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HYPERBOLIC POLYNOMIALS
AND CANONICAL SIGN PATTERNS

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ABSTRACT. A real univariate polynomial is hyperbolic if all its roots are real. By Descartes' rule of signs a hyperbolic polynomial (HP) with all coefficients nonvanishing has exactly c positive and exactly p negative roots counted with multiplicity, where c and p are the numbers of sign changes and sign preservations in the sequence of its coefficients. We discuss the question: If the moduli of all $c + p$ roots are distinct and ordered on the positive half-axis, then at which positions can the p moduli of negative roots be depending on the positions of the positive and negative signs of the coefficients of the polynomial? We are especially interested in the choices of these signs for which exactly one order of the moduli of the roots is possible.

1. Introduction. We consider real univariate polynomials with non-vanishing coefficients. Such a polynomial is *hyperbolic* if all its roots are real. Various problems concerning hyperbolic polynomials (HPs) are exposed in [9].

In this paper we discuss the following question: Suppose that the moduli of all roots of a HP are distinct and ordered on the positive half-axis. Then at which positions can the moduli of the negative roots be depending on the signs of the coefficients of the HP? In this sense we say that we are interested in the possible orders on the positive half-axis of the moduli of roots of HPs with given signs of their coefficients.

Without loss of generality we consider only monic polynomials. A *sign pattern (SP)* is a finite sequence of $(+)$- and/or $(-)$-signs. The SP defined by the polynomial $P := \sum_{j=0}^{d} a_j x^j$, $a_j \in \mathbb{R}^*$, $a_d = 1$, is the vector

$$\sigma(P) := (+, \mathrm{sgn}(a_{d-1}), \mathrm{sgn}(a_{d-2}), \ldots, \mathrm{sgn}(a_0)).$$

Notation 1. When we write $\sigma(P) = \Sigma_{m_1, m_2, \ldots, m_s}$, $m_i \in \mathbb{N}^*$, $m_1 + \cdots + m_s = d + 1$, this means that the SP $\sigma(P)$ begins with a sequence of m_1 signs $+$ followed by a sequence of m_2 signs $-$ followed by a sequence of m_3 signs $+$ etc. The number $s - 1$ is the number of sign changes and the number $d - s + 1$ is the number of sign preservations of the SP $\sigma(P)$.

The classical Descartes' rule of signs says that the polynomial P has not more than $s - 1$ positive roots. When applied to the polynomial $P(-x)$, this rule implies that P has not more than $d - s + 1$ negative roots. Hence if P is hyperbolic, then it has exactly $s - 1$ positive and exactly $d - s + 1$ negative roots (all roots are counted with multiplicity).

Remark 1. Fourier has made Descartes' rule of signs about real (but not necessarily hyperbolic) polynomials more precise by showing that the number of positive roots differs from $s - 1$ by an even integer, see [5]. For such polynomials, Descartes' rule of signs proposes only necessary conditions. Attempts to clarify the question how far from sufficient they are have been carried out in [1], [2], [3], [4], [6], [7] and [8].

Definition 1. *Given a SP (of length $d + 1$ and beginning with $+$) we construct its corresponding* change-preservation pattern (CPP) *(of length d) as follows. For $j \geq 2$, there is a p (resp. a c) in position $j - 1$ of the CPP if in positions $j - 1$ and j of the SP there are two equal (resp. two different) signs. It is clear that the correspondence between SPs beginning with $+$ and CPPs is bijective. Example: for $d = 6$, to the SP $\sigma_0 := (+, +, -, -, +, +, +)$ there corresponds the CPP (p, c, p, c, p, p).*

Definition 2. (1) *Suppose that a degree d HP P is given which defines the SP σ of length $d + 1$, suppose that the moduli of its roots are ordered on the real positive half-line, and suppose that all moduli of roots are distinct. We define formally the* canonical order *of the moduli of roots like this: the CPP corresponding to the given SP σ is read from the back, each p is replaced by an N and each c by a P. For the SP σ_0 from Definition 1 this gives (N, N, P, N, P, N) which means that the moduli of the roots are $0 < \gamma_1 < \cdots < \gamma_6$, where the polynomial has positive roots γ_3 and γ_5, and negative roots $-\gamma_1$, $-\gamma_2$, $-\gamma_4$ and $-\gamma_6$.*

(2) *For a HP P and the SP $\sigma(P)$, we say that the SP $\sigma(P)$ is* realizable *by P.*

Proposition 1. *Every SP σ of length $d + 1$, $d \geq 1$, is realizable by a degree d HP with canonical order of the moduli of its roots.*

P r o o f.   We construct the HP in d steps. At the first step we set $P_1 := x + 1$ if the first component of the CPP is a p and $P_1 := x - 1$ if it is a c. Suppose that the degree k HP P_k is constructed which defines the SP σ_k obtained from σ by deleting its last $d - k$ components. Set $P_{k+1}(x) := P_k(x)(x + \varepsilon)$ if the last two components of σ_{k+1} are equal or $P_{k+1}(x) := P_k(x)(x - \varepsilon)$ if they are different, where $\varepsilon > 0$. One chooses ε so small that:

1) the signs of the first $k + 1$ coefficients of P_{k+1} are the same as the ones of P_k;

2) the number ε is smaller than all the moduli of roots of P_k.

It is clear that for $k = d$, the HP P_d thus obtained defines the SP σ and that the order of the moduli of its roots is the canonical one. □

Remarks 1. (1) The proposition can be generalized for real, but not necessarily hyperbolic polynomials, see Lemmas 14 and 17 in [3]. The way of constructing new polynomials by adding new roots of modulus much smaller than the already existing moduli (which preserves the signs of the first $d + 1$ coefficients) can be called *concatenation of polynomials (or of SPs)*. The construction described in the proof of Proposition 1 extends at each step the SP by adding a $(+)$- or $(-)$-sign at its rear.

(2) One can propose a similar concatenation, i.e. construction of HPs, in which each new root has a modulus much larger than the moduli of the already existing roots. Namely, given a degree d HP $P(x)$ with no vanishing coefficients one considers the HP $(1 \pm \varepsilon x)P(x)$ which for $\varepsilon > 0$ sufficiently small has the

same signs of the last $d + 1$ coefficients as P. Its new root equals $1/(\mp\varepsilon)$. After this one has to multiply the polynomial by $\pm 1/\varepsilon$ to make it monic again. This construction extends the SP by adding a $(+)$- or $(-)$-sign at its front.

Definition 3. *A SP is called* canonical *if it is realizable only by HPs with canonical order of the moduli of their roots.*

Example 1. (1) The following SPs $\Sigma_{m_1,m_2,\ldots,m_s}$ are canonical:

$$\Sigma_{m_1,1}, \Sigma_{1,m_2}, \Sigma_{m_1,1,m_3} \quad \text{and for} \quad m_2 \geq 3, \ \Sigma_{1,m_2,1},$$

see Theorem 1, Corollary 1, Theorem 5 and Theorem 2 in [10] respectively. The SP $\Sigma_{1,2,1}$ is not canonical – by part (1) of Example 2 therein the SP $\Sigma_{1,2,1}$ is realizable by each of the three polynomials $(x+1)(x-1.5)(x-1.6)$, $(x+1)(x-1.5)(x-0.6)$ and $(x+1)(x-0.5)(x-0.6)$.

(2) For $m_1 \geq 2$, $m_2 \geq 2$, the SP $\sigma_0 := \Sigma_{m_1,m_2}$ is not canonical, see Theorem 1 and Corollary 1 in [10].

In the present paper we give sufficient (see Theorem 1, Proposition 3 and Corollary 1) and necessary conditions (see Theorem 2) for a SP to be canonical. In Section 4 we consider non-canonical SPs with two sign variations and we give a lower bound on the number of different orders of the moduli of roots for which these SPs are realizable by HPs.

2. Preliminaries.

Notation 2. (1) We set $\sigma^m(P) = \sigma((-1)^d P(-x))$ and $\sigma^r(P) = \sigma(x^d P(1/x)/P(0))$.

(2) We call *first representation* of a SP the one with signs $(+)$ and/or $(-)$. For a SP in its *second representation* $\Sigma_{m_1,\ldots,m_s}$, if each of its maximal sequences of, say, k consecutive units is replaced by the symbol $[k]$, then one obtains the *third representation* of the SP. E.g. the SP

$$(+,-,-,+,-,+,-,-,-) = \Sigma_{1,2,1,1,1,3}$$

can be represented also in the form $\Sigma_{[1],2,[3],3}$. We call the signs $(+)$ and $(-)$ of the first representation and the numbers m_i of the second one *components* of the SP. The components larger than 1 and the maximal sequences of units in the third representation are called *elements* of the SP.

Remarks 2. (1) The polynomial $x^d P(1/x)$ is the *reverted* of the polynomial P (i.e. read from the back). Its roots are the reciprocals of the roots of P. The roots of $P(-x)$ are the opposite of the roots of P.

(2) The applications

$$\iota_m : \sigma(P) \mapsto \sigma^m(P) \qquad \text{and} \qquad \iota_r : \sigma(P) \mapsto \sigma^r(P)$$

are two commuting involutions. We set

$$\sigma^{mr}(P) := \sigma^m(\sigma^r(P)) = \sigma^r(\sigma^m(P)) =: \sigma^{rm}(P)).$$

For $d \geq 1$, it is always true that $\sigma(P) \neq \sigma^m(P)$ (because their second signs are opposite), but one might have $\sigma(P) = \sigma^r(P)$ or $\sigma(P) = \sigma^{mr}(P)$. Thus the set $\{\sigma(P), \sigma^m(P), \sigma^r(P), \sigma^{mr}(P)\}$ contains either four or two distinct SPs.

(3) The SPs $\sigma(P)$, $\sigma^m(P)$, $\sigma^r(P)$ and $\sigma^{mr}(P)$ are simultaneously canonical or not. With regard to Example 1 one has

$$\sigma^r(\Sigma_{1,m_2}) = \Sigma_{m_2,1} ,$$
$$\sigma^m(\Sigma_{1,m_2}) = \Sigma_{2,[m_2-1]} , \qquad\qquad \sigma^{mr}(\Sigma_{1,m_2}) = \Sigma_{[m_2-1],2} ,$$

$$\sigma^r(\Sigma_{m_1,1,m_3}) = \Sigma_{m_3,1,m_1} ,$$
$$\sigma^m(\Sigma_{m_1,1,m_3}) = \Sigma_{[m_1-1],3,[m_3-1]} , \quad \sigma^{mr}(\Sigma_{m_1,1,m_3}) = \Sigma_{[m_3-1],3,[m_1-1]} ,$$

$$\sigma^r(\Sigma_{1,m_2,1}) = \Sigma_{1,m_2,1} \text{ and} \qquad\qquad \sigma^m(\Sigma_{1,m_2,1}) = \sigma^{mr}(\Sigma_{1,m_2,1}) = \Sigma_{2,[m_2-2],2} .$$

Definition 4. (1) *We say that a SP σ of length $d+1$ is* of type 1 *(notation: $\sigma \in \mathcal{T}_{1,d}$) if either all its even or all its odd positions contain the same sign.*

(2) *We say that a SP σ of length $d + 1$ is* of type 2 *(notation: $\sigma \in \mathcal{T}_{2,d}$) if*

(i) in its second representation the SP σ does not have two consecutive components m_i larger than 1;

(ii) for $2 \leq i \leq s - 1$, one has $m_i \neq 2$ (but $m_1 = 2$ and/or $m_s = 2$ is allowed).

Remark 2. SPs of type 1 are used in the formulation of a result concerning another problem connected with Descartes' rule of signs and formulated for real (not necessarily hyperbolic) polynomials, see Proposition 4 in [3].

Example 2. (1) For $d = 6$ and for the SP

$$\sigma^\dagger := (+,-,-,-,+,-,+) = \Sigma_{1,3,1,1,1} = \Sigma_{[1],3,[3]}$$

one has $\mathcal{T}_{1,6} \ni \sigma^\dagger \in \mathcal{T}_{2,6}$, because there are $(-)$-signs in all odd positions (namely, 1, 3 and 5) and conditions (i) and (ii) from Definition 4 hold true.

(2) The SP σ_0 from Example 1 is neither a type 1 nor a type 2 SP.

(3) One has $\mathcal{T}_{1,7} \not\ni \Sigma_{[1],4,[3]} \in \mathcal{T}_{2,7}$.

(4) The following SPs are of type 1: $\Sigma_{A,[B]}$, $\Sigma_{[A],B}$, $\Sigma_{A,[2B+1],C}$, $\Sigma_{[A],2B+1,[C]}$, A, B, $C \in \mathbb{N}$.

Remark 3. The SPs $\sigma(P)$, $\sigma^m(P)$, $\sigma^r(P)$ and $\sigma^{mr}(P)$ are simultaneously of type 1 or not. E.g. of type 1 are the SPs $\sigma_\bullet := \Sigma_{m_1,[u],m_s}$ for s odd (with $u := d + 1 - m_1 - m_s$),

$$\sigma_\bullet^m = \Sigma_{[m_1-1],u+2,[m_s-1]} \ , \quad \sigma_\bullet^r = \Sigma_{m_s,[u],m_1} \ \text{ and } \ \sigma_\bullet^{mr} = \Sigma_{[m_s-1],u+2,[m_1-1]}.$$

Proposition 2. (1) *One has $\mathcal{T}_{1,d} \subset \mathcal{T}_{2,d}$.*

(2) *One has $\iota_m(\mathcal{T}_{2,d}) = \mathcal{T}_{2,d}$ and $\iota_r(\mathcal{T}_{2,d}) = \mathcal{T}_{2,d}$.*

P r o o f. Part (1). Indeed, if condition (i) of Definition 4 does not hold true, then the SP is of the form

$$(\cdots,+,+,-,-,\cdots) \quad \text{or} \quad (\cdots,-,-,+,+,\cdots)$$

and each of the sequences of signs of even or odd monomials has a sign variation. If condition (ii) does not hold true, then the SP is of the form

$$(\cdots,-,+,+,-,\cdots) \quad \text{or} \quad (\cdots,+,-,-,+,\cdots)$$

and again each of these sequences has at least one sign variation.

Part (2). The inclusion $\iota_r(\mathcal{T}_{2,d}) \subset \mathcal{T}_{2,d}$ follows directly from Definition 4. As ι_r is an involution, this inclusion is an equality. For a SP $\sigma^\triangle \in \mathcal{T}_{2,d}$, its image $\iota_m(\sigma^\triangle)$ is defined by the following rules:

(a) An element $A > 1$ of $\sigma^\triangle$ is replaced by $[A-2]$ if A is not at one of the ends of $\sigma^\triangle$, and by $[A-1]$ if it is.

(b) An element $[B]$ of $\sigma^\triangle$ is replaced by $B+2$ if $[B]$ is not at one of the ends of $\sigma^\triangle$, and by $B+1$ if it is.

One can deduce from rules (a) and (b) that conditions (i) and (ii) of Definition 4 hold true for the SP $\iota_m(\sigma^\triangle)$. Hence $\iota_r(\mathcal{T}_{2,d}) \subset \mathcal{T}_{2,d}$ and as ι_m is an involution, this inclusion is an equality. $\square$

3. Results on canonical sign patterns.

Theorem 1. *Every type 1 SP is canonical.*

P r o o f.   We prove the theorem by induction on d. For $d = 1$, there is nothing to prove. For $d = 2$, one has to consider the SPs $\sigma^\sharp := (+, +, -)$ and $\sigma^\flat := (+, -, -)$. For a HP $P := (x - a)(x + b) = x^2 + g_1 x + g_0$, one has $g_1 = b - a$ which is > 0 if $b > a$ and < 0 for $b < a$ from which for $d = 2$ the theorem follows.

Suppose that $d \geq 3$. Consider a HP P with all roots simple defining a SP σ. In the one-parameter family of polynomials $P_t^* := tP + (1 - t)P'$, $t \in [0, 1]$, every polynomial is hyperbolic with all roots simple and for $t \in (0, 1]$, all roots of P_t^* are nonzero. Moreover, for $t \in (0, 1]$, the polynomial P_t^* defines the SP σ.

For $t = 0$, by inductive assumption, the moduli of the roots of the HP P' define the canonical order. For $t \in (0, 1]$, there is no equality between a modulus of a positive and a modulus of a negative root of P_t^*. Indeed, if P_t^* has roots $\pm\gamma$, $\gamma > 0$, then

(3.1) $$Q_{t,\pm}(\gamma) := P_t^*(\gamma) \pm P_t^*(-\gamma) = 0.$$

This is impossible, because at least one of the two quantities $Q_{t,\pm}(\gamma)$ is a sum of terms of the same sign.

Thus the $d - 1$ largest of the moduli of roots of P_t^* define the same order as the roots of P_0^* (which is the canonical order w.r.t. the SP obtained from σ by deleting its last component). The root of least modulus (for t close to 0) is positive if the last two components of σ are different and negative if they are equal. Thus for $t \in (0, 1]$, the moduli of the roots of P_t^* (hence in particular the ones of P_1^*) define the canonical order. $\square$

Theorem 2. *A canonical SP is a type 2 SP.*

Remark 4. Theorem 2 proposes necessary conditions for a SP to be canonical. It would be interesting to know how far from sufficient these conditions are.

P r o o f.   Suppose that a given SP σ has components $m_j = A > 1$ and $m_{j+1} = B > 1$. The SP $\Sigma_{A,B}$ is not canonical, see part (2) of Example 1. Hence

one can construct two polynomials P and Q defining the SP $\Sigma_{A,B}$ and with different orders of their moduli of roots. To construct two polynomials realizing the SP σ one starts with P and Q and then uses concatenation of polynomials as described in the proof of Proposition 1 and in Remarks 1. At each new concatenation the modulus of the new root is either much smaller or much larger than the moduli of the previously existing roots. Hence the orders of the moduli of the roots of the two polynomials constructed in this way after P and Q remain different.

If the SP σ has a component $m_i = 2$, $2 \leq i \leq s - 1$, then it suffices to consider the case $m_{i-1} = m_{i+1} = 1$. In this case one chooses two polynomials P and Q realizing the SP $\Sigma_{1,2,1}$ with different orders of the moduli of their roots; such polynomials exist, see part (1) of Example 1. After this one again uses the techniques of concatenation to realize the SP σ with two different orders of the moduli of the roots, starting with P and Q respectively. $\square$

Proposition 3. *For $d \geq 5$, the SP $\Sigma_{[1],d-2,[2]}$ is canonical.*

Corollary 1. *For $d \geq 5$, the three SPs $\Sigma_{[2],d-2,[1]} = \sigma^r(\Sigma_{[1],d-2,[2]})$, $\Sigma_{2,[d-4],3} = \sigma^m(\Sigma_{[1],d-2,[2]})$ and $\Sigma_{3,[d-4],2} = \sigma^{mr}(\Sigma_{[1],d-2,[2]})$ are canonical.*

The corollary follows from part (3) of Remarks 2.

Proof of Proposition 3. For $d \geq 5$ odd, the SP is of type 1 and one can apply Theorem 1. For $d = 4$, the SP is not canonical, see Theorem 2. So we assume that $d \geq 6$ (the parity of d is of no importance in the proof). Without loss of generality one can assume that the middle modulus of a positive root of a HP $P := x^d + \sum_{j=0}^{d-1} a_j x^j$ realizing the SP $\Sigma_{[1],d-2,[2]}$ equals 1 (this can be achieved by a linear change of the variable x). So we denote the moduli of positive roots by $0 < \varepsilon < 1 < A$, and by $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{d-3}$ the moduli of negative roots.

Denote by $0 < \delta_1 < \cdots < \delta_{d-3}$ the moduli of negative and by $0 < \varphi < \psi$ the moduli of positive roots of P' (recall that P' defines the SP $\Sigma_{1,d-2,1}$ which is canonical, see part (1) of Example 1). As $\Sigma_{1,d-2,1}$ is canonical, one has $\varphi < \delta_1$, and by Rolle's theorem, $\gamma_j < \delta_{j+1} < \gamma_{j+1}$, $j = 1, \ldots, d - 4$. For δ_1, one has $0 < \delta_1 < \gamma_1$. Thus

$$\varepsilon < \varphi < \delta_1 < \gamma_1.$$

Denote by $0 < \eta_1 < \cdots < \eta_{d-3}$ the moduli of negative and by $0 < \lambda < \mu$ the moduli of positive roots of the HP $P^\dagger := xP' - dP = -a_{d-1}x^{d-1} - 2a_{d-2}x^{d-2} - \cdots - da_0$. The latter defines the SP $\Sigma_{d-2,[2]}$ which is canonical, see part (1) of Example 1. The positive roots of P and $P^\dagger$ interlace, and so do their negative roots as well; we will see below that this is not true about all the roots of P and $P^\dagger$. The leading coefficient of $P^\dagger$ is positive, so the limits at $+\infty$ of P and $P^\dagger$ equal $+\infty$. Their limits at $-\infty$ are opposite. The leftmost root of P equals $-\gamma_{d-3}$. One has $P^\dagger(-\gamma_{d-3}) = -\gamma_{d-3}P'(-\gamma_{d-3})$. Hence

$$\text{either} \quad \lim_{x \to -\infty} P(x) = -\infty , \quad P'(-\gamma_{d-3}) > 0 ,$$

$$\lim_{x \to -\infty} P^\dagger(x) = +\infty , \quad P^\dagger(-\gamma_{d-3}) < 0$$

$$\text{or} \quad \lim_{x \to -\infty} P(x) = +\infty , \quad P'(-\gamma_{d-3}) < 0 ,$$

$$\lim_{x \to -\infty} P^\dagger(x) = -\infty , \quad P^\dagger(-\gamma_{d-3}) > 0 .$$

In both cases the leftmost root $-\eta_{d-3}$ of $P^\dagger$ is $< -\gamma_{d-3}$. By Rolle's theorem and using the fact that the SP $\Sigma_{d-2,[2]}$ is canonical,

$$0 < \lambda < 1 < \mu < \eta_1 < \gamma_2 < \eta_2.$$

One can show that $-\eta_1 < -\gamma_1 < \varepsilon < \lambda$ which means that the interlacing of the roots of P and $P^\dagger$ is interrupted when the variable x passes through 0. The condition $a_{d-1} < 0$ reads:

$$(3.2) \qquad A + 1 + \varepsilon - \sum_{j=1}^{d-3} \gamma_j > 0.$$

As $\gamma_1 > \varepsilon$ and $\gamma_2 > 1 > \varepsilon$, condition (3.2) is possible only if $A > \gamma_{d-3}$. Thus $\varepsilon < \gamma_1 < \cdots < \gamma_{d-3} < A$ and to prove the proposition there remains to show that $1 < \gamma_1$. Set

$$\sigma_1 := \sum_{j=3}^{d-3} 1/\gamma_j, \qquad \sigma_2 := \sum_{3 \le i < j \le d-3} 1/(\gamma_i \gamma_j),$$

$$B := \frac{1}{A} + 1 + \frac{1}{\varepsilon} \quad \text{and} \quad C := \frac{1}{A\varepsilon} + \frac{1}{A} + \frac{1}{\varepsilon}.$$

The conditions $a_0 < 0$, $a_1 > 0$ and $a_2 < 0$ imply

$$B \; - \; \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sigma_1\right) \qquad > \quad 0 \qquad \text{and}$$

(3.3) $\qquad \Phi \; := \; \Lambda \; - \; B\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sigma_1\right) \qquad > \quad 0 \,, \quad \text{where}$

$$\Lambda \; := \; C \; + \; \frac{1}{\gamma_1\gamma_2} + \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\sigma_1 + \sigma_2.$$

Suppose that $\gamma_1 \leq 1$. Then the following inequalities hold true:

(3.4) $$\frac{1}{A\varepsilon} - \frac{1}{\gamma_2\varepsilon} < 0,$$

because $A > \gamma_2$,

(3.5) $$-\frac{1}{\gamma_1}\left(\frac{1}{A} + \frac{1}{\varepsilon}\right) + \frac{1}{A} + \frac{1}{\varepsilon} \leq 0,$$

(3.6) $$-B\sigma_1 + \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\sigma_1 + \sigma_2 < 0,$$

(because $-B\sigma_1 < -(1/\gamma_1 + 1/\gamma_2 + \sigma_1)\sigma_1$, see the first of inequalities (3.3), and one has $\sigma_2 < (\sigma_1)^2$) and as $\gamma_2 > 1$,

(3.7) $$-\frac{1}{\gamma_1} + \frac{1}{\gamma_1\gamma_2} \; < \; 0.$$

The sum of the left-hand sides of inequalities (3.4), (3.5), (3.6) and (3.7) equals

$$\Lambda - \left(\frac{1}{\gamma_2\varepsilon} + \frac{1}{\gamma_1}\left(\frac{1}{A} + \frac{1}{\varepsilon}\right) + B\sigma_1 + \frac{1}{\gamma_1}\right) = \Lambda - B\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sigma_1\right) + \frac{1}{\gamma_2} + \frac{1}{\gamma_2 A}.$$

Thus $\Lambda - B\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sigma_1\right) + \frac{1}{\gamma_2} + \frac{1}{\gamma_2 A} < 0$ which contradicts the second of inequalities (3.3). $\square$

4. On non-canonical sign patterns. The present section deals with SPs with two sign changes, i.e. with $s = 3$, see Notation 1. For $m_1 \geq 2$, $m_2 \geq 2$, $m_3 \geq 2$, such a SP is not canonical, see Theorem 2.

Notation 3. We set $m := m_1$, $n := m_2$, $q := m_3$ and we denote by $0 < \beta < \alpha$ the positive and by $-\gamma_{d-2} < \cdots < -\gamma_1 < 0$ the negative roots of a degree d HP P realizing the SP $\Sigma_{m,n,q}$. By m^*, n^*, q^* we denote the numbers of negative roots of modulus larger than α, between β and α and smaller than β respectively; hence $m^* + n^* + q^* = d - 2$. By $\tau_1 \geq 0$, $\tau_2 \geq 0$, $\delta > 0$, $\ell > 0$ and $r \geq 2$, we denote integers, where $d = \delta + \tau_1 + \tau_2$.

We remind that the canonical order of the roots corresponds to the case $m^* = m - 1$, $n^* = n - 1$, $q^* = q - 1$, see Definition 2.

Theorem 3. (1) *For*

$$r^2 < \delta < (r+1)^2, \quad \delta - r \in 2\mathbb{Z} + 1,$$

$$m \geq (\delta - r + 1)/2, \quad q \geq (\delta - r + 1)/2 \quad \text{and} \quad n = r,$$

the SP $\Sigma_{m,n,q}$ is realizable by HPs with all possible values of m^*, n^*, q^* such that $m^* \geq \tau_1 := m - (\delta - r + 1)/2$ and $q^* \geq \tau_2 := q - (\delta - r + 1)/2$.

(2) *For*

$$r^2 < \delta < (r+1)^2, \quad \delta - r \in 2\mathbb{Z},$$

$$m \geq (\delta - r)/2, \quad q \geq (\delta - r)/2 \quad \text{and} \quad n = r + 1,$$

the SP $\Sigma_{m,n,q}$ is realizable by HPs with all possible values of m^*, n^*, q^* such that $m^* \geq \tau_1 := m - (\delta - r)/2$ and $q^* \geq \tau_2 := q - (\delta - r)/2$.

(3) *For $\delta = r^2$, the SP*

$$(\tau_1 + r(r-1)/2 + 1, r, \tau_2 + r(r-1)/2) \ (\text{resp.} \ (\tau_1 + r(r-1)/2, r, \tau_2 + r(r-1)/2 + 1))$$

is realizable by HPs with all possible values of m^*, n^*, q^* such that $m^* \geq \tau_1 + 1$ and $q^* \geq \tau_2$ (resp. $m^* \geq \tau_1$ and $q^* \geq \tau_2 + 1$).

Remarks 3. (1) Consider the case $\tau_1 = \tau_2 = 0$. Hence $d = \delta$ and all possible orders of the moduli of the $d - 2$ negative and 2 positive roots are

realizable. The number of these orders is

$$\sum_{k=0}^{d-2}\sum_{j=0}^{d-2-k} 1 = \sum_{k=0}^{d-2}(d-1-k) = d(d-1)/2$$

(here k and j are the numbers of moduli of negative roots larger than α and between β and α respectively). At the same time $d \sim r^2$, i.e. $d \sim n^2$. Thus the theorem guarantees the possibility to realize the SP $\Sigma_{m,n,q}$ by $\sim n^4/2$ HPs with different orders of the moduli of their roots when m and q are (almost) equal. The latter condition is essential – for $q = 1$, the number of different orders is $\sim 2n$, see Theorem 4 in [10].

(2) The theorem gives only sufficient conditions for realizability of certain SPs with two sign changes by HPs with different orders of the moduli of their roots. It would be interesting to obtain necessary conditions as well.

In order to prove the theorem we need a technical lemma.

Notation 4. We set $P_\ell(x) := (x-1)^2(x+1)^\ell$, $\ell \geq 2$. This polynomial contains either 0 or 2 vanishing coefficients, see Lemma 1. By $\Sigma(\ell)$ we denote its SP which, in the case when there are 2 vanishing coefficients, we represent in the form $(v, 0, n, 0, w)$. This means that $\Sigma(\ell)$ begins with $v = m - 1$ signs $(+)$ followed by a zero followed by $n = n(\Sigma(\ell))$ signs $(-)$ followed by a zero followed by $w = q - 1$ signs $(+)$. If there are no vanishing coefficients, then we write $\Sigma(\ell) = (v, n, w)$ in which case $v = m$ and $w = q$.

Lemma 1. (1) *For $r^2 - 2 < \ell < (r+1)^2 - 2$ and $\ell - r \in 2\mathbb{Z} + 1$,*

$$\Sigma(\ell) = ((\ell - r + 3)/2, r, (\ell - r + 3)/2)),$$

so $n(\Sigma(\ell)) = r$.

(2) *For $r^2 - 2 < \ell < (r+1)^2 - 2$ and $\ell - r \in 2\mathbb{Z}$,*

$$\Sigma(\ell) = ((\ell - r + 2)/2, r + 1, (\ell - r + 2)/2),$$

so $n(\Sigma(\ell)) = r + 1$.

(3) *For $\ell = r^2 - 2$, the SP $\Sigma(\ell)$ equals*

$$\Sigma(\ell) = (r(r-1)/2, 0, r-1, 0, r(r-1)/2).$$

Hence $n(\Sigma(\ell)) = r - 1$.

Proof. Clearly $P_\ell = \sum_{j=0}^{\ell+2} c_j x^j$, where $c_j = \binom{\ell}{j} - 2\binom{\ell}{j-1} + \binom{\ell}{j-2}$.
The condition $c_j = 0$ is equivalent to

$$4j^2 - 4(\ell+2)j + (\ell+1)(\ell+2) = 0$$

which yields

$$j = j_\pm(\ell) := (\ell + 2 \pm \sqrt{\ell+2})/2.$$

For $\ell = r^2 - 2$, one gets $j = (r^2 \pm r)/2$ from which part (3) follows (both numbers $(r^2 \pm r)/2$ are natural).

When ℓ is not of the form $r^2 - 2$ the condition $c_j = 0$ does not provide a natural solution. Hence no coefficient of P_ℓ vanishes. The formula expressing $j_\pm(\ell)$ implies that $c_j > 0$ for $j \leq [(\ell + 2 - (r+1))/2]$ while $c_{[(\ell+2-(r+1))/2]+1} < 0$; here [.] stands for the integer part of. If ℓ and r are of different parity, then

$$[(\ell + 2 - (r+1))/2] = (\ell - r + 1)/2$$

which proves part (1). If ℓ and r are of one and the same parity, then

$$[(\ell + 2 - (r+1))/2] = (\ell - r)/2$$

which proves part (2). □

Proof of Theorem 3. To prove parts (1) and (2) of Theorem 3 we use parts (1) and (2) of Lemma 1 respectively. We consider first the case $\tau_1 = \tau_2 = 0$. In this case the conditions

$$m \geq (\delta - r + 1)/2, \quad q \geq (\delta - r + 1)/2 \quad \text{and} \quad n = r \qquad \text{from part (2) or}$$

$$m \geq (\delta - r)/2, \qquad q \geq (\delta - r)/2 \qquad \text{and} \quad n = r + 1 \quad \text{from part (3)}$$

are possible only if

$$m = (\delta - r + 1)/2, \quad q = (\delta - r + 1)/2 \quad \text{and} \quad n = r \qquad \text{or}$$

$$m = (\delta - r)/2, \qquad q = (\delta - r)/2 \qquad \text{and} \quad n = r + 1$$

respectively, because $m + n + q = \delta + 1$.

Set $d = \delta := \ell + 2$. We deform the polynomial P_ℓ corresponding to part (1) or (2) of Lemma 1 so that the moduli of the roots are all distinct and define

any possible order (fixed in advance) on the positive half-axis. The positive roots $\beta < \alpha$ of the deformed polynomial (denoted by $\tilde{P}_\ell$) remain close to 1 and the ℓ negative roots remain close to -1. Hence the signs of the coefficients of $\tilde{P}_\ell$ are the same as the signs of the coefficients of P_ℓ and

$$\sigma(\tilde{P}_\ell) = ((\delta - r + 1)/2, \ r, (\delta - r + 1)/2) \quad \text{in the case of part (2) or}$$

$$\sigma(\tilde{P}_\ell) = ((\delta - r)/2, r + 1, (\delta - r)/2) \qquad \text{in the case of part (3).}$$

This proves the theorem for $\tau_1 = \tau_2 = 0$.

In the general case, i.e. for $\tau_1 \geq 0$ and $\tau_2 \geq 0$, one first constructs the polynomial $\tilde{P}_\ell$ as above. Then one performs τ_1 concatenations of $\tilde{P}_\ell$ with polynomials of the form $1 + \varepsilon_j x$, $j = 1, \ldots, \tau_1$, as explained in part (2) of Remarks 1, where

$$0 < \varepsilon_{\tau_1} \ll \varepsilon_{\tau_1-1} \ll \cdots \ll \varepsilon_1 \ll 1.$$

This adds τ_1 negative roots whose moduli are larger than α. After this one performs τ_2 concatenations, see part (1) of Remarks 1, with polynomials of the form $x + \varepsilon_j$, $j = \tau_1 + 1, \ldots, \tau_1 + \tau_2$, where

$$0 < \varepsilon_{\tau_1+\tau_2} \ll \varepsilon_{\tau_1+\tau_2-1} \ll \cdots \ll \varepsilon_{\tau_1+1} \ll \varepsilon_{\tau_1}.$$

This adds τ_2 negative roots whose moduli are smaller than β.

Part (3). Consider first the case $\tau_1 = \tau_2 = 0$. We use Lemma 1 with $\ell = r^2 - 3$. Hence one can apply part (2) of Lemma 1 with $r - 1$ substituted for r (because $\ell - (r - 1) \in 2\mathbb{Z}$). This implies that the polynomial P_{r^2-3} realizes the SP $(r(r-1)/2, r, r(r-1)/2)$. Setting $P_{r^2-3} := \sum_{j=0}^{d} a_j x^j$ one deduces that

$$a_{r(r-1)/2} < 0, \qquad a_{r(r-1)/2-1} > 0, \qquad a_{r(r-1)/2} + a_{r(r-1)/2-1} = 0,$$

$$a_{r(r+1)/2+1} > 0, \qquad a_{r(r+1)/2} < 0, \qquad a_{r(r+1)/2+1} + a_{r(r+1)/2} = 0.$$

The two equalities to 0 result from the polynomial $P_{r^2-2} = (x+1)P_{r^2-3}$ having vanishing coefficients of $x^{r(r-1)/2}$ and $x^{r(r+1)/2}$, see part (3) of Lemma 1. Hence for the SPs defined by the polynomials $P_\pm := (x + 1 \pm \varepsilon)P_{r^2-3}$, $\varepsilon > 0$, one has

$$\sigma(P_+) = (r(r-1)/2+1, r, r(r-1)/2) \ \text{ and } \ \sigma(P_-) = (r(r-1)/2, r, r(r-1)/2+1).$$

Then one perturbs the roots of P_{r^2-3} (the perturbed negative roots must keep away from the root $-1 \mp \varepsilon$ of $P_\pm$). In the case of P_+ (resp. P_-) the largest (resp. the smallest) of the moduli of perturbed roots is the one of the negative root $-1-\varepsilon$ (resp. $-1+\varepsilon$) and the order of the remaining $d-3$ negative and 2 positive roots can be arbitrary. This proves part (3) for $\tau_1 = \tau_2 = 0$. In the general case, i.e. for $\tau_1 \geq 0$ and $\tau_2 \geq 0$, the proof is finished in the same way as for parts (1) and (2). $\square$

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