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# METRIC DIMENSION OF ULTRAMETRIC SPACES*

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Abstract. For an arbitrary finite metric space $(X, d)$ a subset $A$, $A \subset X$, is called a resolving set if for any two points $x$ and $y$ from the space $X$ there is an element $a$ from subset $A$, such that distances $d(a, x)$ and $d(a, y)$ are different. The metric dimension $md(X)$ of the space $X$ is the minimum cardinality of a resolving set.

It is well known that the problem of finding the metric dimension of a metric space is NP-complete [7]. In this paper, the metric dimension for finite ultrametric spaces is completely characterized. It is proved that for any finite ultrametric space there exists a polynomial-time algorithm for determining the metric dimension of this spaces.

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*Key words:* metric dimension, ultrametric space, rooted tree, polynomial-time algorithm.

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**1. Introduction.** Let $(X, d)$ be a finite metric space. A non-empty subset $A$ of the set $X$ is a *resolving set* of $X$ if for any $x$ and $y$ from $X$ there is an element $a$ from $A$ such that the equality $d(x, a) = d(y, a)$ holds if and only if $x = y$. A *metric basis* of $X$ is a resolving set $A$ with minimal cardinality. The *metric dimension* $md(X)$ of the metric space $(X, d)$ is the cardinality of its metric basis.

The concept of the metric dimension of metric spaces was introduced in 1953 by L.M. Blumenthal [2]. 20 years later, F. Harary, R. A. Melter, and P. J. Slater applied it to the metric spaces defined by the graphs [10, 19]. In the last case, for a graph $G = (V, E)$ the *distance* $d_G$ between two vertices $u$ and $v$ is defined as the length of the shortest path between $u$ and $v$. Metric dimension has a wide range of applications, particularly in robotics, chemistry, biology, combinatorial analysis, and others [19, 14, 18].

The metric dimension problem is to find the metric dimension of a metric space. In general case, the metric dimension problem is NP-complete [7]. But for some families of graphs, for example, trees [14], chain graphs [6], and outerplanar graphs [4] there are polynomial algorithms for finding the metric dimension.

In 2013, S. Bau and F. Beardon [1] expanded on Blumenthal's ideas regarding the metric dimension of metric spaces. In particular, they calculated the metric dimension of a sphere in $k$-dimensional Euclidean space. Later, M. Heydarpour and S. Maghsoudi calculated the metric dimension of geometric spaces [12]. The metric dimension of metric transforms was described in [17].

A metric space $(X, d)$ is called ultrametric if and only if for arbitrary $x$, $y$, $z$ from the set $X$, the following inequality holds:

$$d(x, y) \leq \max(d(x, z), d(y, z)).$$

Ultrametrics are important special metric spaces which are connected with various branches of mathematics (see [11, 3, 8, 15]). The main result of this paper is a characterization of the metric dimension of finite ultrametric spaces.

**Theorem 1.** *Let $(X, d)$ be a finite ultrametric space. Then there is a polynomial-time algorithm for determining the metric dimension of $(X, d)$.*

We use a well-known correspondence between trees and ultrametric (see [8, 13, 9, 16]), but we construct a simple algorithm that takes an arbitrary ultrametric space $X$ and produces a tree isometric with $X$. The algorithm runs in a time $O(n^5)$, where $n = |X|$.

**2. Metric dimension of rooted trees.** A tree $T$ is called *rooted* if it has a fixed vertex $v_0$, where $v_0$ is called the *root of the tree* $T$. A vertex $v$ is called a vertex of the $m$th level, if the length of path between $v$ and $v_0$ is equal to $m$. We denote the set of all vertices of the level $m$ of the tree $T$ by $L_m$. Vertices of degree 1 are called *leaves* or *end vertices.* A vertex $v$ is called *inner* if the degree of $v$ is greater than 2. For unspecified notions in graph theory we refer to [5].

Denote by $T_m$ a finite rooted tree, such that every leaf $v$ of $T_m$ is a vertex of the level $m$, i.e. $v \in L_m$.

We say that an inner vertex $v$ is *close* to leaf $l$ if there are no other inner vertices in the path between them. In other words, $v$ and $l$ are connected by the path with vertices of degree 2. The number of leaves that are close to the inner vertex $v$ will be denoted by $n_v$.

**Lemma 1** ([19]). *If $T$ is a tree but is not a path, then*

$$md(T) = \sum_{v \in V'(G), n_v \geq 1} (n_v - 1), \tag{1}$$

*where $V'(G)$ is the set of all inner vertices and basic vertices are leaves of tree $T$.*

**Proposition 1.** *Let $T_m$ be a rooted tree with $m$ levels and all vertices of the level $(m-1)$ are inner. Then the metric dimension of the rooted tree $T_m$ is equal to $|L_m| - |L_{m-1}|$.*

Proof. First, note that any leaf $l$ of this tree is located on the level $m$. Inner vertices that are close to leaves are located on the level $(m-1)$. This means that for each inner vertex $v_i$ there are $n_{v_i}$ leaves close to it, where $1 \leq i \leq k$ and $k$ being the number of vertices on the $m-1$ level of the rooted tree $T_m$. According to Lemma 1 we have that:

$$md(T_m) = \sum_{v \in V(G), n_v \geq 1} (n_v - 1) = (n_{v_1} + \ldots + n_{v_k}) - (1 + \ldots + 1) =$$

$$= \sum_{i=1}^{k} (n_{v_i}) - \sum_{i=1}^{k} 1 = |L_m| - |L_{m-1}| \qquad \square$$

Let $(\alpha_0, \ldots, \alpha_m)$ be a finite sequence of real numbers, such that

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{m-1} < \alpha_m.$$

Define a *metric* $\rho$ on subset $L_m$ of vertices of $T_m$ in the following way: for any vertices $u, v$ from $L_m$, the distance $\rho(u, v)$ is equal to $\alpha_{m-k}$ if and only if the joint part of paths connecting the root $v_0$ with $v$ and $u$ has the length $k$, $0 \leq k \leq m$.

Note that the diameter of the metric space $(L_m, \rho)$ is equal to $\alpha_{m-r}$, where $r$ is the greatest number such that the level $r$ of $T_m$ contains exactly one vertex.

The sequence $(\alpha_0, \ldots, \alpha_m)$ is called the *defining set* of the tree $T_m$ [16].

**Lemma 2.** *For all vertices $u$, $v$ from $L_m$ the equality $\rho(u, v) = \alpha_{m-k}$ holds if and only if $d_G(u, v) = 2(m - k)$, $0 \leq k \leq m$.*

Proof. Let $u, v \in L_m$ and $\rho(u, v) = \alpha_{m-k}$, $0 \leq k \leq m$. By definition of the metric $\rho$, the joint part of paths connecting the root $v_0$ with $v$ and $u$ has the length $k$, where $0 \leq k \leq m$. This implies the existence of a vertex $z \in L_k$, such that there exists path from the vertex $u$ to the vertex $v$ through the vertex $z$. As vertices $u$ and $v$ are $m$th level vertices, $d_G(u, v) = 2(m - k)$. □

**Theorem 2.** *Let $T_m$ be a rooted tree with defining set $(\alpha_0, \ldots, \alpha_m)$. If there exists an inner vertex $v_1$ of the level $r$ which is close to two leaves $l_1$ and $l_2$ and there are no another inner vertices on levels $0, \ldots, r$, then $md(T_m) - 1 = md(L_m, \rho)$. In other cases the metric dimension of the space $(L_m, \rho)$ is equal to the metric dimension of the graph $T_m$.*

Proof. Let $W = \{w_1, \ldots, w_l\}$ be a metric basis of tree $T_m$. Lemma 1 implies that $W \subset L_m$.

First, we show that $W$ is a resolving set of $(L_m, \rho)$. Let $u$, $v$ be vertices of $L_m$. As $W$ is the metric basis of the graph $T_m$, there exists a vertex $w_j \in W$, $0 \leq j \leq l$, such that $d_G(u, w_j) \neq d_G(v, w_j)$. Then by Lemma 2 $\rho(u, w_j) \neq \rho(v, w_j)$. So, vertices $u$,$v$ are resolved by $w_j$. Hence, $W$ is a resolving set of the space $(L_m, \rho)$.

Assume that there is no inner vertex $v_1$ of the level $r$ which is close to two leaves $l_1$ and $l_2$, such that there are no another inner vertices on levels $0, \ldots, r$. We need to show that the set $W$ is an inclusion-wise minimal resolving set. Indeed, let $W' = W \setminus \{w_i\}$ for some $i$, $0 \leq i \leq l$. As $W$ is a basis of the tree $T_m$, there exists a pair of leaves $u$, $v$ that is resolved by $w_i$ and is not resolved by any other $w_j$, $i \neq j$, $0 \leq j \leq l$. In other words, $d_G(u, w_j) = d_G(v, w_j)$ for all $w_j \in W'$. So, by Lemma 2 the equality $\rho(u, w_j) = \rho(v, w_j)$ holds for all $w_j \in W'$, $i \neq j$, $0 \leq j \leq l$. Hence, $W'$ is not a resolving set of $(L_m, \rho)$. Therefore, $W$ is a resolving set with minimum cardinality of $(L_m, \rho)$ and $md(L_m, \rho) = |W| = md(T_m)$.

Let now in tree $T_m$ there exists an inner vertex $v_1$ of the level $r$ which is close to two leaves $l_1$ and $l_2$ and there are no another inner vertices on levels $0, \ldots, r$. Note that the root $v_0$ is vertex of degree 2 in this case. There exists an inner vertex $v_1$ of the level $r$ which is close to two leafs $l_1$ and $l_2$ and there are no another inner vertices on levels $0, \ldots, r$. Without loss of generality we may assume that $w_1 = l_1$. Denote $\hat{W} = \{w_2, \ldots, w_l\}$. The set $\hat{W}$ is a metric basis of $(L_m, \rho)$. Indeed, for any leaf $u$ of tree $T_m$, $u \neq l_1$, $\rho(l_1, u) = diam\ L_m$. So, $l_1$ don't resolve any pair of points of $L_m$. Hence, $\hat{W}$ is resolving set of $(L_m, \rho)$. The proof of minimallity of cardinality $\hat{W}$ is similar to the reasoning above. This completes the proof of Theorem 2. □

**Remark 1.** Note that if the assertion in Proposition 1 for tree $T_m$ is true, then

$$md(L_m, \rho) = |L_m| - |L_{m-1}|.$$

## 3. Metric dimension of ultrametric spaces.

Let us recall well-known properties of ultrametric spaces, that are derived directly from the definition.

(A) Every point inside a ball of an ultrametric space is its center.

(B) The intersection of two balls of an ultrametric space equals exactly one of the balls or empty space.

**Theorem 3.** *Let $(X, d)$ be a finite ultrametric space, $|X| = n > 3$. There exists a algorithm of construction of rooted tree $T_m$ and a defining set $\alpha_0, \alpha_1, \ldots, \alpha_m$ with a complexity $O(n^5)$ such that metric spaces $(X, d)$ and $(L_m, \rho)$ are isometric.*

Proof. For proof of this theorem we provide an Algorithm 1 of construction of rooted tree $T_m$ and calculate its complexity. In the algorithm we define the ultrametric space $(X, d)$ by distance matrix $M_X$. On step 2 we determine a defining set of tree $T_m$. In steps 3–4 we define an empty rooted tree $T_m$ and represent the space $X$ as a root of the tree $T_m$. After that, we decompose the space $X$ into a disjoint union of balls of radius $\alpha_{m-1}$. $L^l_{m-1}$ denotes the $l$-th vertex of the 1-st level of tree $T_m$. Any vertex on the 1-st level is defined by a

**Input**: finite ultrametric space $(X, d)$ that is defined by distance matrix $M_X$.
**Output**: tree $T_m$ that is defined as a label tree.

From matrix $M_X$ get a set $D$ of all different distances in the space $(X, d)$;
Order the elements of the set $D$: $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = diamX$;
Define an empty rooted tree $T_m$;
$L_0^1 := X$, $k_0 = 1$. Define a $L_0^1$ as a root of tree $T$;
**for** $q = 0$ *to* $m - 1$ **do**
  $k_{q+1} := 0$ ;
  **for** $i := 1$ *to* $k_q$ **do**
    Define $R := L_q^i$;
    **if** $|R| = 1$ **then**
      $k_{q+1} := k_{q+1} + 1$;
      $L_{q+1}^{k_{q+1}} := L_q^i$;
      Define $L_{q+1}^{k_{q+1}}$ as a vertex of $q + 1$-th level of tree $T$. Connect by edge $L_q^i$ and $L_{q+1}^{k_{q+1}}$;
    **else**
      $u_q := 0$;
      **while** $R$ *is not empty* **do**
        Fixed $x_i^q \in R$;
        Define $L_{q+1}^{i+u_q} = \{a \in L_q^i \mid d(x_i^q, a) < \alpha_{m-q}\}$;
        $R := R \setminus L_{q+1}^{i+u_q}$;
        $u_q := u_q + 1$;
        $k_{q+1} := k_{q+1} + 1$;
        Define $L_{q+1}^{i+u_q}$ as a vertex of $q + 1$-th level of tree $T$. Connect by edge $L_q^i$ and $L_{q+1}^i$;
      **end**
    **end**
  **end**
  $q := q + 1$;
**end**

Algorithm 1. Construction of rooted tree

closed ball with radius $\alpha_{m-l}$ of space $X$. Property (B) implies that this definition is correct. We connect the root $X$ with vertices $L_{m-1}^l$ by edges. After that, we decompose any vertex of the 1-st level to a union of balls of radius $\alpha_{m-2}$ and connect new vertices by edges, and so on.

We would like to construct a tree $T_m$ with leaves of the level $m$. So, in steps 9–13 we check the number of elements in ball $L_q^i$. If $L_q^i$ consists only of one element, we define the vertex of the level $(i+1)$ as ball $L_q^i$.

As a result of the algorithm, we will get a tree $T$. All leaf nodes (vertices of the level $m$) of this tree $T$ are represented by one-element subsets of the set $X$.

Let us show that $(L_m, \rho)$ and $(X, d)$ are isometric. Define a function $f : \{x\} \to x$ from $(L_m, \rho)$ to $(X, d)$. The function $f$ is an isometry. Indeed, $f$ is a bijection, and the distance between nodes $\{x\}$ and $\{y\}$ equals $\alpha_l$ if and only if the joint part of paths connecting the root $v_0$ with $\{x\}$ and $\{y\}$ has the length $m - l$. But this means that there exists a closed ball $B(x, \alpha_l)$ in space $X$, such that $y \in B(x, \alpha_l)$, but $y \notin B(x, \alpha_{l-1})$. So, the distance between points $x$ and $y$ in the space $X$ equals $\alpha_l$ and $f$ is an isometry between $L_m$ and $X$.

Let $|X| = n$. First, note that $m < \dfrac{n(n-1)}{2}$. The complexity of steps 1–4 is $O(n^2)$. There are three loops in steps 4–23 of the algorithm. The number of iterations for these loops is bounded by $m$, $n$ and $n^2$ respectively. So, the final complexity won't exceed $O(n^3 \cdot m)$ or $O(n^5)$. Therefore, the algorithm has polynomial complexity. □

We say that the root tree $T_m$ *represents* the metric space $(X, d)$.

**Corollary 1.** *Let $(X, d)$ be a finite ultrametric space, $T_m$ be the rooted tree representing it. Assume that for any $x \in X$ there exists $y \in X$ such that $d(x, y) = \alpha_1$. Then the metric dimension of the space $X$ can be calculated by the formula:*

$$md(X) = |L_m| - |L_{m-1}|.$$

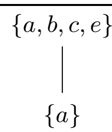
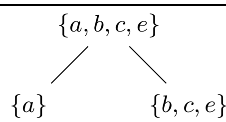
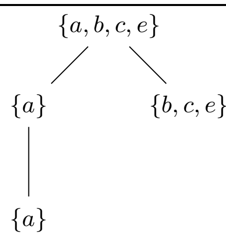
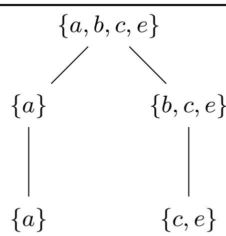
P r o o f. From Theorem 3, it follows that there exists a tree $T_m$ such that the finite ultrametric spaces $(X, d)$ and $(L_m, \rho)$ are isometric. As for any $x \in X$ there exists $y \in X$ such that $d(x, y) = \alpha_1$, all vertices of the $(m-1)$th level of tree $T_m$ are inner. Then by Remark 1 $md(L_m, \rho) = |L_m| - |L_{m-1}|$. Therefore, $md(X) = |L_m| - |L_{m-1}|$. □

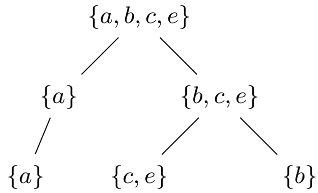
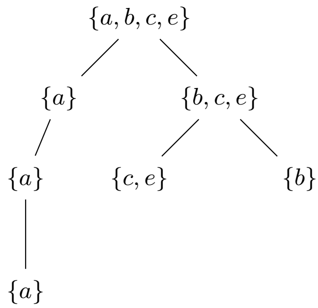
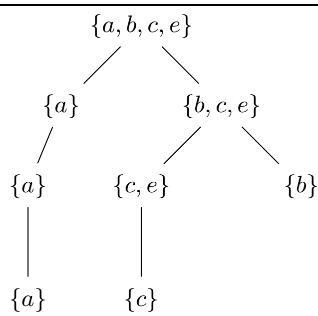
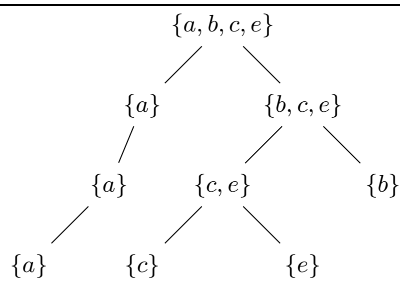
**Example 1.** Assume, that we have the ultrametric space $(X, d)$, which can be represented by distance matrix

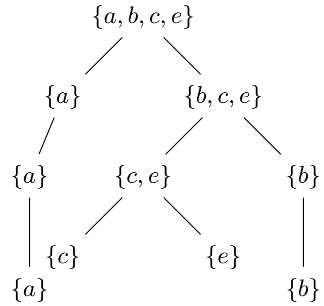
$$M_X = \begin{pmatrix} & a & b & c & d \\ a & 0 & 3 & 3 & 3 \\ b & 3 & 0 & 2 & 2 \\ c & 3 & 2 & 0 & 1 \\ d & 3 & 2 & 1 & 0 \end{pmatrix}$$

Let us construct its graph representation using the proposed algorithm and find its metric dimension.

At the beginning, we will have only the root node $L_0^1 = \{a, b, c, e\}$. The next iterations will be represented by the following table:

| $q$ | $i$ | $k_q$ | $u_q$ | $R$ | $x_i^q$ | $L_{q+1}^{i+u_q}$ | $R \setminus L_{q+1}^{i+u_q}$ | $k_{q+1}$ | Graph |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | $\{a, b, c, e\}$ | $a$ | $L_1^1 = \{a\}$ | $\{b, c, e\}$ | 1 |  |
| 0 | 1 | 1 | 1 | $\{b, c, e\}$ | $b$ | $L_1^2 = \{b, c, e\}$ | $\{\emptyset\}$ | 2 |  |
| 1 | 1 | 2 | 0 | $\{a\}$ | – | $L_2^1 = \{a\}$ | – | 1 |  |
| 1 | 2 | 2 | 0 | $\{b, c, e\}$ | $c$ | $L_2^2 = \{c, e\}$ | $\{b\}$ | 2 |  |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | $\{b\}$ | $b$ | $L_2^3 = \{b\}$ | $\{\emptyset\}$ | 3 |   |
| 2 | 1 | 3 | 0 | $\{a\}$ | – | $L_3^1 = \{a\}$ | – | 1 |   |
| 2 | 2 | 3 | 0 | $\{c, e\}$ | $c$ | $L_3^2 = \{c\}$ | $\{e\}$ | 2 |   |
| 2 | 2 | 3 | 1 | $\{e\}$ | $d$ | $L_3^3 = \{e\}$ | $\{\emptyset\}$ | 3 |   |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 3 | 0 | $\{b\}$ | – | $L_3^4 = \{b\}$ | – | 4 |   |

According to Theorem 3 a metric dimension of the metric space $(X, d)$ will be $md(X, d) = 1$.

P r o o f o f T h e o r e m 1. From Theorem 3, it follows that the finite ultrametric space $(X, d)$ is isometric to the metric space $(L_m, \rho)$ defined on a tree $T_m$. Moreover, the tree $T_m$ and the metric space $(L_m, \rho)$ can be constructed in polynomial time. Following Theorem 2, finding the metric dimension of a metric space $(L_m, \rho)$ is equivalent to finding the metric dimension of the tree $T_m$. However, the metric dimension of any tree can be found in polynomial time (see [18]). Therefore, for any ultrametric space, the finding of its metric dimension is polynomial. □

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