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QUASI-MINIMAL LORENTZ SURFACES IN PSEUDO-EUCLIDEAN 4-SPACE WITH NEUTRAL METRIC*

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Abstract. A Lorentz surface in the pseudo-Euclidean 4-space with neutral metric is called quasi-minimal if its mean curvature vector is lightlike at each point. We prove that any quasi-minimal Lorentz surface whose Gauss curvature K and normal curvature $\varkappa$ satisfy the condition $K^2 - \varkappa^2 \neq 0$ at every point is determined (up to a rigid motion) by five geometric functions satisfying a system of four partial differential equations.

1. Introduction. In the present paper we study Lorentz surfaces in the pseudo-Euclidean space with neutral metric $\mathbb{E}^4_2$. A submanifold in a Riemannian or pseudo-Riemannian manifold is called *minimal* if its mean curvature vector vanishes identically. The study of minimal surfaces is one of the main topics

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in differential geometry both in Euclidean and pseudo-Euclidean spaces, since pseudo-Riemannian geometry has many important applications in Physics. In [4] B.-Y. Chen classified minimal Lorentz surfaces in $\mathbb{C}^2_1$. Several classification results for minimal Lorentz surfaces in indefinite space forms are obtained in [6].

A natural extension of minimal surfaces is the class of quasi-minimal surfaces. A surface in a pseudo-Riemannian manifold is called *quasi-minimal* if its mean curvature vector is lightlike at each point of the surface [22]. When the ambient space is the Lorentz-Minkowski space $\mathbb{E}^4_1$, the quasi-minimal submanifolds are also called *marginally trapped* – a notion borrowed from General Relativity. The concept of trapped surfaces was first introduced by Roger Penrose [21] in connection with the theory of cosmic black holes. In Physics, a surface in a 4-dimensional spacetime is called *marginally trapped* if it is closed, embedded, spacelike and its mean curvature vector is lightlike at each point of the surface. In the mathematical literature, it is customary to call a surface marginally trapped if its mean curvature vector H is lightlike at each point, removing the other hypotheses, i.e. the surface does not need to be closed or embedded. In the Minkowski space, marginally trapped surfaces satisfying some extra conditions have recently been intensively studied in connection with the rapid development of the theory of black holes in Physics. For example, marginally trapped surfaces with positive relative nullity were classified by B.-Y. Chen and J. Van der Veken in [11]. They also classified marginally trapped surfaces with parallel mean curvature vector in Lorentz space forms [12]. Marginally trapped surfaces in Lorentz-Minkowski 4-space which are invariant under boost transformations were studied by S. Haesen and M. Ortega in [16], and marginally trapped surfaces which are invariant under spacelike rotations were classified in [17]. A classification of marginally trapped surfaces which are invariant under the group of screw rotations (a group of Lorenz rotations with an invariant lightlike direction) is obtained in [18]. A classification of marginally trapped surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_1$ is obtained independently in [19] and [23]. In [14] G. Ganchev and the second author studied marginally trapped surfaces in $\mathbb{E}^4_1$ in terms of a geometrically determined moving frame field. They proved a fundamental theorem of Bonnet-type stating that each marginally trapped surface in $\mathbb{R}^4_1$ is determined up to a motion by seven invariant functions.

Quasi-minimal surfaces in the pseudo-Euclidean 4-space $\mathbb{E}^4_2$ have also been studied actively in the last few years. In [2] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}^4_2$ and gave a complete classification of biharmonic quasi-minimal surfaces. Quasi-minimal surfaces with constant Gauss curvature in $\mathbb{E}^4_2$

were classified in [3, 10]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [7] and [9]. A classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ is obtained in [8], quasi-minimal rotational surfaces of elliptic, hyperbolic or parabolic type are classified in [15]. The complete classification of quasi-minimal Lorentz surfaces in $\mathbb{E}^4_2$ with pointwise 1-type Gauss map is given in [20]. The quasi-minimal surfaces in the 4-dimensional de Sitter space with 1-type Gauss map are studied in [24]. Recently, quasi-minimal surfaces with positive relative nullity in the pseudo-Euclidean space $\mathbb{E}^4_2$ and the pseudo-sphere $\mathbb{S}^4_2$ were classified [13].

In the present paper we study quasi-minimal surfaces in $\mathbb{E}^4_2$. Introducing a geometrically determined moving frame field at each point of the surface we obtain five geometric functions and prove that any quasi-minimal Lorentz surface whose Gauss curvature K and normal curvature $\varkappa$ satisfy the condition $K^2 - \varkappa^2 \neq 0$ at every point is determined (up to a rigid motion) by these five functions satisfying a system of four partial differential equations (Theorem 3.2).

2. Preliminaries. Let $\mathbb{E}^4_2$ be the pseudo-Euclidean 4-space endowed with the canonical pseudo-Euclidean metric of index 2 given in local coordinates by

$$g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2,$$

where x_1, x_2, x_3, x_4 are rectangular coordinates of the points of $\mathbb{E}^4_2$. As usual, we denote by $\langle \, , \rangle$ the indefinite inner scalar product associated with g_0. Since g_0 is an indefinite metric, a vector $v \in \mathbb{E}^4_2$ can have one of the three casual characters: *spacelike*, if $\langle v, v \rangle > 0$ or $v = 0$, *timelike* if $\langle v, v \rangle < 0$, and *lightlike* if $\langle v, v \rangle = 0$ and $v \neq 0$. This terminology is inspired by General Relativity and the Minkowski 4-space $\mathbb{E}^4_1$.

A surface M^2_1 in $\mathbb{E}^4_2$ is called *Lorentz* if the induced metric g on M^2_1 is Lorentzian. So, at each point $p \in M^2_1$ we have the following decomposition

$$\mathbb{E}^4_2 = T_pM^2_1 \oplus N_pM^2_1$$

with the property that the restriction of the metric onto the tangent space $T_pM^2_1$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_pM^2_1$ is of signature $(1, 1)$.

We denote by ∇ and $\widetilde{\nabla}$ the Levi Civita connections of M^2_1 and $\mathbb{E}^4_2$, respectively. For vector fields X, Y tangent to M^2_1 and a vector field ξ normal to M^2_1, the formulas of Gauss and Weingarten, giving a decomposition of the vector fields $\widetilde{\nabla}_X Y$ and $\widetilde{\nabla}_X \xi$ into tangent and normal components, are given respectively by:

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y);\\ \widetilde{\nabla}_X \xi &= -A_\xi X + D_X \xi.\end{aligned}$$

These formulas define the second fundamental form σ, the normal connection D, and the shape operator A_ξ with respect to ξ. In general, A_ξ is not diagonalizable. The shape operator and the second fundamental form are related by the formula

$$\langle \sigma(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

for X, Y tangent to M^2_1 and ξ normal to M^2_1.

At a given point $p \in M^2_1$, the *first normal space* of M^2_1 in $\mathbb{E}^4_2$, denoted by $\operatorname{Im}\sigma_{\rm p}$, is the subspace given by

$$\operatorname{Im}\sigma_p = \operatorname{span}\{\sigma(X,Y) : X, Y \in T_p M^2_1\}.$$

The mean curvature vector field H of M^2_1 in $\mathbb{E}^4_2$ is defined as $H = \dfrac{1}{2}\operatorname{tr}\sigma$. The surface is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. The surface is called *quasi-minimal* if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$. Obviously, quasi-minimal surfaces are always non-minimal.

3. Fundamental theorem for quasi-minimal surfaces in $\mathbb{E}^4_2$. It is known that for each Lorentz surface M^2_1 in $\mathbb{E}^4_2$ locally there exists a coordinate system (u,v) such that the metric tensor is given by

$$g = -f^2(u,v)(du \otimes dv + dv \otimes du), \tag{1}$$

fore some positive function $f(u,v)$ [5]. Let $z = z(u,v), (u,v) \in \mathcal{D}\,(\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on M^2_1 such that (1) holds. Then

$$\langle z_u, z_u \rangle = 0; \quad \langle z_u, z_v \rangle = -f^2(u,v); \quad \langle z_v, z_v \rangle = 0.$$

Denoting $x = \dfrac{z_u}{f}$ and $y = \dfrac{z_v}{f}$, we obtain a pseudo-orthonormal frame field $\{x, y\}$ of the tangent bundle of M^2_1 such that $\langle x, x\rangle = 0$, $\langle x, y\rangle = -1$, $\langle y, y\rangle = 0$. Hence, the mean curvature vector field H is given by

$$H = -\sigma(x, y).$$

Now, let M^2_1 be quasi-minimal, i.e. its mean curvature vector is lightlike at each point. Then there exists a pseudo-orthonormal frame field $\{n_1, n_2\}$ of the normal bundle such that $n_1 = -H$, $\langle n_1, n_1\rangle = 0$, $\langle n_1, n_2\rangle = -1$, $\langle n_2, n_2\rangle = 0$.

By a direct computation we obtain the following derivative formulas:

$$\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x + \lambda_1 n_1 + \mu_1 n_2, & \widetilde{\nabla}_y x = -\gamma_2 x + n_1, \\
\widetilde{\nabla}_x y = -\gamma_1 y + n_1, & \widetilde{\nabla}_y y = \gamma_2 y + \lambda_2 n_1 + \mu_2 n_2, \\
\widetilde{\nabla}_x n_1 = -\mu_1 y + \beta_1 n_1, & \widetilde{\nabla}_y n_1 = -\mu_2 x + \beta_2 n_1, \\
\widetilde{\nabla}_x n_2 = -x - \lambda_1 y - \beta_1 n_2, & \widetilde{\nabla}_y n_2 = -\lambda_2 x - y - \beta_2 n_2,
\end{array} \tag{2}$$

for some smooth functions $\lambda_1, \lambda_2, \mu_1, \mu_2$, β_1, β_2, where $\gamma_1 = \dfrac{f_u}{f^2}$ and $\gamma_2 = \dfrac{f_v}{f^2}$.

Note that the pseudo-orthonormal frame field $\{x, y, n_1, n_2\}$ is geometrically determined as follows: x and y are the two lightlike directions in the tangent bundle of the surface, $-n_1$ is the mean curvature vector field, n_2 is uniquely determined by the conditions $\langle n_2, n_2\rangle = 0$, $\langle n_1, n_2\rangle = -1$. We call this pseudo-orthonormal frame field a *geometric frame field* of the surface. The functions $\lambda_1, \lambda_2, \mu_1, \mu_2$, β_1, β_1 are determined by the geometric frame field as follows:

$$\lambda_1 = -\langle \widetilde{\nabla}_x x, n_2\rangle, \quad \lambda_2 = -\langle \widetilde{\nabla}_y y, n_2\rangle,$$

$$\mu_1 = -\langle \widetilde{\nabla}_x x, n_1\rangle, \quad \mu_2 = -\langle \widetilde{\nabla}_y y, n_1\rangle,$$

$$\beta_1 = -\langle \widetilde{\nabla}_x n_1, n_2\rangle, \quad \beta_2 = -\langle \widetilde{\nabla}_y n_1, n_2\rangle.$$

We call these functions *geometric functions* of the surface since they are determined by the geometric frame field $\{x, y, n_1, n_2\}$.

It follows from (2) that the Gaussian curvature K and the normal curvature $\varkappa$ of M^2_1 are expressed by

$$\begin{aligned} K &= -R(x,y,y,x) = x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2, \\ \varkappa &= -R^D(x,y,n_1,n_2) = x(\beta_2) - y(\beta_1) + \gamma_1\beta_2 - \gamma_2\beta_1, \end{aligned} \tag{3}$$

where R and R^D are the curvature tensors associated with the connections ∇ and D, respectively.

Remark 1. Note that, if M^2_1 is a quasi-minimal surface with parallel mean curvature vector field, i.e. $D_xH = 0, D_yH = 0$, then M^2_1 has flat normal connection ($\varkappa = 0$).

Using the equations of Gauss and Codazzi, from formulas (2) we obtain the following integrability conditions:

$$\begin{aligned} & x(\lambda_2) = -\lambda_2\beta_1 - 2\lambda_2\gamma_1 + \beta_2, \\ & x(\mu_2) = \mu_2\beta_1 - 2\mu_2\gamma_1, \\ & y(\lambda_1) = -\lambda_1\beta_2 - 2\lambda_1\gamma_2 + \beta_1, \\ & y(\mu_1) = \mu_1\beta_2 - 2\mu_1\gamma_2, \\ & x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 = \lambda_1\mu_2 + \mu_1\lambda_2, \\ & x(\beta_2) - y(\beta_1) - \beta_1\gamma_2 + \beta_2\gamma_1 = \lambda_1\mu_2 - \mu_1\lambda_2. \end{aligned} \tag{4}$$

Equalities (3) and (4) imply that the Gauss curvature K and the normal curvature $\varkappa$ are expressed as follows:

$$\begin{aligned} K &= \lambda_1\mu_2 + \mu_1\lambda_2, \\ \varkappa &= \lambda_1\mu_2 - \mu_1\lambda_2. \end{aligned} \tag{5}$$

Remark 2. If $\mu_1 = \mu_2 = 0$, then the first equality of (5) implies $K = 0$, i.e. M^2_1 is a flat surface. Moreover, M^2_1 has flat normal connection. In [2, Theorem 4.1], B.-Y. Chen obtained a complete classification of flat quasi-minimal surfaces in $\mathbb{E}^4_2$. Following the proof of B.-Y. Chen, one can see that a quasi-minimal surface in $\mathbb{E}^4_2$ satisfying $\mu_1 = \mu_2 = 0$ is congruent to a surface parametrized by

$$z(u,v) = \left(\theta(u,v), \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}, \theta(u,v)\right)$$

for a smooth function θ.

The flat quasi-minimal surfaces can be characterized in terms of the so called null allied mean curvature vector field. The *allied vector field* of a normal vector field ξ of an n-dimensional submanifold M^n of $(n+m)$-dimensional Riemannian manifold $\widetilde{M}^{n+m}$ is defined by B.-Y. Chen [1] by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^{m} \{\mathrm{tr}(A_1 \circ A_k)\} \xi_k,$$

where $\{\xi_1 = \dfrac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is a local orthonormal frame of the normal bundle of M^n, and $A_i = A_{\xi_i},\ i = 1, \dots, m$ is the shape operator with respect to ξ_i. The allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M^n in $\widetilde{M}^{n+m}$. The notion of allied mean curvature vector field is extended by S. Haesen and M. Ortega [18] to the case when the normal space is a two-dimensional Lorenz space and the mean curvature vector field is lightlike as follows. Denote by $\{H, H^\perp\}$ a pseudo-orthonormal basis of the normal space such that $\langle H, H \rangle = 0;\ \langle H^\perp, H^\perp \rangle = 0;\ \langle H, H^\perp \rangle = -1$. The *null allied mean curvature vector field* is defined as

$$a(H) = \frac{1}{2}\,\mathrm{tr}(A_H \circ A_{H^\perp})\, H^\perp. \tag{6}$$

Now, if M^2_1 is a quasi-minimal surface, then using equalities (2) we get

$$A_H = A_{-n_1} = \begin{pmatrix} 0 & -\mu_1 \\ -\mu_2 & 0 \end{pmatrix}; \qquad A_{H^\perp} = A_{-n_2} = \begin{pmatrix} -1 & -\lambda_1 \\ -\lambda_2 & -1 \end{pmatrix}.$$

Hence, formulas (6) and (5) imply that the null allied mean curvature vector field of M^2_1 is expressed as follows:

$$a(H) = \frac{K}{2}\, n_2.$$

Thus we obtain the following result.

Proposition 3.1. *Let M^2_1 be a quasi-minimal surface in $\mathbb{R}^4_2$. Then M^2_1 is a flat surface if and only if M^2_1 has vanishing null allied mean curvature vector field.*

The same proposition holds for marginally trapped surfaces in $\mathbb{R}^4_1$ [14].

Now, let us consider again equalities (4). In the case $\lambda_1\mu_1\lambda_2\mu_2 \neq 0$ we can express the functions β_1 and β_2 as follows:

$$\beta_1 = x\left(\ln|\mu_2 f^2|\right); \quad \beta_2 = y\left(\ln|\mu_1 f^2|\right).$$

Then, the first and the third equality of (4) imply that the functions $\lambda_1, \lambda_2, \mu_1, \mu_2$ satisfy the conditions

$$\begin{aligned} x\left(\ln|\lambda_2\mu_2 f^4|\right) &= \frac{1}{\lambda_2}\, y\left(\ln|\mu_1 f^2|\right); \\ y\left(\ln|\lambda_1\mu_1 f^4|\right) &= \frac{1}{\lambda_1}\, x\left(\ln|\mu_2 f^2|\right). \end{aligned}$$

Using that $x = \dfrac{z_u}{f}$, $y = \dfrac{z_v}{f}$, from the last two equalities in (4) we get

$$\begin{aligned} &2ff_{uv} - 2f_uf_v = f^4(\lambda_1\mu_2 + \mu_1\lambda_2); \\ &\left(\ln|\frac{\mu_1}{\mu_2}|\right)_{uv} = f^2(\lambda_1\mu_2 - \mu_1\lambda_2). \end{aligned}$$

In the local theory of surfaces in Euclidean space a statement of significant importance is a theorem of Bonnet-type giving the natural conditions under which the surface is determined up to a motion. A theorem of this type was proved for surfaces with flat normal connection in Euclidean space by B.-Y. Chen in [1]. In [14] the fundamental existence and uniqueness theorem for marginally trapped surfaces in $\mathbb{E}^4_1$ is proved in terms of seven invariant functions. Now we shall prove a similar theorem for the class of quasi-minimal surfaces in $\mathbb{E}^4_2$.

Theorem 3.2. *Let* $\lambda_1(u,v)$, $\mu_1(u,v)$, $\lambda_2(u,v), \mu_2(u,v)$, $f(u,v)$ *be smooth* *functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$
\begin{array}{ll}
 & f > 0; \qquad \lambda_1\mu_1\lambda_2\mu_2 \neq 0, \quad (u,v) \in \mathcal{D}; \\
 & \left(\ln|\lambda_1\mu_1 f^4|\right)_v = \dfrac{1}{\lambda_1}\left(\ln|\mu_2 f^2|\right)_u; \\
(7) & \left(\ln|\lambda_2\mu_2 f^4|\right)_u = \dfrac{1}{\lambda_2}\left(\ln|\mu_1 f^2|\right)_v; \\
 & 2ff_{uv} - 2f_u f_v = f^4(\lambda_1\mu_2 + \mu_1\lambda_2); \\
 & \left(\ln\left|\dfrac{\mu_1}{\mu_2}\right|\right)_{uv} = f^2(\lambda_1\mu_2 - \mu_1\lambda_2).
\end{array}
$$

Let $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ be a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_2$. Then, there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique quasi-minimal Lorentz surface $M^2_1 : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ passing through the point p_0.

P r o o f. We introduce functions β_1 and β_2 by the formulas:

$$\beta_1 = \frac{1}{f}\left(\ln|\mu_2 f^2|\right)_u; \quad \beta_2 = \frac{1}{f}\left(\ln|\mu_1 f^2|\right)_v.$$

Let us consider the following system of partial differential equations for the unknown vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$ in $\mathbb{R}^4_2$:

$$
(8) \quad
\begin{array}{ll}
x_u = (\ln f)_u\, x + f\lambda_1\, n_1 + f\mu_1\, n_2 & x_v = -(\ln f)_v\, x + f\, n_1 \\
y_u = -(\ln f)_u\, y + f\, n_1 & y_v = (\ln f)_v\, y + f\lambda_2\, n_1 + f\mu_2\, n_2 \\
(n_1)_u = -f\mu_1\, y + f\beta_1\, n_1 & (n_1)_v = -f\mu_2\, x + f\beta_2\, n_1 \\
(n_2)_u = -f\, x - f\lambda_1\, y - f\beta_1\, n_2 & (n_2)_v = -f\lambda_2\, x - f\, y - f\beta_2\, n_2
\end{array}
$$

We denote

$$
Z = \begin{pmatrix} x \\ y \\ n_1 \\ n_2 \end{pmatrix}; \qquad
A = \begin{pmatrix}
(\ln f)_u & 0 & f\lambda_1 & f\mu_1 \\
0 & -(\ln f)_u & f & 0 \\
0 & -f\mu_1 & f\beta_1 & 0 \\
-f & -f\lambda_1 & 0 & -f\beta_1
\end{pmatrix};
$$

$$B = \begin{pmatrix} -(\ln f)_v & 0 & f & 0 \\ 0 & (\ln f)_v & f\lambda_2 & f\mu_2 \\ -f\mu_2 & 0 & f\beta_2 & 0 \\ -f\lambda_2 & -f & 0 & -f\beta_2 \end{pmatrix}.$$

Using matrices A and B we can rewrite system (8) in the form:

$$\begin{aligned} Z_u &= A\,Z \\ Z_v &= B\,Z \end{aligned} \tag{9}$$

The integrability conditions of system (9) are

$$Z_{uv} = Z_{vu},$$

i.e.

$$\frac{\partial a_i^k}{\partial v} - \frac{\partial b_i^k}{\partial u} + \sum_{j=1}^{4}(a_i^j\, b_j^k - b_i^j\, a_j^k) = 0, \quad i,k = 1,\dots,4, \tag{10}$$

where a_i^j and b_i^j are the elements of the matrices A and B. By direct computation, using (7) we obtain that equalities (10) are fulfilled. Hence, there exist a subset $\mathcal{D}_1 \subset \mathcal{D}$ and unique vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$, $(u,v) \in \mathcal{D}_1$, which satisfy system (8) and the initial conditions

$$x(u_0,v_0) = x_0, \quad y(u_0,v_0) = y_0, \quad n_1(u_0,v_0) = (n_1)_0, \quad n_2(u_0,v_0) = (n_2)_0.$$

We shall prove that for each $(u,v) \in \mathcal{D}_1$ the vectors $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudo-orthonormal frame in $\mathbb{R}^4_2$. Let us consider the following functions:

$$\begin{array}{lll} \varphi_1 = \langle x, x\rangle; & \varphi_5 = \langle x, y\rangle + 1; & \varphi_8 = \langle y, n_1\rangle; \\ \varphi_2 = \langle y, y\rangle; & \varphi_6 = \langle x, n_1\rangle; & \varphi_9 = \langle y, n_2\rangle; \\ \varphi_3 = \langle n_1, n_1\rangle; & \varphi_7 = \langle x, n_2\rangle; & \varphi_{10} = \langle n_1, n_2\rangle + 1; \\ \varphi_4 = \langle n_2, n_2\rangle; & & \end{array}$$

defined for $(u,v) \in \mathcal{D}_1$. Since $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ satisfy (8), for the derivatives of φ_i we obtain the following system:

$$\begin{aligned} \frac{\partial \varphi_i}{\partial u} &= \alpha_i^j \, \varphi_j, \\ \frac{\partial \varphi_i}{\partial v} &= \beta_i^j \, \varphi_j; \end{aligned} \qquad i = 1, \dots, 10,$$

where $\alpha_i^j, \beta_i^j, \ i,j = 1, \dots, 10$ are functions of $(u,v) \in \mathcal{D}_1$. This is a linear system of partial differential equations for the functions $\varphi_i(u,v), \ i = 1, \dots, 10, \ (u,v) \in \mathcal{D}_1$, satisfying the initial conditions $\varphi_i(u_0,v_0) = 0, \ i = 1, \dots, 10$. Hence, $\varphi_i(u,v) = 0, \ i = 1, \dots, 10$ for each $(u,v) \in \mathcal{D}_1$. Consequently, $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudo-orthonormal frame in $\mathbb{R}^4_2$ for each $(u,v) \in \mathcal{D}_1$.

Now we consider the following system of partial differential equations for the vector function $z = z(u,v)$:

$$\begin{aligned} z_u &= f\, x \\ z_v &= f\, y \end{aligned} \tag{11}$$

By use of (8) we get that $z_{uv} = z_{vu}$, which means that the integrability conditions of system (11) are fulfilled. Hence, there exist a subset $\mathcal{D}_0 \subset \mathcal{D}_1$ and a unique vector function $z = z(u,v)$, defined for $(u,v) \in \mathcal{D}_0$, such that $z(u_0,v_0) = p_0$.

Now, we consider the surface $M^2_1 : z = z(u,v), \ (u,v) \in \mathcal{D}_0$. Using that $z(u,v)$ is a solution of system (11) and $x(u,v)$, $y(u,v)$ satisfy (8), we obtain

$$\begin{aligned} \widetilde{\nabla}_x x &= \frac{f_u}{f^2}\, x + \lambda_1 \, n_1 + \mu_1 \, n_2, \\ \widetilde{\nabla}_x y &= -\frac{f_u}{f^2}\, y + n_1, \\ \widetilde{\nabla}_y x &= -\frac{f_v}{f^2}\, x + n_1, \\ \widetilde{\nabla}_y y &= \frac{f_v}{f^2}\, y + \lambda_2 \, n_1 + \mu_2 \, n_2. \end{aligned}$$

Hence, M^2_1 is a Lorentz surface in $\mathbb{R}^4_2$ with mean curvature vector field $H = -n_1$. So, M^2_1 is quasi-minimal.

Consequently, the surface $M^2_1 : z = z(u,v), \ (u,v) \in \mathcal{D}_0$ satisfies the assertion of the theorem. □

It follows from (5) that $K^2 - \varkappa^2 = 4\lambda_1\mu_1\lambda_2\mu_2$. So, Theorem 3.2 states that the quasi-minimal surfaces in $\mathbb{R}^4_2$ with $K^2 - \varkappa^2 \neq 0$ are determined (up to a rigid motion) by five geometric functions satisfying a system of four partial differential equations. The class of quasi-minimal surfaces satisfying the condition $K^2 - \varkappa^2 = 0$, i.e. $K = \pm\varkappa$, will be studied separately.

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