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LEIBNIZ ALGEBRAS WHOSE SUBALGEBRAS ARE LEFT IDEALS

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Abstract. In this paper we have described the Leibniz algebras, whose subalgebras are left ideals.

1. Introduction. Let L be an algebra over a field F with binary operations $+$ and $[,]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*), if it satisfies the (left) Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]] \quad \text{for all} \quad a, b, c \in L.$$

We will also use another form of this identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Leibniz algebras appeared first in the papers of A. M. Bloh [3], in which he called them the D-algebras. However, in that time these works were not in

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demand, and they have not been properly developed. Only after two decades, a real interest to Leibniz algebras rose. It happened thanks to the J.-L. Loday work [12] (see also [13, Section 10.6]), who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Gottfried Wilhelm Leibniz who discovered and proved the *Leibniz rule* for differentiation of functions. Recently, the theory of Leibniz algebras has been developing quite intensively (see, for example, [1]).

Note that the Lie algebras are the partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra, in which $[a, a] = 0$ for every element $a \in L$, then L is a Lie algebra. Thus, Lie algebras can be characterized as anticommutative Leibniz algebras. In this regard, a parallel comes to mind with associative structures, such as, for example, groups and associative rings. There we see a significant difference between abelian and non-abelian groups, between commutative and non-commutative rings. These differences are expressed not only in the results, but also in ideas, approaches, methods. Of course, anticommutativity is not such a good property as commutativity, but this is a rather strong restriction. We see that the theory of Lie algebras is one of the most developed among other non-associative structures. Therefore, it is natural to expect that the differences between Lie algebras and Leibniz algebras will be very significant. This can be seen immediately by looking at the cyclic Lie algebras and at the cyclic Leibniz algebras. The structure of cyclic Lie algebras is very simple, they have dimension 1. But the structure of cyclic Leibniz algebras is more complicated, it can be seen from the results of the paper [4], where cyclic Leibniz algebras have been described. The significant difference between Lie algebras and Leibniz algebras, which are distinguished by the same restriction, can be traced in [8, 9, 10, 14]. In particular, such a situation takes place for the Leibniz algebras whose subalgebras are ideals. Again, in the case of Lie algebras, we arrive at the abelian algebras, while in the case of Leibniz algebras, the following interesting type of algebras arises here.

The *left* (respectively, *right*) *center* $\zeta^{\text{left}}(L)$ (respectively, $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\text{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\text{right}} = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}).$$

The *center* $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

A Leibniz algebra L is called an *extraspecial algebra*, if it satisfies the following condition:

- $\zeta(L)$ is non-trivial and has dimension 1;
- $L/\zeta(L)$ is abelian.

Leibniz algebras, whose subalgebras are ideals, were described in the paper [8]. Such an algebra L has the following structure: $L = E \oplus Z$, where Z is a subalgebra of the center of L and E is an extraspecial algebra such that $[x, x] \neq 0$ for each element $x \notin \zeta(E)$.

An extraspecial algebra E is called *a strong extraspecial algebra*, if $[x, x] \neq 0$ for each element $x \notin \zeta(E)$.

In the paper [8], the strong extraspecial algebras were associated with a specific bilinear form, in a sense similar to positively defined bilinear forms.

As for associative rings, in Leibniz algebras right and left ideals arise. (This also shows their difference from Lie algebras.)

It will not be superfluous to recall their definitions.

A subalgebra A of a Leibniz algebra L is called a *left* (respectively, *right*) ideal of L, if $[y, x] \in A$ (respectively, $[x, y] \in A$) for every elements $x \in A$, $y \in L$.

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, that is $[x, y] \in A$ and $[y, x] \in A$ for every elements $x \in A$, $y \in L$.

If A is an ideal of L, we can consider the *factor-algebra* L/A. It is not hard to see that this factor-algebra is a Leibniz algebra.

If A, B are subspaces of L, then $[A, B]$ will denote the subspace generated by all elements $[a, b]$, where $a \in A$, $b \in B$.

If M is a non-empty subset of L, then $\langle M \rangle$ denotes the subalgebra of L generated by M.

The Leibniz algebra L is called *abelian*, if $[a, b] = 0$ for any $a, b \in L$.

Every Leibniz algebra L possesses the following specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace, generated by all elements $[a, a]$, $a \in L$. Then $\mathrm{Leib}(L)$ is an ideal of L. The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of the algebra L.

We note that the factor-algebra $L/\operatorname{Leib}(L)$ is a Lie algebra, and conversely if H is an ideal of L such that the factor-algebra L/H is a Lie algebra, then $\operatorname{Leib}(L) \leqslant H$.

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\operatorname{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$, so that $L/\zeta^{\mathrm{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L, and in general, the left and right centers are different. Moreover, they even may have different dimensions. See [6] for some corresponding examples.

In the current paper we consider the Leibniz algebras whose subalgebras are left ideals. Our first result describes the cyclic Leibniz algebras with this property.

Everywhere below the phrase "Let L be a Leibniz algebra means Let L be a Leibniz algebra, which is not a Lie algebra".

Theorem A. *Let L be a Leibniz algebra over a field F, and let all subalgebras of L be left ideals. Suppose that L is cyclic and not abelian.*

(*i*) *If L is nilpotent, then $L = Fa \oplus Fb$, where $b = [a, a]$, $[b, a] = [a, b] = [b, b] = 0$.*

(*ii*) *If L is not nilpotent, then $L = \langle a \rangle = Fd \oplus Fb$, where $b = [a, a]$, $d = a - \lambda^{-1}b$, $[d, d] = [a, d] = [b, b] = [b, a] = [b, d] = 0$, $[d, a] = b$ and $[a, b] = [d, b] = \lambda b$, where λ is a non-zero element of the field F.*

(*iii*) *If L is not nilpotent, $L = Fd \oplus Fb$, where $[d, d] = 0 = [b, b]$, then L is cyclic.*

Conversely, if L is a Leibniz algebra of the above types (*i*) *or* (*ii*), *then every subalgebra of L is a left ideal.*

The general case naturally splits into two subcases: locally nilpotent Leibniz algebras, and not locally nilpotent Leibniz algebras.

Theorem B. *Let L be a Leibniz algebra over a field F, whose subalgebras are left ideals.*

(*i*) *If L is locally nilpotent, then $L = S \oplus D$, where S is a strong extraspecial subalgebra, $D \leqslant \zeta(L)$. In particular, $\zeta(S) = \operatorname{Leib}(L) = [L, L]$, L is central-by--abelian and every subalgebra of L is an ideal.*

(*ii*) *If L is not locally nilpotent, then $L = \operatorname{Leib}(L) \oplus Fv$ and there exists a non-zero element $\sigma \in F$ such that $[v, a] = \sigma a$ for every element $a \in \operatorname{Leib}(L)$.*

Conversely, if L is a Leibniz algebra of types (i) or (ii), then every subalgebra of L is a left ideal.

We see that in a locally nilpotent Leibniz algebra, whose subalgebras are left ideals, every subalgebra is a two-sided ideal.

As we can see, the difference between Lie algebras and Leibniz algebras, whose subalgebras are ideals, lies in the presence of the latter strong extraspecial algebra.

Our last result indicates some fields over which every strong extraspecial algebra has finite dimension.

We say that the field F is 2-closed if the polynomial $X^2 + \alpha \in F[X]$ has a root in the field F for every non-zero element $\alpha \in F$. This means that the multiplicative group $\mathbf{U}(F)$ of the field F is 2-divisible.

In particular, every finite field F of characteristic 2 is 2-closed. Indeed, $|F| = 2^n$ for some positive integer n, so that $|\mathbf{U}(F)| = 2^n - 1$ is odd. It follows that $\mathbf{U}(F)$ is 2-divisible.

Theorem C. *Let E be a strong extraspecial algebra over a field F. If field F is 2-closed, then E has finite dimension at most 3.*

2. On the structure of cyclic Leibniz algebras whose subalgebras are left ideals.

Lemma 2.1. *Let L be a Leibniz algebra over a field F. Suppose that every subalgebra of L is a right (respectively left) ideal of L. Then the factor-algebra $L/\operatorname{Leib}(L)$ is abelian.*

Proof. Indeed, the factor-algebra $L/\operatorname{Leib}(L)$ is a Lie algebra. In particular, $L/\operatorname{Leib}(L)$ is anticommutative. Therefore every left (respectively right) ideal is a two-sided ideal. In other words, every subalgebra of $L/\operatorname{Leib}(L)$ is an ideal. We note now that a Lie algebra with this property is abelian. □

Let L be a Leibniz algebra over a field F, M be non-empty subset of L, and H be a subalgebra of L. Define

$$\operatorname{Ann}_H^{\text{left}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle\},$$

$$\operatorname{Ann}_H^{\text{right}}(M) = \{a \in H \mid [M, a] = \langle 0 \rangle\}.$$

The subset $\operatorname{Ann}_H^{\text{left}}(M)$ is called the *left annihilator* or the *left centralizer* of M in the subalgebra H. The subset $\operatorname{Ann}_H^{\text{right}}(M)$ is called the *right annihilator* or the *right centralizer* of M in the subalgebra H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle \ = [M, a]\}$$

is called the *annihilator* or the *centralizer* of M in the subalgebra H.

It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_H^{\mathrm{left}}(M)$, $b \in M$. Then

$$[[a, x], b] = [a, [x, b]] - [x, [a, b]] = 0 - [x, 0] = 0, \text{ and}$$

$$[[x, a], b] = [x, [a, b]] - [a, [x, b]] = [x, 0] - 0 = 0.$$

If M is an ideal of L, then $\mathrm{Ann}_L^{\mathrm{right}}(M)$ is a left ideal. Indeed, let x be an arbitrary element of L, $d \in \mathrm{Ann}_L^{\mathrm{right}}(M)$, $a \in M$. Then

$$[a, [x, d]] = [[a, x], d]] + [x, [a, d]]] = 0 + [x, 0] = 0.$$

Furthermore, $\mathrm{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_H(M)$, $b \in M$. Using the above arguments, we obtain that

$$[[a, x], b] = [[x, a], b] = 0.$$

Further,

$$[b, [a, x]] = [[b, a], x]] + [a, [b, x]]] = [0, x] + 0 = 0,$$

and

$$[b, [x, a]] = [[b, x], a] + [x, [b, a]]] = 0 + [x, 0] = 0.$$

Lemma 2.2. *Let L be a Leibniz algebra over a field F, and let A be an abelian ideal of L. Suppose that every subalgebra of A is a left ideal of L. Then the factor-algebra $L/\mathrm{Ann}_L^{\mathrm{left}}(A)$ has dimension 1 and for every element $x \in L$ there exists an element $\sigma_x \in F$ such that $[x, a] = \sigma_x a$ for every element $a \in A$.*

Proof. For every element $x \in L$ consider the mapping $\ell_x \colon A \longrightarrow A$, defined by the rule $\ell_x(a) = [x, a]$, $a \in A$. As remarked above, ℓ_x is a derivation of A, and the set $\{\ell_x \mid x \in L\}$ is a subalgebra of the algebra $\mathrm{Der}(A)$ of all derivations of A.

Let $a \in A$. Since the subalgebra $\langle a \rangle = Fa$ is a left ideal, $[x, a] = \alpha a$ for some element $\alpha \in F$. By similar reasons, for another element $c \in A$ we obtain $[x, c] = \gamma c$ for some $\gamma \in F$. We have $[x, a - c] = [x, a] - [x, c] = \alpha a - \gamma c$. On the other hand, $a - c \in A$, so that $F(a - c)$ must be a left ideal of L, i.e. $[x, a - c] = \eta(a - c) = \eta a - \eta c$. It follows that $\alpha a - \gamma c = \eta a - \eta c$, hence $\alpha = \eta = \gamma$.

In other words, for every element $x \in L$ there exists an element $\sigma_x \in F$ such that $\ell_x(a) = [x, a] = \sigma_x a$ for all $a \in A$.

Consider the mapping $\delta\colon L \longrightarrow F$ defined by the rule: $\delta(x) = \sigma_x$ for each element $x \in L$. For elements $x, y \in L$ we have

$$[x + y, a] = [x, a] + [y, a] = \sigma_x a + \sigma_y a = (\sigma_x + \sigma_y)a,$$

and

$$[\beta x, a] = \beta[x, a] = \beta(\sigma_x a) = (\beta\sigma_y)a,$$

which shows that $\sigma_{x+y} = \sigma_x + \sigma_y$ and $\sigma_{\beta x} = \beta\sigma_x$ for all $x, y \in L$, $\beta \in F$. It follows that the mapping δ is linear. Furthermore, $\mathrm{Ker}(\delta) = \{x \in L \mid \delta(x) = \sigma_x = 0\}$. This means that $[x, a] = 0$ for every element $x \in A$. In other words, $\mathrm{Ker}(\delta) \leqslant \mathrm{Ann}_L^{\mathrm{left}}(A)$. The converse inclusion is obvious, so that $\mathrm{Ker}(\delta) = \mathrm{Ann}_L^{\mathrm{left}}(A)$. As we remarked above, $\mathrm{Ann}_L^{\mathrm{left}}(A)$ is a two-sided ideal of L, so we obtain that $L/\,\mathrm{Ann}_L^{\mathrm{left}}(A)$ is isomorphic to F, in particular, this factor-algebra has dimension 1. □

We start the study of the structure of cyclic algebras, whose subalgebras are left ideals.

Let L be a Leibniz algebra. Define the *lower central series*

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \geqslant \gamma_\delta(L)$$

of L by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots] \ldots]]$ is the *left normed commutator* of k copies of L.

As usual, we say that a Leibniz algebra L is *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\mathrm{ncl}(L)$.

Lemma 2.3. *Let L be a Leibniz algebra over a field F, whose subalgebras are left ideals. Suppose that a is an element of L such that $[a, a] \neq 0$.*

(i) If the subalgebra $\langle a\rangle$ is nilpotent, then $\langle a\rangle = Fa \oplus Fb$, where $b = [a, a]$, $[b, a] = [a, b] = 0$.

(ii) If the subalgebra $\langle a\rangle$ is not nilpotent, then $\langle a\rangle = Fd \oplus Fb$, where $b = [a, a]$, $d = a - \lambda^{-1}b$, $[d, d] = [a, d] = [b, b] = [b, a] = [b, d] = 0$, $[d, a] = b$ and $[a, b] = [d, b] = \lambda b$, where λ is a non-zero element of a field F.

Proof. We note that every subalgebra of the left center of L is a right ideal. Being a left ideal, it is a two-sided ideal. Let $b = [a, a]$. If $b \in Fa$, then $[a, a] = \gamma a$ for some $0 \neq \gamma \in F$. In this case we have $0 = [[a, a], a] = [\gamma a, a] = \gamma[a, a]$, which implies that $[a, a] = 0$, and we obtain a contradiction. This contradiction shows that the elements a, b are linearly independent. We have $b = [a, a] \in \mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$. As remarked above, it follows that the subalgebra $\langle b\rangle$ is an ideal of L. Then $[a, b] \in \langle b\rangle$, that is $[a, b] = \lambda b$ for some element $\lambda \in F$. If $\lambda = 0$, then the subalgebra $\langle b\rangle$ coincides with the center of $\langle a\rangle$, and the subalgebra $\langle a\rangle$ is nilpotent.

Suppose that $\lambda \neq 0$, and consider the element $d = a - \lambda^{-1}b$. By its choice, $d \notin Fb$, so that $Fd \cap Fb = \langle 0\rangle$. We have

$$[d, d] = [a - \lambda^{-1}b, a - \lambda^{-1}b] = [a, a] - [a, \lambda^{-1}b] = b - \lambda^{-1}[a, b] = b - b = 0.$$

It follows that $\langle d\rangle = Fd$. Thus $\langle a\rangle = \langle b\rangle \oplus \langle d\rangle = Fb \oplus Fd$. On the other hand,

$$[a, d] = [a, a - \lambda^{-1}b] = 0, [d, a] = [a - \lambda^{-1}b, a] = b.$$

Since $[b, d] = 0$, $\langle d\rangle$ is a left ideal of $\langle a\rangle$. From $[a, d] = b \notin \langle d\rangle$ we can see that $\langle d\rangle$ is not an ideal of $\langle a\rangle$. □

Lemma 2.4. *Let L be a Leibniz algebra over a field F, whose subalgebras are left ideals. If A, B are subalgebras of L, then $A + B$ is a subalgebra, and hence a left ideal of L.*

Proof. Let $a_1, a_2 \in A$, $b_1, b_2 \in B$, we have

$$[a_1 + b_1, a_2 + b_2] = [a_1, a_2] + [a_1, b_2] + [b_1, a_2] + [b_1, b_2].$$

Since A, B are left ideals, $[a_1, a_2]$, $[b_1, a_2] \in A$, $[a_1, b_2]$, $[b_1, b_2] \in B$, so that $[a_1 + b_1, a_2 + b_2] \in A + B$. It shows that $A + B$ is a subalgebra of L. $\square$

The cyclic algebra of type (ii) of Lemma 2.3 is a sum of two one-dimensional subalgebras. Therefore, the following natural question appears here: When the sum of two one-dimensional subalgebras is a cyclic subalgebra?

Lemma 2.5. *Let L be a Leibniz algebra over a field F, whose subalgebras are left ideals. Suppose that a, b are non-zero elements of L such that a, b are linearly independent and $[a, a] = 0 = [b, b]$. Then either $[a, b] = [b, a] = 0$, or $\langle a, b\rangle$ is a cyclic subalgebra of type (ii) of Lemma 2.3.*

Proof. By Lemma 2.4 $\langle a, b\rangle = \langle a\rangle + \langle b\rangle = Fa \oplus Fb$. Since $\langle a\rangle$ is a left ideal, $[b, a] = \lambda a$ for some element $\lambda \in F$. By the same reason $[a, b] = \mu b$ for some element $\mu \in F$. Let $c = \gamma a + \sigma b$ be an arbitrary element of $\langle a, b\rangle$. We have

$$[c, c] = [\gamma a + \sigma b, \gamma a + \sigma b] = \gamma^2[a, a] + \gamma\sigma[a, b] + \gamma\sigma[b, a] + \sigma^2[b, b]$$
$$= \gamma\sigma\mu b + \gamma\sigma\lambda a == \gamma\sigma(\lambda a + \mu b).$$

Since γ, σ are arbitrary elements of F, we obtain that $\lambda a + \mu b \in \mathrm{Leib}(L)$. The inclusion $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$ and the fact that $\zeta^{\mathrm{left}}(L)$ is an abelian subalgebra imply that the subalgebra $\langle \lambda a + \mu b\rangle$ is abelian. It follows that

$$0 = [\lambda a + \mu b, \lambda a + \mu b] = \lambda^2[a, a] + \lambda\mu[a, b] + \lambda\mu[b, a] + \mu^2[b, b]$$
$$= \lambda\mu[a, b] + \lambda\mu[b, a] = \lambda\mu^2 + \lambda^2\mu = \lambda\mu(\mu + \lambda).$$

If $\lambda = \mu = 0$, then the subalgebra $\langle a, b\rangle$ is abelian. Suppose that $(\lambda, \mu) \neq (0, 0)$.

Since $\langle \lambda a+\mu b\rangle$ is a left ideal of L, $[a, \lambda a+\mu b] \in \langle \lambda a+\mu b\rangle$ and $[b, \lambda a+\mu b] \in \langle \lambda a + \mu b\rangle$. Thus we obtain

$$[a, \lambda a + \mu b] = \lambda[a, a] + \mu[a, b] = \mu^2 b = \nu(\lambda a + \mu b) \text{ for some element } \nu \in F,$$
$$[b, \lambda a + \mu b] = \lambda[b, a] + \mu[b, b] = \lambda^2 a = \rho(\lambda a + \mu b) \text{ for some element } \rho \in F.$$

It follows that $\nu\lambda a = \mu^2 b - \nu\mu b = \mu(\mu-\nu)b$ and $\rho\mu b = \lambda^2 a - \rho\lambda a = \lambda(\lambda-\rho)a$. Since the elements a, b are linearly independent, we obtain that $\nu\lambda = 0$, $\mu(\mu - \nu) = 0$, $\rho\mu = 0$, $\lambda(\lambda - \rho) = 0$.

Suppose that $\lambda \neq 0$, then $\nu = 0$, and it follows that $\mu = 0$. Thus we obtain that $[a, b] = 0$. In this case for $u = \lambda^{-1}a + b$, we obtain

$$[u, u] = [\lambda^{-1}a + b, \lambda^{-1}a + b] = \lambda^{-2}[a, a] + \lambda^{-1}[a, b] + \lambda^{-1}[b, a] + [b, b]$$
$$= \lambda^{-1}[b, a] = \lambda^{-1}\lambda a = a.$$

It follows that $\langle a, b\rangle = \langle u\rangle$, and, moreover, $\langle u\rangle$ is a subalgebra, which has been described in Lemma 2.3.

If we assume that $\mu \neq 0$, then we obtain that $[b, a] = 0$, and again we obtain a subalgebra, which was described in Lemma 2.3. □

3. Proof of Theorem A. If L is a Leibniz algebra, whose subalgebras are left ideals, then we can apply Lemmas 2.3 and 2.5.

Conversely, let L be an algebra of type (i). Let x be an arbitrary element of L, then $x = \gamma a + \sigma b$ for some elements $\gamma, \sigma \in F$. If $\gamma = 0$, then $\langle x\rangle = Fx = Fb = \zeta(L)$, so that the subalgebra $\langle x\rangle$ is an ideal of L. If $\gamma \neq 0$, then

$$[x, x] = [\gamma a + \sigma b, \gamma a + \sigma b] = \gamma^2[a, a] = \gamma^2 b \neq 0.$$

It follows that $Fb \leqslant \langle x\rangle$. By $\gamma \neq 0$ we obtain that $x \notin Fb$. This means that $\langle x\rangle = L$.

Let L be an algebra of type (ii), and again x be an arbitrary element of L. Then $x = \gamma d + \sigma b$ for some elements $\gamma, \sigma \in F$. If $\gamma = 0$, then $\langle x\rangle = Fx = Fb = \mathrm{Leib}(L)$, so that the subalgebra $\langle x\rangle$ is an ideal of L. If $\sigma = 0$, then $\langle x\rangle = Fd = \langle d\rangle$, so that the subalgebra $\langle x\rangle$ is an ideal of L. If $\gamma \neq 0$, $\sigma \neq 0$, then

$$[x, x] = [\gamma d + \sigma b, \gamma d + \sigma b] = \gamma^2[d, d] + \gamma\sigma[d, b] = \gamma\sigma\lambda b \neq 0.$$

It follows that $Fb \leqslant \langle x\rangle$. By $\gamma \neq 0$, we have that $x \notin Fb$. This means that $\langle x\rangle = L$. □

4. The structure of Leibniz algebras whose subalgebras are left ideals. The general case.

Lemma 4.1. *Let L be a locally nilpotent Leibniz algebra over a field F, whose subalgebras are left ideal. Then $L = S \oplus D$, where S is a strong extraspecial subalgebra, $D \leqslant \zeta(L)$. In particular, every subalgebra of L is an ideal.*

Proof. Let $Z = \zeta^{\text{left}}(L)$. Every subalgebra of Z is a right ideal. Therefore, every subalgebra of Z, being a left ideal, is a two-sided ideal of L. In particular, every one-dimensional subalgebra of Z is a two-sided ideal of L. Since

L is locally nilpotent, the center of L includes every one-dimensional subalgebra of Z [11, Theorem E]. It follows that the left center coincides with the center. In particular, the center of L includes the Leibniz kernel. Let $K = \mathrm{Leib}(L)$. By Lemma 2.1, the factor-algebra L/K is abelian.

Let y be an element of L such that $[y, y] = 0$. If $y \in K$, then, as above noted, $y \in \zeta(L)$, so that $[x, y] = 0$ for every element $x \in L$. Suppose that $y \notin K$. Since the factor-algebra L/K is abelian, $[x, y] \in K$ for every element $x \in L$. On the other hand, the subalgebra $\langle y\rangle = Fy$ is a left ideal of L, therefore $[x, y] \in \langle y\rangle$, so that $[x, y] \in K \cap \langle y\rangle = \langle 0\rangle$. Thus $[x, y] = 0$ for every element $x \in L$. This means that the right center of L includes every one-dimensional subalgebra of L.

Since L is not a Lie algebra, there exists an element a such that $c = [a, a] \neq 0$. Suppose that there exists an element b such that $d = [b, b] \neq 0$ and the elements c, d are linearly independent. As was proved above $c, d \in \zeta(L)$, and therefore $[a, c] = 0 = [b, d]$. It follows that $\langle a\rangle = Fa \oplus Fc$ and $\langle b\rangle = Fb \oplus Fd$. The fact that L/K is abelian and the subalgebra $\langle a\rangle$ is a left ideal imply that $[b, a] = \lambda c$ for some element $\lambda \in F$. By the same reason, $[a, b] = \mu d$ for some element $\mu \in F$.

First suppose that $\lambda = \mu = 0$. Let $x = a + b$. Then

$$[x, x] = [a + b, a + b] = [a, a] + [b, b] = c + d.$$

As above $\langle x\rangle = Fx \oplus F[x, x]$ and $\langle x\rangle \cap \zeta(L) = F(c + d)$. On the other hand, $[a, x] = [a, a + b] = [a, a] + [a, b] = c$, $[b, x] = d$. Since the subalgebra $\langle x\rangle$ is a left ideal, $c, d \in \langle x\rangle \cap K = F(c+d)$, and we obtain a contradiction. This contradiction shows that $(\lambda, \mu) \neq (0, 0)$.

Suppose now that $\lambda \neq 0$ and $\mu = 0$. For $b_1 = \lambda^{-1}b$ we obtain

$$[b_1, a] = [\lambda^{-1}b, a] = \lambda^{-1}[b, a] = \lambda^{-1}\lambda a = a,$$

$$[a, b_1] = [a, \lambda^{-1}b] = \lambda^{-1}[a, b] = 0,$$

$$[b_1, b_1] = [\lambda^{-1}b, \lambda^{-1}b] = \lambda^{-2}d,$$

so that $\langle b_1\rangle = Fb_1 \oplus F(\lambda^{-2}d)$, and the elements c, $\lambda^{-2}d$ are linearly independent and $\langle a, b\rangle = \langle a, b_1\rangle$. Therefore, without loss of generality we may assume further that $\lambda = 1$, i.e. $[b, a] = c$ and $[a, b] = 0$.

Suppose first that $\mathrm{char}(F) = 2$. Then for $u = a + b$ we have

$$v = [u, u] = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = c + c + d = d.$$

Again, $v \in \zeta(L)$, so that $\langle u \rangle = Fu \oplus Fd$. The subalgebra $\langle u \rangle = Fu \oplus Fd$ is a left ideal, so that $[a, u] \in \langle u \rangle$ and $[b, u] \in \langle u \rangle$. We have

$$[a, u] = [a, a + b] = [a, a] + [a, b] = c,$$
$$[b, u] = [b, a + b] = [b, a] + [b, b] = c + d.$$

On the other hand, the fact that L/K is abelian implies that $[a, u] \in K$, and $[b, u] \in K$, that is $[a, u] \in K \cap \langle u \rangle = \langle d \rangle$, and $[b, u] \in K \cap \langle u \rangle = \langle d \rangle$. Thus, we obtain a contradiction.

Suppose now that $\mathrm{char}(F) \neq 2$. For $u = a - b$ we have

$$v = [u, u] = [a - b, a - b] = [a, a] - [a, b] - [b, a] + [b, b] = c - c + d = d.$$

Again, $v \in \zeta(L)$, so that $\langle u \rangle = Fu \oplus Fd$. The subalgebra $\langle u \rangle = Fu \oplus Fd$ is a left ideal, so that $[a, u] \in \langle u \rangle$, and $[b, u] \in \langle u \rangle$. We have

$$[a, u] = [a, a - b] = [a, a] - [a, b] = c,$$
$$[b, u] = [b, a - b] = [b, a] - [b, b] = c - d.$$

Since $c \notin \langle d \rangle, c - d \notin \langle d \rangle$, we again obtain a contradiction.

Finally suppose that $\lambda \neq 0$ and $\mu \neq 0$. As above, without loss of generality, we may assume that $\lambda = 1$, i.e. $[b, a] = c$, and $[a, b] = \mu d$.

Suppose that $\mathrm{char}(F) = 2$, and again consider the element $u = a + b$. We have

$$v = [u, u] = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = c + \mu d + c + d = (\mu + 1)d.$$

Again, $v \in \zeta(L)$, so that $\langle u \rangle = Fu \oplus Fd$. The subalgebra $\langle u \rangle = Fu \oplus Fd$ is a left ideal, so that $[a, u] \in \langle u \rangle$, and $[b, u] \in \langle u \rangle$. We have

$$[a, u] = [a, a + b] = [a, a] + [a, b] = c + \mu d,$$
$$[b, u] = [b, a + b] = [b, a] + [b, b] = c + d.$$

We can see that $c + d \notin \langle d \rangle$, so we obtain a contradiction.

Suppose that $\mathrm{char}(F) \neq 2$ and consider the element $u = a - b$. We have

$$v = [u, u] = [a - b, a - b] = [a, a] - [a, b] - [b, a] + [b, b] = c - \mu d - c + d = (1 - \mu)d.$$

Again, $v \in \zeta(L)$, so that $\langle u \rangle = Fu \oplus Fd$. A subalgebra $\langle u \rangle = Fu \oplus Fd$ is a left ideal, so that $[a, u] \in \langle u \rangle$, and $[b, u] \in \langle u \rangle$. We have

$$[a, u] = [a, a - b] = [a, a] - [a, b] = c - \mu d,$$

$$[b, u] = [b, a - b] = [b, a] - [b, b] = c - d.$$

Since $c - d \notin \langle d \rangle$, we obtain a contradiction.

Thus, in every case, we found a subalgebra, which is not a left ideal. This contradiction shows that for every element b such that $[b, b] \neq 0$ it must be $[b, b] \in F[a, a]$. It follows that $\mathrm{Leib}(L) = F[a, a]$.

Let R be the right center of L. As we have noted above, $K \leqslant \zeta(L)$, so that $K \leqslant R$. Since the factor-algebra L/K is abelian, $L/K = R/K \oplus S/K$ for some ideal S of L. Let x be an arbitrary element of S such that $x \notin K$. If we suppose that $[x, x] = 0$, then as proved above, $x \in R$, so that $x \in S \cap R = K$, and we obtain a contradiction. This contradiction shows that $[x, x] \neq 0$ for each element $x \in S \setminus K$. By above proved $K \leqslant \zeta(L)$. The construction of S yields that $\zeta(S) = K$, so that S is an extraspecial algebra.

Since R is abelian, $R = K \oplus D$ for some subalgebra D of R. Then $L = S \oplus D$. Since D is a subalgebra of the right center, $[S, R] = \langle 0 \rangle$.

Suppose that S contains an element $h \notin K$ and D contains an element d such that $[d, h] \neq 0$. The choice of the element h implies that $[h, h] = \sigma a$ for some non-zero element $\sigma \in F$. Since L/K is an abelian algebra, $[d, h] \in K$. This means that $[d, h] = \tau a$ for some non-zero element $\tau \in F$. Then for $z = h - \sigma\tau^{-1}d$ we have

$$[z, z] = [h - \sigma\tau^{-1}d, h - \sigma\tau^{-1}d] = [h, h] - \sigma\tau^{-1}[d, h] = \sigma a - \sigma\tau^{-1}\tau a = \sigma a - \sigma a = 0.$$

As it was proved, $z = h - \sigma\tau^{-1}d \in \zeta^{\mathrm{right}}(L) = R$. By $d \in R$ we obtain that $h \in R$. Then $h \in S \cap R = K$, and we obtain a contradiction. This contradiction proves that $[D, S] = \langle 0 \rangle$. The equalities $L = S \oplus D$, $[D, S] = \langle 0 \rangle = [S, D]$ imply that the center of L includes D. Using Theorem A of the paper [8], we obtain that every subalgebra of L is an ideal. □

Lemma 4.2. *Let L be a not locally nilpotent Leibniz algebra over a field F, whose subalgebras are left ideals. Then $L = \mathrm{Leib}(L) \oplus Fv$, and there exists a non-zero element $\sigma \in F$ such that $[v, a] = \sigma a$ for every element $a \in \mathrm{Leib}(L)$.*

Proof. Let $K = \mathrm{Leib}(L)$. By Lemma 2.1, the factor-algebra L/K is abelian. It follows that the ideal K is not central in L, that is, there exists an element $x \in L$ such that $[x, K] \neq \langle 0 \rangle$.

Let $Z = \zeta^{\text{left}}(L)$. Every subalgebra of Z is a right ideal, therefore, being a left ideal, every subalgebra of Z is a two-sided ideal of L. In particular, every one-dimensional subalgebra of Z is a two-sided ideal of L. By Lemma 2.2, the factor-algebra $L/\operatorname{Ann}_L^{\text{left}}(Z)$ has dimension 1. Then $L = \operatorname{Ann}_L^{\text{left}}(Z) \oplus Fv$ for some element $v \in L$. Moreover, there exists an element $\sigma \in F$ such that $[v, a] = \sigma a$ for each element $a \in Z$. In particular, $[v, a] = \sigma a$ for each element $a \in K$. Since K is not central in L, $\sigma \neq 0$. Let b be an arbitrary element of K. As we have noted above, the subalgebra Fb is an ideal of L. If a is an arbitrary element of Fb, then $a = \lambda b$ for some $\lambda \in F$. Then

$$a = \lambda(\sigma^{-1}\sigma)b = \lambda\sigma^{-1}(\sigma b) = \lambda\sigma^{-1}\sigma b = \lambda\sigma^{-1}[v, b] = [v, \lambda\sigma^{-1}b] \in [v, Fb].$$

It follows that $[v, Fb] = Fb$. Since it is true for every one-dimensional subalgebra of K, $[v, K] = K$.

Let x be an arbitrary element of L. The fact that L/K is abelian implies that $[v, x] = c \in K$. By proved above, K contains an element d such that $c = [v, d]$. Hence $[v, x] = [v, d]$, and therefore $[v, x - d] = 0$. This means that $x - d \in \operatorname{Ann}_L^{\text{right}}(v)$ or $x \in K + \operatorname{Ann}_L^{\text{right}}(v)$. Since x is an arbitrary element of L, we obtain the equality $L = K + \operatorname{Ann}_L^{\text{right}}(v)$. Let $a \in K \cap \operatorname{Ann}_L^{\text{right}}(v)$, and suppose that $a \neq 0$. Then $[v, a] = 0$. On the other hand, $[v, a] = \sigma a$, where σ is not zero, and we obtain a contradiction. This contradiction proves that $K \cap \operatorname{Ann}_L^{\text{right}}(v) = \langle 0 \rangle$. Put $V = \operatorname{Ann}_L^{\text{right}}(v)$, then $L = K \oplus V$. The isomorphism $V \cong L/K$ implies that a subalgebra V is abelian.

We have $\operatorname{Ann}_L^{\text{right}}(Z) = L$, therefore $\operatorname{Ann}_L^{\text{left}}(Z) = \operatorname{Ann}_L(Z)$. The inclusion $K \leqslant Z$ implies that $\operatorname{Ann}_L(Z) \leqslant \operatorname{Ann}_L(K)$. On the other hand, K is not central in L, so that $L \neq \operatorname{Ann}_L(K)$. Since $\operatorname{Ann}_L(Z)$ has codimension 1, we obtain the equality $\operatorname{Ann}_L(Z) = \operatorname{Ann}_L(K)$. We have $\operatorname{Ann}_L(K) = K \oplus (\operatorname{Ann}_L(K) \cap V)$. The fact that V is abelian, implies that $\operatorname{Ann}_L(K) \cap V$ is also abelian. By its choice, $[K, \operatorname{Ann}_L(K) \cap V] = [\operatorname{Ann}_L(K) \cap V, K] = \langle 0 \rangle$. It follows that $\operatorname{Ann}_L(K)$ is an abelian ideal. Lemma 2.2 shows that there exists an element $\tau \in F$ such that $[v, u] = \tau u$ for all elements $u \in \operatorname{Ann}_L(K)$. Since $K \leqslant \operatorname{Ann}_L(K)$ and $[v, a] = \sigma a$ for all elements $a \in K$, we obtain that $\tau = \sigma$. Suppose that $\operatorname{Ann}_L(K) \cap V \neq \langle 0 \rangle$. If $0 \neq u \in \operatorname{Ann}_L(K) \cap V$, then by remarked above, we obtain that $[v, u] = \sigma u \neq 0$. On the other hand, the subalgebra V is abelian, so that $[v, u] = 0$, and we obtain a contradiction. This contradiction proves that $\operatorname{Ann}_L(K) \cap V = \langle 0 \rangle$. Since

$\mathrm{Ann}_L(K) \cap V$ has a codimension 1 in V, we obtain that $\dim_F(V) = 1$, that is $V = Fv$. Thus $L = K \oplus Fv$. $\square$

5. Proof of Theorem B. If L is a Leibniz algebra, whose subalgebra are left ideals, then we can apply Lemmas 4.1 and 4.2.

Conversely, let L be an algebra of type (i). Let x be an arbitrary element of L. If $x \in \zeta(L)$, then the subalgebra $\langle x \rangle = Fx$ is an ideal. Suppose that $x \notin \zeta(L)$, then $x = y + d$, where $y \in S$, $d \in D$. From $x \notin \zeta(L)$ we derive that $y \notin \zeta(S)$. Then

$$[x, x] = [y + d, y + d] = [y, y] \neq 0.$$

Since $\mathrm{Leib}(L)$ has dimension 1, $\mathrm{Leib}(L) \leqslant \langle x \rangle$. Then the fact that $L/\mathrm{Leib}(L)$ is abelian implies that the subalgebra $\langle x \rangle$ is an ideal. Since every cyclic subalgebra of L is an ideal, every subalgebra of L is an ideal.

Let L be an algebra of type (ii) and let $K = \mathrm{Leib}(L)$. At once, we note that every cyclic subalgebra of K has dimension 1 and is an ideal. It follows that every subalgebra of K is an ideal. We note also that the subalgebra $\langle v \rangle = Fv$ is a left ideal. Indeed, let x be an arbitrary element of L. Then $x = a + \beta v$ for some $a \in K$ and $\beta \in F$. Then

$$[x, v] = [a + \beta v, v] = [a, v] + \beta[v, v] = 0,$$

because $a \in \zeta^{\mathrm{left}}(L)$. Moreover, we can see that $\langle v \rangle$ is the right center of L. Furthermore, suppose that $x \notin K$ and $x \notin \langle v \rangle$. Then $a \neq 0$ and $\beta \neq 0$, and

$$[x, x] = [a + \beta v, a + \beta v] = [a, a] + \beta[v, a] + \beta[a, v] + \beta^2[v, v] = \beta\sigma a.$$

It follows that $\langle x \rangle = Fx \oplus Fa$.

Now let y be an another arbitrary element of L, $y = b + \gamma v$ for some elements $b \in K$ and $\gamma \in F$. Then

$$[y, x] = [b + \gamma v, a + \beta v] = [b, a] + \gamma[v, a] + \beta[b, v] + \gamma\beta[v, v] = \gamma\sigma a \in Fa \leqslant \langle x \rangle.$$

Thus we can see that every cyclic subalgebra of L is a left ideal, and therefore every subalgebra of L is a left ideal. $\square$

6. The structure of nilpotent Leibniz algebras whose subalgebras are left ideals. The general case. In this section we obtain some details of the structure of a strong extraspecial algebra E such that $[x, x] \neq 0$ for every element $x \notin \zeta(E)$.

In the paper [8], some properties of strong extraspecial algebras were obtained. In particular, the following assertion has been proved.

Proposition 6.1. *Let E be a finite dimensional strong extraspecial algebra over a field F. Then E has a basis $\{z, a_1, \dots, a_n\}$ such that $[z, a_j] = [a_j, z] = 0$ for all $j \in \{1, \dots, n\}$ and $[a_j, a_k] = 0$ whenever $j > k$.*

Lemma 6.1. *Let E be a finite dimensional strong extraspecial algebra over a field F. Then E has a basis $\{z, c_1, \dots, c_n\}$ such that $[z, c_j] = [c_j, z] = 0$ for all $j \in \{1, \dots, n\}$, $[c_1, c_1] = z, [c_j, c_1] = 0$ whenever $j > 1$ and $[c_1, c_j] = 0$ whenever $j > 2$.*

P r o o f. Suppose that $\dim_F(E) = n + 1$. Choose an arbitrary basis $\{z, a_1, \dots, a_n\}$, where $z = [a_1, a_1]$. Let $[a_j, a_1] = \sigma_j z$, $2 \leqslant j \leqslant n$ and let $b_j = a_j - \sigma_j a_1$, $2 \leqslant j \leqslant n$. In particular, if $\sigma_j = 0$, then $b_j = a_j$. We have

$$[b_j, a_1] = [a_j - \sigma_j a_1, a_1] = [a_j, a_1] - \sigma_j[a_1, a_1] = \sigma_j z - \sigma_j z = 0, \quad 2 \leqslant j \leqslant n.$$

Suppose that there is a number j such that $[a_1, b_j] \neq 0$. Without loss of generality we may assume that $j = 2$, that is $[a_1, b_2] = \alpha z \neq 0$. It follows that $\alpha \neq 0$. Let $[a_1, b_j] = \alpha_j z$, $3 \leqslant j \leqslant n$. Put $c_j = \alpha_j \alpha^{-1} b_2 - b_j$, $3 \leqslant j \leqslant n$. Then

$$[a_1, c_j] = [a_1, \alpha_j \alpha^{-1} b_2 - b_j] = \alpha_j \alpha^{-1}[a_1, b_2] - [a_1, b_j] = \alpha_j \alpha^{-1} \alpha z - \alpha_j z = 0,$$

$$3 \leqslant j \leqslant n.$$

We note that $[c_j, a_1] = 0$, $3 \leqslant j \leqslant n$. If $[a_1, b_j] = 0$ for all $j \in \{2, \dots, n\}$, then we assume that $c_j = b_j$.

Thus we construct the basis $\{z, c_1 = a_1, c_2 = b_2, c_3, \dots, c_n\}$ satisfying the following conditions:

$$[c_j, c_1] = 0 \text{ whenever } 2 \leqslant j \leqslant n, \text{ and } [c_1, c_j] = 0 \text{ whenever } 3 \leqslant j \leqslant n. \quad \square$$

7. Proof of Theorem C. Let $Z = \zeta(E)$. Suppose that $\dim_F(E) > 3$. In the factor-algebra E/Z we choose the cosets $a_1 + Z$, $a_2 + Z$, $a_3 + Z$ such that the subset $\{a_1 + Z, a_2 + Z, a_3 + Z\}$ is free. Since the factor-algebra E/Z is abelian, the subalgebra S/Z, generated by the cosets $a_1 + Z, a_2 + Z, a_3 + Z$, is $F(a_1 + Z) \oplus F(a_2 + Z) \oplus F(a_3 + Z)$. Since E is a strong extraspecial algebra, $z = [a_1, a_1] \neq 0$. It follows that $Z = Fz$. The subalgebra S is strong extraspecial and has a basis $\{z, a_1, a_2, a_3\}$. Using Lemma 6.1 we obtain that S has a basis $\{z, c_1, c_2, c_3\}$ such that $z = [c_1, c_1]$ and $[c_1, c_3] = 0 = [c_3, c_1]$. Let $[c_3, c_3] = \gamma z$, then $\gamma \neq 0$. Let $x = \alpha c_1 + \beta c_3$, where α, β are the non-zero elements of the field F. We have

$$\begin{aligned}[x, x] &= [\alpha c_1 + \beta c_3, \alpha c_1 + \beta c_3] = \alpha^2[c_1, c_1] + \alpha\beta[c_1, c_3] + \beta\alpha[c_3, c_1] + \beta^2[c_3, c_3] \\ &= \alpha^2 z + \beta^2\gamma z = (\alpha^2 + \beta^2\gamma)z.\end{aligned}$$

Suppose that $\alpha^2 + \beta^2\gamma = 0$. By our choice, $\beta \neq 0$. Let $\lambda = \alpha\beta^{-1}$. We obtain $\lambda^2 + \gamma = 0$. Since the field F is 2-closed, the polynomial $X^2 + \gamma$ has a root σ. Now, for $y = \sigma c_1 + c_3$, we have $y \notin Z$, and $[y, y] = (\sigma^2 + \gamma)z = 0$. On the other hand, since E is a strong extraspecial algebra, $[y, y]$ must be non-zero, and we obtain a contradiction. This contradiction shows that $\dim_F(E) \leqslant 3$.

Corollary C1. *Let E be a strong extraspecial algebra over a field F. If the field F is algebraically closed, then E has finite dimension at most* 3.

Corollary C2. *Let E be a strong extraspecial algebra over a field F. If F is a finite field of characteristic* 2*, E has finite dimension at most* 3.

Corollary C3. *Let E be a strong extraspecial algebra over a field F. If F is a locally finite field of characteristic* 2*, E has finite dimension at most* 3.

Corollary C4. *Let E be a strong extraspecial algebra over a field F. If F is a* 2*-closed field and* $\mathrm{char}(F) \neq 2$*, then E has finite dimension* 2.

Proof. By Theorem C, $\dim_F(E) \leqslant 3$. Suppose that $\dim_F(E) = 3$ and let $Z = \zeta(E)$. Then the factor-algebra E/Z has dimension 2. Let $\{a_1+Z, a_2+Z\}$ be a basis of E/Z. Since E is a strong extraspecial algebra, $z = [a_1, a_1] \neq 0$. It follows that $Z = Fz$. Then E has a basis $\{z, a_1, a_2\}$. We have $[a_2, a_1] = \beta z$. If $\beta \neq 0$, then for $b_2 = \beta a_1 - a_2$ we have

$$[b_2, a_1] = [\beta a_1 - a_2, a_1] = \beta[a_1, a_1] - [a_2, a_1] = \beta z - \beta z = 0.$$

If $\beta = 0$, then let $b_2 = a_2$ and $b_1 = a_1$. Clearly, the subset $\{z, b_1, b_2\}$ is a basis of E. We have $[b_1, b_2] = \gamma z$ and $[b_2, b_2] = \sigma z$, for some elements γ, σ of a field F.

Let $x = \lambda b_1 + \mu b_2$, where λ, μ are non-zero elements of a field F. We have

$$[x, x] = [\lambda b_1 + \mu b_2, \lambda b_1 + \mu b_2] = \lambda^2 [b_1, b_1] + \lambda\mu [b_1, b_2] + \mu^2 [b_2, b_2] = \lambda^2 z + \lambda\mu\gamma z + \mu^2 \sigma z.$$

Consider the polynomial $f(X) = X^2 + \gamma X + \sigma$. Since $\mathrm{char}(F) \neq 2$, there exists an element ν in the field F such that $\gamma = 2\nu$. Then

$$f(X) = X^2 + 2\nu X + \sigma = X^2 + 2\nu X + \nu^2 + (\sigma - \nu^2) = (X + \nu)^2 + (\sigma - \nu^2).$$

Since the field F is 2-closed, the polynomial $f(X)$ has a root τ in the field F. In other words, $\tau^2 + \tau\gamma + \sigma = 0$. For $\mu = 1$, $\lambda = \tau$ we obtain that $[x, x] = 0$. Clearly, $x \notin Z$, and we obtain a contradiction, which proves the result.

We note that for an arbitrary field the assertion of Theorem C is not true. The following simple example shows it.

Let $E_j = \mathbb{Q}a_j \oplus \mathbb{Q}b_j$ be a nilpotent Leibniz algebra over the field $\mathbb{Q}$ of rational numbers, $j \in \mathbb{N}$. In other words,$[a_j, a_j] = b_j$, $[b_j, a_j] = [a_j, b_j] = [b_j, b_j] = 0$, $j \in \mathbb{N}$.

Then for $L = \bigoplus_{j \in \mathbb{N}} E_j$, $B = \bigoplus_{j \in \mathbb{N}} \mathbb{Q}b_j$, we have that L is a Leibniz algebra and $B \leqslant \zeta(L)$. Denote by A the subspace of B, generated by the elements $b_{j+1} - b_j$, $j \in \mathbb{N}$. Clearly, A is an ideal of L. If $E = L/A$, $Z = B/A$, then $Z = \mathbb{Q}c$, where $c = b_1 + A$, and E has a basis $\{c, u_j \mid j \in \mathbb{N}\}$, where $u_j = a_j + A$, $j \in \mathbb{N}$. We have $Z = \zeta(E) = \mathrm{Leib}(L) = [E, E]$, so that E is an extraspecial algebra. More precisely,

$$[u_j, u_j] = c \text{ for all } j \in \mathbb{N}, \quad [u_j, u_k] = [u_k, u_j] = [c, u_j] = [u_j, c] = 0, \quad j, k \in \mathbb{N}.$$

Let $x = \gamma c + \lambda_1 u_1 + \cdots + \lambda_k u_k$ be an arbitrary element of E, $\gamma, \lambda_1, \ldots, \lambda_k \in F$, and assume that $x \notin Z$. We have

$$[x, x] = [\gamma c + \lambda_1 u_1 + \cdots + \lambda_k u_k, \gamma c + \lambda_1 u_1 + \ldots + \lambda_k u_k] = (\lambda_1^2 + \cdots + \lambda_k^2)c.$$

Since $x \notin Z$, there exists at least one number j such that $\lambda_j \neq 0$. It follows that $\lambda_1^2 + \cdots + \lambda_k^2 \neq 0$, therefore $[x, x] \neq 0$. Thus E is a strong extraspecial algebra. We note that E has an infinite dimension. □

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