ has infinite order, then $Z(R[x;\sigma]) \subseteq R^\sigma$, but clearly, $R^\sigma \subseteq Z(R[x;\sigma])$. Suppose that σ has finite order, say, v. If $\sigma^i \neq i_R$, i.e., if $v \nmid i$, then $p_i = 0$, hence $p(x) = p_0 + p_1x^v + p_2(x^v)^2 + \cdots + p_t(x^v)^t$. Since $p(x)$ commutes with x, then $\sigma(p_i) = p_i$ for every $0 \le i \le t$. This proves that $Z(R[x;\sigma]) \subseteq R^\sigma[x^v]$. But, $R^\sigma[x^v] \subseteq Z(R[x;\sigma])$ since every element of the form $r(x^v)^t$, with $r \in R^\sigma$ and $t \ge 0$, commutes with every element $s \in R$ and with x. □

2. Main theorem. We start with the following easy proposition. We include the proof for completeness.

Proposition 2.1. *Let A be a right Ore domain.*

(i) *If $\frac{p}{q} \in Z(Q_r(A))$ then*

  (a) $pq = qp$.

  (b) *For every $s \in A - \{0\}$, $psq = qsp$.*

  (c) *$p \in Z(A)$ if and only if $q \in Z(A)$.*

(ii) *Let $p \in A$. Then, $\frac{p}{1} \in Z(Q_r(A))$ if and only if $p \in Z(A)$. Thus, $Z(A) \hookrightarrow Z(Q_r(A))$.*

(iii) *If K is a field and A is a K-algebra such that $Z(Q_r(A)) = K$, then $Q(Z(A)) = K = Z(Q_r(A))$.*

Proof. (i) (a) We have $\frac{p}{q}\frac{q}{1} = \frac{q}{1}\frac{p}{q}$, so $\frac{p}{1} = \frac{qp}{q}$, whence $\frac{p}{1}\frac{q}{1} = \frac{qp}{q}\frac{q}{1}$, i.e., $\frac{pq}{1} = \frac{qp}{1}$, thus $pq = qp$.

(b) For $s = 0$ is clear. Let $s \in A - \{0\}$, then $\frac{p}{q} = \frac{ps}{qs}$, so by (a), $psqs = qsps$, whence $psq = qsp$.

(c) If $\frac{p}{q} = 0$, then $\frac{p}{q} = \frac{0}{1}$ and the claimed trivially holds. We can assume that $p \neq 0$; by (b), for every $s \in A - \{0\}$, $psq = qsp$, hence if $p \in Z(A)$, then $sqp = qsp$, whence $sq = qs$, i.e., $q \in Z(A)$. On the other hand, since $\frac{q}{p} \in Z(Q_r(A))$, then if $q \in Z(A)$ we get $p \in Z(A)$.

(ii) If $\frac{p}{1} \in Z(Q_r(A))$, then by (i), $ps = sp$ for every $s \neq 0$, whence $p \in Z(A)$. Conversely, let $p \in Z(A) - \{0\}$ (for $p = 0$, $\frac{p}{1} \in Z(Q_r(A))$), then for every $\frac{a}{s} \in Q_r(A)$ we have $\frac{p}{1}\frac{a}{s} = \frac{pa}{s}$ and $\frac{a}{s}\frac{p}{1} = \frac{ac}{r} = \frac{acp}{rp}$, where $sc = pr = rp$, with $c, r \neq 0$. From this we get $\frac{acp}{rp} = \frac{apc}{sc} = \frac{ap}{s} = \frac{pa}{s}$, i.e., $\frac{p}{1} \in Z(Q_r(A))$.

(iii) From (ii), $K \subseteq Z(A) \subseteq Z(Q_r(A)) = K$, so $Z(A) = K$, and hence $Q(Z(A)) = K = Z(Q_r(A))$. $\square$

The next example illustrates the part (iii) of Proposition 2.1.

Example 2.2. We consider the quantum plane $A := K_q[x, y]$, where q is not a root of unity. We will show that $Z(Q_r(A)) = K$. Let $\frac{p}{s} \in Z(Q_r(A)) - \{0\}$, where $p := \sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i}$ and $s := \sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j}$, with $r_i, u_j \in K - \{0\}$. From $px_1s = sx_1p$ and since q is not a root of unity, we get $\beta_i + \beta_i\theta_j = \gamma_j + \gamma_j\alpha_i$ for every $1 \leq i \leq t$ and $1 \leq j \leq l$. Similarly, from $px_2s = sx_2p$ we obtain $\theta_j\beta_i + \theta_j = \alpha_i\gamma_j + \alpha_i$ for all i, j, whence $\beta_i + \alpha_i = \gamma_j + \theta_j$, so fixing i and then fixing j we conclude that p and s are homogeneous of the same degree (this condition is not enough since $\frac{x_1}{x_2} \notin Z(Q_r(A))$). Now,

$$\frac{p}{s}\frac{x_1}{1} = \frac{x_1}{1}\frac{p}{s}, \text{ i.e., } \frac{\sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i}}{\sum_{j=1}^{l} u_j x_1^{\theta_j - 1} x_2^{\gamma_j}} = \frac{\sum_{i=1}^{t} r_i x_1^{\alpha_i + 1} x_2^{\beta_i}}{\sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j}},$$

hence there exist $c := x_1^m p_m(x_2) + \cdots + p_0(x_2), d := x_1^k q_k(x_2) + \cdots + q_0(x_2) \in A - \{0\}$ such that

$$\left(\sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i}\right) c = \left(\sum_{i=1}^{t} r_i x_1^{\alpha_i + 1} x_2^{\beta_i}\right) d,$$

$$\left(\sum_{j=1}^{l} u_j x_1^{\theta_j - 1} x_2^{\gamma_j}\right) c = \left(\sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j}\right) d.$$

Since p and q are homogeneous, we can assume $\alpha_1 > \cdots > \alpha_t$ and $\theta_1 > \cdots > \theta_l$, whence $\beta_1 < \cdots < \beta_t$ and $\gamma_1 < \cdots < \gamma_l$. Then,

$$(r_1 x_1^{\alpha_1} x_2^{\beta_1})(x_1^m p_m(x_2)) = r_1 q^{m\beta_1} x_1^{\alpha_1+m} x_2^{\beta_1} p_m(x_2),$$

$$r_1 x_1^{\alpha_1+1} x_2^{\beta_1} x_1^k q_k(x_2) = r_1 q^{k\beta_1} x_1^{\alpha_1+1+k} x_2^{\beta_1} q_k(x_2),$$

whence $\alpha_1 + m = \alpha_1 + 1 + k$, i.e., $m = k + 1$. Moreover, let p_m be the leader coefficient of $p_m(x_2)$ and q_k be the leader coefficient of $q_k(x_2)$, then $q^{\beta_1} p_m = q_k$. Similarly, we can prove that $q^{\gamma_1} p_m = q_k$, but since q is not a root of unity, $\beta_1 = \gamma_1$. From $\alpha_1 + \beta_1 = \theta_1 + \gamma_1$ we get that $\alpha_1 = \theta_1$ (considering instead the identity $\frac{p}{s}\frac{x_2}{1} = \frac{x_2}{1}\frac{p}{s}$ we obtain the same result). Thus, we have

$$\alpha_1 = \theta_1 \quad \text{and} \quad \beta_1 = \gamma_1.$$

Notice that

$$\frac{p}{s} = r_1 u_1^{-1} + \frac{p - r_1 u_1^{-1} s}{s}, \quad \text{with} \quad r_1 u_1^{-1} \in K \subseteq Z(Q_r(A)),$$

hence $\frac{p - r_1 u_1^{-1} s}{s} \in Z(Q_r(A))$. But observe that $\frac{p - r_1 u_1^{-1} s}{s} = 0$, contrary, we could repeat the previous procedure and find that there exists, either $i \geq 2$ such that $\alpha_i = \theta_1 = \alpha_1, \beta_i = \gamma_1 = \beta_1$, or $j \geq 2$ such that $\theta_j = \theta_1, \gamma_j = \gamma_1$, a contradiction. Thus, $\frac{p}{s} = r_1 u_1^{-1} \in K$, and hence, $Z(Q_r(A)) = K$.

The previous example shows that the proof of the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$ by direct computation of the center of the total ring of fractions is tedious. An alternative more practical method is given by the following theorem.

Theorem 2.3. *Let K be a field and A be a right Ore domain. If A is a finitely generated K-algebra such that* $\mathrm{GKdim}(A) < \mathrm{GKdim}(Z(A)) + 1$*, then*

$$Z(Q_r(A)) = \{\frac{p}{q} \mid p, q \in Z(A), q \neq 0\} \cong Q(Z(A)).$$

P r o o f. We divide the proof in three steps.

Step 1. As in the proof of Theorem 4.12 in [5], we will show that

$$Q_r(A) \cong A(Z(A)_0)^{-1} = \left\{\frac{p}{q} \mid p \in A, q \in Z(A)_0\right\}, \quad \text{with} \quad Z(A)_0 := Z(A) - \{0\}.$$

First observe that $Z(A)_0$ is a right Ore set of A, so $A(Z(A)_0)^{-1}$ exists. From the canonical injection $Q(Z(A)) \hookrightarrow A(Z(A)_0)^{-1}$, $\frac{p}{q} \mapsto \frac{p}{q}$, we get that $A(Z(A)_0)^{-1}$ is a vector space over $Q(Z(A))$, moreover, $A(Z(A)_0)^{-1} = AQ(Z(A))$. We will show that the dimension of this vector space is finite. Let V be a frame that generates A. Since $\{V^n\}_{n\geq 0}$ is a filtration of A, then $A = \bigcup_{n\geq 0} V^n$ and $AQ(Z(A)) = \bigcup_{n\geq 0} V^n Q(Z(A))$. Arise two possibilities: Either there exists $n \geq 0$ such that $V^n Q(Z(A)) = V^{n+1}Q(Z(A))$, or else

$$Q(Z(A)) \subsetneq VQ(Z(A)) \subsetneq V^2 Q(Z(A)) \subsetneq \cdots$$

In the first case $AQ(Z(A)) = V^n Q(Z(A))$ and we get the claimed. In the second case,

$$\dim_{Q(Z(A))} Q(Z(A)) \lneq \dim_{Q(Z(A))} VQ(Z(A)) \lneq \dim_{Q(Z(A))} V^2 Q(Z(A)) \lneq \cdots$$

and we will show that this produces a contradiction. In fact, for every $n \geq 0$,

$$\dim_{Q(Z(A))} V^n Q(Z(A)) \geq n + 1;$$

let $u_1, \ldots, u_{d(n)}$ be a $Q(Z(A))$-basis of $V^n Q(Z(A))$, thus, $d(n) \geq n+1$; we can assume that $u_i \in V^n$ for every $1 \leq i \leq d(n)$; let W be an arbitrary K-subspace of $Z(A)$ of finite dimension, then

$$(V + W)^{2n} \supseteq V^n W^n \supseteq u_1 W^n \oplus \cdots \oplus u_{d(n)} W^n$$

(the sum is direct since the elements u_i are linearly independent over $Z(A)$); from this we get

$$\dim_K (V + W)^{2n} \geq d(n) \dim_k(W^n) \geq (n + 1) \dim_K(W^n),$$

but since $V + W$ is a frame of A, then $\mathrm{GKdim}(A) \geq 1 + \mathrm{GKdim}(Z(A))$, false.

Now we can prove the claimed isomorphism. For this consider the canonical injective homomorphism $g : A \to A(Z(A)_0)^{-1}$, $a \mapsto \frac{a}{1}$. If $a \in A - \{0\}$, then $\frac{a}{1}$ is invertible in $A(Z(A)_0)^{-1}$. In fact, the map $h : A(Z(A)_0)^{-1} \to A(Z(A)_0)^{-1}$, $\frac{p}{q} \mapsto \frac{a}{1}\frac{p}{q}$, is an injective $Q(Z(A))$-homomorphism since A is a domain, but as was observed above, $A(Z(A)_0)^{-1}$ is finite-dimensional over $Q(Z(A))$, therefore h

is surjective, whence, there exists $\frac{p}{q} \in A(Z(A)_0)^{-1}$ such that $\frac{a}{1}\frac{p}{q} = \frac{1}{1}$. Observe that $\frac{p}{q}\frac{a}{1} = \frac{1}{1}$: In fact, since $p \neq 0$, there exists $\frac{p'}{q'}$ in $A(Z(A)_0)^{-1}$ such that $\frac{p}{1}\frac{p'}{q'} = \frac{1}{1}$, and since $\frac{p}{q} = \frac{1}{q}\frac{p}{1}$, then

$$\frac{a}{1}\frac{1}{q}\frac{p}{1}\frac{p'}{q'} = \frac{1}{1}\frac{p'}{q'}, \text{ i.e., } \frac{a}{1}\frac{1}{q} = \frac{p'}{q'}, \text{ so } \frac{a}{1}\frac{1}{q}\frac{q}{1} = \frac{p'}{q'}\frac{q}{1},$$

$$\text{whence } \frac{a}{1} = \frac{p'}{q'}\frac{q}{1}, \text{ so } \frac{p}{q}\frac{a}{1} = \frac{p}{q}\frac{p'}{q'}\frac{q}{1} = \frac{1}{q}\frac{p}{1}\frac{p'}{q'}\frac{q}{1} = \frac{1}{q}\frac{q}{1} = \frac{1}{1}.$$

In order to conclude the proof of the isomorphism $A(Z(A)_0)^{-1} \cong Q_r(A)$, observe that any element $\frac{p}{q} \in A(Z(A)_0)^{-1}$ can be written as $\frac{p}{q} = g(p)g(q)^{-1}$.

Step 2. Let $C := \{\frac{p}{q} \mid p, q \in Z(A), q \neq 0\}$. If $\frac{p}{q} \in Z(Q_r(A))$, then by the first step we can assume that $q \in Z(A)_0$, and from the part (i)-(c) of Proposition 2.1, we get $p \in Z(A)$. Therefore, $Z(Q_r(A)) \subseteq C$. Conversely, let $\frac{p}{q} \in C$, then $p, q \in Z(A)$, with $q \neq 0$, whence, by the part (ii) of Proposition 2.1, $\frac{p}{1}, \frac{q}{1} \in Z(Q_r(A))$, so $\frac{1}{q} \in Z(Q_r(A))$, and hence, $\frac{p}{q} = \frac{p}{1}\frac{1}{q} \in Z(Q_r(A))$. Thus, $C \subseteq Z(Q_r(A))$.

Step 3. According to the part (ii) of Proposition 2.1, we have the canonical injective homomorphism $Z(A) \xrightarrow{\iota} Z(Q_r(A))$, $p \mapsto \frac{p}{1}$, that sends invertible elements of $Z(A)$ in invertible elements of $Z(Q_r(A))$, moreover, by the step 2, every element $\frac{p}{q} \in Z(Q_r(A))$ can be written $\frac{p}{q} = \iota(p)\iota(q)^{-1}$. This proves the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$. □

3. Examples. Next we present many examples of K-algebras that satisfy the hypotheses of Theorem 2.3. Most of them within the skew PBW extensions.

Example 3.1. (i) Any domain A such that $\dim_K A < \infty$. For example, the real algebra $\mathbb{H}$ of quaternions since $\dim_{\mathbb{R}}(\mathbb{H}) = 4$.

(ii) Any right Ore domain A finitely generated as $Z(A)$-module.

Example 3.2. Applying Propositions 1.4 and 1.5, we will check next that the following skew PBW extensions are K-algebras that satisfy the hypotheses of Theorem 2.3. The precise definition of any of these algebras can be found in [8].

(i) Consider a skew polynomial ring $A := R[x;\sigma]$, with R a commutative domain R that is a K-algebra generated by a subspace V of finite dimension such that $\sigma(V) \subseteq V$, σ is K-linear of finite order m, $R^\sigma = K$ and $\mathrm{GKdim}(R) = 0$. Then, $\mathrm{GKdim}(A) = 1$, and from Proposition 1.6, $Z(A) = K[x^m]$, and hence, $\mathrm{GKdim}(Z(A)) = 1$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = K(x^m)$. A particular case of this general example is $A := \mathbb{C}[x;\sigma]$ as $\mathbb{R}$-algebra, with $\sigma(z) := \overline{z}$, $z \in \mathbb{C}$ (here $\mathbb{C}$ and $\mathbb{R}$ are the fields of complex and real numbers, respectively). In this case the order of σ is two and $\mathrm{GKdim}(\mathbb{C}) = 0$ since $\dim_{\mathbb{R}}(\mathbb{C}) = 2$.

(ii) Let $\mathrm{char}(K) = p > 0$ and $A := S_h := K[t][x_h;\sigma_h]$ be the algebra of shift operators. Then, $\mathrm{GKdim}(A) = 2$. Moreover, for every $k \geq 0$, $\sigma_h^k(t) = t - kh$, then $\sigma_h^p(t) = t$, i.e., the order of σ_h is p, therefore, $Z(A) = K[t]^{\sigma_h}[x_h^p]$. Since $K[t^p] \subseteq K[t]^{\sigma_h} \subseteq K[t]$ and $K[t]$ is finitely generated over $K[t^p]$, then $\mathrm{GKdim}(K[t^p]) = \mathrm{GKdim}(K[t]) = 1$, whence $\mathrm{GKdim}(K[t]^{\sigma_h}) = 1$. Therefore, $\mathrm{GKdim}(Z(A)) = 2$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = Q(K[t]^{\sigma_h})(x_h^p)$.

(iii) Let $\mathrm{char}(K) = p > 0$ and $A := K[t][x;\frac{d}{dt}][x_h;\sigma_h]$ be the algebra of shift differential operators. Then, $\mathrm{GKdim}(A) = 3$ and can be proved that $Z(A) = K[x^p, x_h^p, t^{p^2} - t^p]$. Since $\mathrm{GKdim}(Z(A)) = 3$, then $Z(Q_r(A)) \cong Q(Z(A)) = K(x^p, x_h^p, t^{p^2} - t^p)$.

(iv) Let $\mathrm{char}(K) = p > 0$ and $A := A_n(K)$ be the Weyl algebra. Since $\mathrm{GKdim}(A) = 2n$ and $Z(A) = K[t_1^p, \ldots, t_n^p, x_1^p, \ldots, x_n^p]$ (see [6], ejemplo 1.3.), then $\mathrm{GKdim}(Z(A)) = 2n$. Therefore, $Z(Q_r(A)) \cong Q(Z(A)) = K(t_1^p, \ldots, t_n^p, x_1^p, \ldots, x_n^p)$.

(v) Let $\mathrm{char}(K) = p > 0$ and $A := \mathcal{J} := K\{x, y\}/\langle yx - xy - x^2\rangle$ be the Jordan algebra. Since $\mathrm{GKdim}(\mathcal{J}) = 2$ and $Z(A) = K[x^p, y^p]$ (see Theorem 2.2 in [13]), then $\mathrm{GKdim}(Z(A)) = 2$, whence $Z(Q_r(A)) \cong Q(Z(A)) = K(x^p, y^p)$.

(vi) Consider the quantum plane $A := K_q[x, y]$, with $q \neq 1$ a root of unity of degree $m \geq 2$. Then $\mathrm{GKdim}(A) = 2$ and $Z(A) = K[x^m, y^m]$ (see [13]). Therefore, $Z(Q_r(A)) \cong Q(Z(A)) = K(x^m, y^m)$.

(vii) The previous example can be extended to the quantum polynomials $A := K_q[x_1, \ldots, x_n]$, where $n \geq 2$ and $q \in K - \{0, 1\}$, defined by

$$x_j x_i = q x_i x_j, \text{ with } 1 \leq i < j \leq n.$$

If q is a root of unity of degree $m \geq 2$, then can be proved that if n even, then $Z(A) = K[x_1^m, \ldots, x_n^m]$. Therefore, $\mathrm{GKdim}(Z(A)) = n = \mathrm{GKdim}(A)$ and hence

$$Z(Q_r(A)) \cong Q(Z(A)) = K(x_1^m, \ldots, x_n^m).$$

(viii) Let A_q be the quantum Weyl algebra generated by x, y with rule of multiplication $yx = qxy + a$, where $q, a \in K - \{0\}$. If q is a primitive root of unity of degree $m \geq 2$, then $Z(A_q) = K[x^m, y^m]$ (see [2]). Since $\text{GKdim}(A_q) = 2$, then $Z(Q_r(A_q)) \cong Q(Z(A)) = K(x^m, y^m)$.

(ix) In [9] has been computed the center of the following algebras. In every example we assume that the parameters q's are root of unity of degree $l \geq 2$, or $l_i \geq 2$, appropriately:

(a) Algebra of q-differential operators, then $Z(A) = K[x^l, y^l]$ and $\text{GKdim}(A) = 2$, so $Z(Q_r(A)) \cong Q(Z(A)) = K(x^l, y^l)$.

(b) Additive analogue of the Weyl algebra, $Z(A) = K[x_1^{l_1}, \dots, x_n^{l_n}, y_1^{l_1}, \dots, y_n^{l_n}]$ and $\text{GKdim}(A) = 2n$, so $Z(Q_r(A)) \cong Q(Z(A)) = K(x_1^{l_1}, \dots, x_n^{l_n}, y_1^{l_1}, \dots, y_n^{l_n})$.

(c) Algebra of linear partial q-dilation operators, in this case we have $\text{GKdim}(A) = 2n$ and $Z(A) = K[t_1^l, \dots, t_n^l, H_1^l, \dots, H_n^l]$. Therefore, $Z(Q_r(A)) \cong Q(Z(A)) = K(t_1^l, \dots, t_n^l, H_1^l, \dots, H_n^l)$.

(d) Algebra of linear partial q-differential operators, in this case we have $\text{GKdim}(A) = 2n$ and $Z(A) = K[t_1^l, \dots, t_n^l, D_1^l, \dots, D_n^l]$. Hence, $Z(Q_r(A)) \cong Q(Z(A)) = K(t_1^l, \dots, t_n^l, D_1^l, \dots, D_n^l)$.

(x) Let $\mathfrak{sl}(n, K)$ be the Lie algebra of 2×2 matrices with null trace with K-basis e, f, h. If $\text{char}(K) = 2$, then $Z(\mathcal{U}(\mathfrak{sl}(2, K))) = K[e^2, f^2, h]$ (see [6], p. 147). Moreover, $\text{GKdim}(\mathcal{U}(\mathfrak{sl}(2, K))) = 3$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = K(e^2, f^2, h)$.

Remark 3.3. As occurs for the Gelfand-Kirillov conjecture (see [4]), if the hypotheses of Theorem 2.3 fail, then the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$ could hold or fail. Thus, the hypotheses are not necessary conditions. For example,

(a) $\mathbb{H}$ is not finitely generated as $\mathbb{Q}$-algebra, however $Q_r(\mathbb{H}) = \mathbb{H}$ and $Z(Q_r(\mathbb{H})) \cong \mathbb{R} \cong Q(Z(\mathbb{H}))$.

(b) Let K be a field with $\text{char}(K) = 0$, and let $\mathcal{G}$ be a three-dimensional completely solvable Lie algebra with basis x, y, z such that $[y, x] = y$, $[z, x] = \lambda z$ and $[y, z] = 0$, $\lambda \in K - \{0\}$ (see Example 14.4.2 in [10]). If $\lambda \in K - \mathbb{Q}$, then $Z(U(\mathcal{G})) = K$ and $\text{GKdim}(\mathcal{U}(\mathcal{G})) = 3$, thus, in this case $\text{GKdim}(\mathcal{U}(\mathcal{G})) \not\leq \text{GKdim}(Z(\mathcal{U}(\mathcal{G}))) + 1$, and 14.4.7 in [10] shows that $Z(Q_r(\mathcal{U}(\mathcal{G}))) \ncong Q(Z(\mathcal{U}(\mathcal{G})))$.

(c) Let $A := U_q^+(sl_m)$ be the quantum enveloping algebra of the Lie algebra of strictly superior triangular matrices of size $m \times m$ over a field K, where $q \in K - \{0\}$ is not a root of unity. In [1] was proved that $Z(A)$ is the classical commutative polynomial algebra over K in n variables, with $m = 2n$

or $m = 2n+1$, whence, $\mathrm{GKdim}(Z(A)) = n$. On the other hand, according to [1], p. 236, A is an iterated skew polynomial ring of K of $\dfrac{m(m-1)}{2}$ variables, hence $\mathrm{GKdim}(A) = \dfrac{m(m-1)}{2}$. Thus, $\mathrm{GKdim}(A) \not\leq \mathrm{GKdim}(Z(A)) + 1$, however, $Z(Q_r(A)) \cong Q(Z(A))$.

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