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THE NON-EXISTENCE OF $[383, 5, 286]$ AND $[447, 5, 334]$ QUATERNARY LINEAR CODES

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Abstract. It is known that $n_4(5, 286) = 383$ or 384 and $n_4(5, 334) = 447$ or 448, where $n_q(k, d)$ is the minimum length n for which an $[n, k, d]_q$ code exists. We prove the non-existence of $[383, 5, 286]_4$ and $[447, 5, 334]_4$ codes, which determine the exact value of $n_4(5, d)$ for $d = 286, 334$.

1. Introduction. We denote by $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. An $[n, k, d]_q$ code $\mathcal{C}$ is a linear code over $\mathbb{F}_q$ of length n and dimension k with minimum Hamming distance d. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. We only consider *non-degenerate* codes having no coordinate which is identically zero. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists. This problem is sometimes called the

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optimal linear codes problem, see [3, 4]. A well-known lower bound on $n_q(k,d)$, called the Griesmer bound, says:

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k. For quaternary linear codes, $n_4(k,d)$ is known for $k \leq 4$ for all d, see [1, 10, 16].

Theorem 1.1 ([8]). *$n_4(4,d) = g_4(4,d)+1$ for $d = 3,4,7,8,13\text{-}16,23\text{-}32,37\text{-}44,77\text{-}80$ and $n_4(4,d) = g_4(4,d)$ for any other d.*

As for the case $k = 5$, the value of $n_4(5,d)$ is unknown for many integer d although the Griesmer bound is attained for all $d \geq 369$, see [16]. It is known that $n_4(5,d) = g_4(5,d)$ or $g_4(5,d)+1$ for $d = 286, 334$. Our purpose is to prove the following theorems.

Theorem 1.2. *There exists no $[383,5,286]_4$ code and $n_4(5,286) = 384$.*

Theorem 1.3. *There exists no $[447,5,334]_4$ code and $n_4(5,334) = 448$.*

If there exists an $[n+1,k,d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called extendable and $\mathcal{C}'$ is an extension of $\mathcal{C}$. Yoshida and Maruta [20] proved the following Theorem.

Theorem 1.4 ([20]). *Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $k \geq 3$, $d \equiv -2 \pmod 4$. $\mathcal{C}$ is extendable if*

$$\sum_{i \equiv -3 \pmod 4} A_i = 0 \text{ and } \sum_{i \equiv -1 \pmod 4} A_i > 0.$$

But this theorem cannot be applied for quaternary codes which have no codeword with weight i congruent to -1 modulo 4. Other theorem for quaternary codes with three weights modulo 4 need some restriction on A_i, such as $\sum_{4|i} A_i = 4^{k-2}$ [20]. We give the following theorem which does not have such a restriction.

Theorem 1.5. *Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $k \geq 3$, $d \equiv -2 \pmod{4^2}$, such that $A_i = 0$ for all $i \not\equiv 0, -2 \pmod{4^2}$. Then $\mathcal{C}$ is extendable.*

Our application of Theorem 1.5 proves Theorem 1.2 and 1.3.

2. Preliminaries. In this section, we give the geometric method through $\mathrm{PG}(r, q)$, the projective geometry of dimension r over $\mathbb{F}_q$, and preliminary results to prove the main results. A j-flat is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. The 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats in $\mathrm{PG}(r, q)$ are called *points, lines, planes, secundums* and *hyperplanes*, respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r, q)$ and denote by θ_j the number of points in a j-flat, i.e. $\theta_j = (q^{i+1} - 1)/(q - 1)$. We set $\theta_{-1} = 0$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\mathcal{P} = \mathrm{PG}(k-1, q)$, denoted by $\mathcal{M}_\mathcal{C}$. An *i-point* is a point of $\mathcal{P}$ which has multiplicity i in $\mathcal{M}_\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from $\mathcal{P}$ in $\mathcal{M}_\mathcal{C}$ and let $\mathcal{P}_i$ be the set of i-points in $\mathcal{P}$, $0 \le i \le \gamma_0$. For any subset S of $\mathcal{P}$, *the multiplicity of* S *with respect to* $\mathcal{M}_\mathcal{C}$, denoted by $m_\mathcal{C}(S)$, is defined as

$$m_\mathcal{C}(S) = \sum_i i \cdot |S \cap \mathcal{P}_i|,$$

where $|T|$ denotes the number of elements in a set T. A line ℓ with $t = m_\mathcal{C}(\ell)$ is called a *t-line*. A *t-plane* and so on are defined similarly. Then we obtain the partition $\mathcal{P} = \bigcup_{i=0}^{\gamma_0} \mathcal{P}_i$ such that

$$\begin{aligned} n &= m_\mathcal{C}(\mathcal{P}), \\ n - d &= \max\{m_\mathcal{C}(H) \mid H \in \mathcal{H}\}, \end{aligned}$$

where $\mathcal{H}$ denotes the set of hyperplanes of $\mathcal{P}$. For a t-flat Π in $\mathcal{P}$, we define

$$\gamma_j(\Pi) = \max\{m_\mathcal{C}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\} \text{ for } 0 \le j \le t.$$

We denote simply by γ_j instead of $\gamma_j(\mathcal{P})$. Then $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. For a Griesmer $[n, k, d]_q$ code, it is known (see [13]) that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \le j \le k-1. \tag{2.1}$$

We denote by λ_s the number of s-points in $\mathcal{P}$. When $\gamma_0 = 2$, we have

$$\lambda_2 = \lambda_0 + n - \theta_{k-1}. \tag{2.2}$$

Lemma 2.1 ([14]). *$\mathcal{C}$ is extendable if and only if there exists a point $P \in \mathcal{P}$ such that $m_{\mathcal{C}}(H) < n-d$ for all hyperplanes H through P.*

Let $\mathcal{P}^*$ be the dual space of $\mathcal{P}$ (Considering $\mathcal{H}$ as the set of points of $\mathcal{P}^*$). Then Lemma 2.1 is equivalent to the following.

Lemma 2.2 ([14]). *$\mathcal{C}$ is extendable if and only if there exists a hyperplane H of $\mathcal{P}^*$ such that*

$$H \subset \{\pi \in \mathcal{H} \mid m_{\mathcal{C}}(\pi) < n-d\}.$$

Denote by a_i the number of i-hyperplanes in $\mathcal{P}$. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of $\mathcal{P}$ to distinguish from the spectrum of $\mathcal{C}$. Let θ_j be the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$. Simple counting arguments yield the following.

Lemma 2.3 ([11]). (a) $\sum_{i=0}^{n-d} a_i = \theta_{k-1}$. (b) $\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}$.
(c) $\sum_{i=2}^{n-d} \binom{i}{2} a_i = \binom{n}{2}\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0}\binom{s}{2}\lambda_s$.

When $\gamma_0 \le 2$, the above three equalities yield the following:

$$\sum_{i=0}^{n-d-2}\binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{2.3}$$

Lemma 2.4 ([19]). *Let H be an i-hyperplane through a t-secundum δ. Then*

(a) $t \le \gamma_{k-2} - (n-i)/q = (i + q\gamma_{k-2} - n)/q$.

(b) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

(c) $\gamma_{k-3}(H) = \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ *does not exist.*

(d) *Let* c_j *be the number of* j*-hyperplanes through* δ *other than* H. *Then* $\sum_j c_j = q$ *and*

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \tag{2.4}$$

(e) *For a* γ_{k-2}*-hyperplane* H_0 *with spectrum* $(b_0, \cdots, b_{\gamma_{k-3}})$, $b_t > 0$ *holds if* $i + q\gamma_{k-2} - n - qt < q$.

Lemma 2.5 ([9]). *Let* H *be an* i*-hyperplane and let* $\mathcal{C}_H$ *be an* $[i, k-1, d_0]$ *code generated by* $\mathcal{M}_\mathcal{C}(H)$. *If any* γ_{k-2}*-hyperplane has no* t*-secundum with* $t = \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$, *then* $d_0 \geq i - t + 1$.

3. Proof of Theorem 1.5.

In this section, we prove Theorem 1.5. For a secundum σ, it is easy to see that $\sum_{H \in \mathcal{H}, H \supset \sigma} m_\mathcal{C}(H) = n + q m_\mathcal{C}(\sigma)$ for an $[n, k, d]_q$ code $\mathcal{C}$. This equality is generalized to the following.

Lemma 3.1 ([7]). *Let* S *be an* s*-flat in* $\mathcal{P}$. *For hyperplanes* H *through* S, *it holds that*

$$\sum_{H \in \mathcal{H}, H \supset S} m_\mathcal{C}(H) = n\theta_{k-s-3} + q^{k-s-2} m_\mathcal{C}(S). \tag{3.1}$$

From now on, we only consider the case when $q = 4$. For an $[n, k, d]_4$ code $\mathcal{C}$, let

$$M_i = \{H \in \mathcal{H} \mid m_\mathcal{C}(H) \equiv n - i \pmod 4\}.$$

Since the spectrum of $\mathcal{C}$ satisfies $a_i = A_{n-i}/3$ for $0 \leq i \leq n - d$, we have

$$|M_i| = \frac{1}{3} \sum_{0 < j \equiv i \pmod 4} A_j.$$

P r o o f o f T h e o r e m 1.5. Let $\mathcal{C}$ be an $[n, k, d]_4$ code, $d \equiv -2 \pmod{16}$, whose weights of codewords are congruent to $0, -2$ modulo 16. Then M_0 and M_2 are defined by

$$M_0 = \{H \in \mathcal{H} \mid m_\mathcal{C}(H) \equiv n \pmod{16}\},$$

Table 1. The possible φ_{0t}-flats with spectra [18].

φ_0	$c^{(t)}_{\theta_{t-1}}$	$c^{(t)}_{\theta_{t-2}}$	$c^{(t)}_{\theta_{t-2}+2\cdot 4^{t-2}}$
θ_t	θ_t	0	0
θ_{t-1}	1	$\theta_t - 1$	0
$\theta_{t-1} + 2 \cdot 4^{t-1}$	3	2	$\theta_t - \theta_1$

$$M_2 \quad = \quad \{H \in \mathcal{H} \mid m_{\mathcal{C}}(H) \equiv n+2 \pmod{16}\}.$$

Let π be a plane of $\mathcal{P}^*$, which is a $(k-4)$-flat of $\mathcal{P}$. Then the following equations

$$\begin{aligned} |\pi \cap M_0| + |\pi \cap M_2| &= 21, \\ n|\pi \cap M_0| + (n+2)|\pi \cap M_2| &\equiv n\theta_1 \pmod{16} \end{aligned}$$

imply that $|\pi \cap M_0| = 21, 5, 13$. Then, for a t-flat Π, the possible spectra of $M_0 \cap \Pi$ are as in Table 1, where $c^{(t)}_j$ denotes the number of $(t-1)$-flats Δ in Π with $|\Delta \cap M_0| = j$, see [18].

From Table 1, $\mathcal{P}^*$ has a hyperplane which is contained in M_0. Hence $\mathcal{C}$ is extendable by Lemma 2.2. □

4. Non-existence of $[383, 5, 286]_4$ codes and $[447, 5, 334]_4$ codes. In this section, we demonstrate how to apply Theorem 1.5 to prove that $n_4(5, 286) = 384$ and $n_4(5, 334) = 448$. Since there exist a $[384, 5, 286]_4$ code and a $[448, 5, 334]_4$ code, we need to prove the non-existence of $[383, 5, 286]_4$ codes and $[447, 5, 334]_4$ codes to show that $n_4(5, d) = g_4(5, d) + 1$ for $d = 286, 334$. We use the following lemmas.

Lemma 4.1 ([12])**.** *The spectrum of a* $[97, 4, 72]_4$ *code satisfies*
$a_i = 0$ *for all* $i \notin \{9, 13, 17, 21, 25\}$.

Lemma 4.2 ([17])**.** *The spectrum of a* $[113, 4, 84]_4$ *code satisfies*
$a_i = 0$ *for all* $i \notin \{1, 9, 13, 17, 21, 25, 27, 29\}$.

Landjev and Rousseva [12] proved the non-existence of a $[384, 5, 287]_4$ code. Moreover their result [17] shows that there exists no $[448, 5, 335]_4$ code. So it suffices to prove that a putative $[383, 5, 286]_4$ code and a $[447, 5, 334]_4$ are extendable.

Table 2. The spectra of some $[n,4,d]_4$ codes

parameters	possible spectra	reference
$[75,4,56]_4$	$(a_{15},a_{19})=(10,75)$	[11]
$[82,4,61]_4$	$(a_{18},a_{20},a_{21})=(5,48,32)$	[2]
	$(a_{18},a_{19},a_{20},a_{21})=(1,12,36,36)$	[2]
$[83,4,62]_4$	$(a_{19},a_{20},a_{21})=(5,32,48)$	[2]
$[84,4,63]_4$	$(a_{20},a_{21})=(21,64)$	[2]
$[85,4,64]_4$	$(a_{21})=(85)$	[2]

Proof of Theorem 1.2. Let $\mathcal{C}$ be a putative $[383,5,286]_4$ code, where $383=g_4(5,286)$. Then, we have $\gamma_0=2$, $\gamma_1=7$, $\gamma_2=25$, $\gamma_3=97$ from (2.1). Let Δ_{97} be a 97-hyperplane. From Lemma 4.1, a j-secundum on Δ_{97} satisfies $j\in\{9,13,17,21,25\}$. If there exists a 1-hyperplane, then one can find a 1-secundum in the hyperplane. Setting $(w,t)=(1,1)$, any solution of (2.4) satisfies $c_{97}>0$, which contradicts the fact that a 97-hyperplane has no 1-secundum. If there exists a 74-hyperplane, then one can find a 19-secundum there as well. But (2.4) has no solution for $(w,t)=(74,19)$, a contradiction. If there exists a 30-hyperplane, then it corresponds to $[30,5,22]_4$ code by Lemma 2.4 (a), which does not exist by [16]. In the same manner, we can show $a_j=0$ for all $i\notin$ {31, 47-49, 63-65, 75, 79-85, 95-97} using Lemmas 2.4, 2.5, the Griesmer bound and [16], since an i-hyperplane Δ_i can not meet Δ_{97} in t-secundum with $t\notin\{9,13,17,21,25\}$. This procedure is called the **first sieve** in the proofs of the non-existence results [9].

From (2.3), we get

$$(4.1)\qquad 64\lambda_2=\sum_{i=31}^{95}\binom{97-i}{2}a_i+1371.$$

For any i-solid through a t-plane, (2.4) gives

$$(4.2)\qquad \sum_j(97-j)c_j=i+5-4t$$

with $\sum_j c_j=4$. Since an i-hyperplane Δ_i can not meet Δ_{97} in a t-solid with $t\notin\{9,13,17,21,25\}$ for $i=75$ by Table 2 and Lemma 4.1, we have $a_{75}=0$.

Suppose $a_{82} > 0$ and let Δ_{82} be a 82-hyperplane. Setting $w = 82$, the equation (4.2) has no solution for $t = 19$ since a 97-hyperplane contains no 19-solid. Then, the spectrum of Δ_{82} is just $(\tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}) = (5, 0, 48, 32)$ from Table 2. The maximum possible contributions of c_j's to the LHS of (4.1) are $(c_{85}, c_{96}) = (1, 3)$ for $t = 18$, $(c_{95}, c_{96}) = (3, 1)$ for $t = 20$, $(c_{95}, c_{96}, c_{97}) = (1, 1, 2)$ for $t = 21$, respectively. Estimating the LHS of (4.1) for the spectrum of Δ_{82}, we get

$$\lambda_2 \leq \frac{66\tau_{18} + 3\tau_{20} + 1\tau_{21} + 105 + 1371}{64} = \frac{1982}{64} = 30.9688\ldots.$$

This contradicts that $\lambda_2 \geq 42$ from (2.2). Hence $a_{82} = 0$.

Hence, the weights of codewords of $\mathcal{C}$ are congruent to $0, -1, -2$ modulo 4. If $\mathcal{C}$ has a codewords whose weight is congruent to -1 modulo 4, then $\mathcal{C}$ is extendable by Theorem 1.4, which contradicts the non-existence of a $[383, 5, 286]_4$ code. Hence $A_i = 0$ for $i \equiv -1 \pmod 4$, and we have $a_j = 0$ for all $j \notin \{31, 47, 49, 63, 65, 75, 79, 81, 83, 85, 95, 97\}$.

Suppose $a_{85} > 0$ and let Δ_{85} be a 85-hyperplane. Then, the spectrum of Δ_{85} is $\tau_{21} = 85$ from Table 2. Setting $w = 85$, the maximum possible contribution of c_j's to the LHS of (4.1) is $(c_{95}, c_{97}) = (3, 1)$ for $t = 21$. Estimating the LHS of (4.1) for the spectrum of Δ_{85}, we get

$$\lambda_2 \leq \frac{3\tau_{21} + 66 + 1371}{64} = \frac{1692}{64} = 26.4375\ldots.$$

This contradicts that $\lambda_2 \geq 42$ from (2.2). Hence $a_{85} = 0$.

Suppose $a_{83} > 0$ and let Δ_{83} be a 83-hyperplane. Then, the spectrum of Δ_{83} is $(\tau_{19}, \tau_{20}, \tau_{21}) = (5, 32, 48)$ from Table 2. Setting $w = 83$, the equation (4.2) has no solution for $t = 19$ since a 97-hyperplane contains no 19-plane by Lemma 4.1, a contradiction. Hence $a_{83} = 0$.

Now, $a_i = 0$ for all $i \notin \{31, 47, 49, 63, 65, 79, 81, 95, 97\}$. Hence the weight distribution of $\mathcal{C}$ satisfies that $A_i = 0$ for all $i \not\equiv 0, -2 \pmod{16}$. Thus, $\mathcal{C}$ is extendable by Theorem 1.5, which contradicts the non-existence of a $[383, 5, 286]_4$ code. Thus Theorem 1.2 is proved. □

Proof of Theorem 1.3. Let $\mathcal{C}$ be a putative $[447, 5, 334]_4$ code, where $447 = g_4(5, 334)$. Then, we have $\gamma_0 = 2$, $\gamma_1 = 8$, $\gamma_2 = 29$, $\gamma_3 = 113$ from (2.1). Let Δ_{113} be a 113-hyperplane. From Lemma 4.2, a j-secundum on Δ_{113} satisfies $j \in \{1, 9, 13, 17, 21, 25, 27, 29\}$. By the first sieve, we have $a_j = 0$ for all

$i \notin \{1, 31, 47\text{-}49,\ 63\text{-}65,\ 75,\ 79\text{-}85,\ 95\text{-}97,\ 103,\ 111\text{-}113\}$. From (2.3), we get

$$64\lambda_2 = \sum_{i=47}^{111} \binom{113-i}{2} a_i + 4291. \tag{4.3}$$

For any i-solid through a t-plane, (2.4) gives

$$\sum_j (113-j)c_j = i + 5 - 4t \tag{4.4}$$

with $\sum_j c_j = 4$. Since an i-hyperplane Δ_i can not meet Δ_{113} in a t-solid with $t \notin \{1, 9, 13, 17, 21, 25, 27, 29\}$ for $i = 75$ by Table 2 and Lemma 4.2, we have $a_{75} = 0$.

Suppose $a_{82} > 0$ and let Δ_{82} be a 82-hyperplane. Setting $i = 82$, the equation (4.4) has no solution for $t = 19$. Then, the spectrum of Δ_{82} is just $(\tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}) = (5, 0, 48, 21)$ from Table 2. The maximum possible contributions of c_j's to the LHS of (4.3) are $(c_{103}, c_{111}, c_{112}) = (1, 2, 1)$ for $t = 18$, $(c_{111}, c_{112}) = (3, 1)$ for $t = 20$, $(c_{111}, c_{112}, c_{113}) = (1, 1, 2)$ for $t = 21$, respectively. Estimating the LHS of (4.3) for the spectrum of Δ_{82}, we get

$$\lambda_2 \le \frac{47\tau_{18} + 3\tau_{20} + 1\tau_{21} + 465 + 4291}{64} = 80.7344\ldots.$$

This contradicts that $\lambda_2 \ge 106$ from (2.2). Hence $a_{82} = 0$.

Hence, the weights of codewords of $\mathcal{C}$ are congruent to $0, -1, -2$ modulo 4. If $\mathcal{C}$ has a codeword whose weight is congruent to -1 modulo 4, then $\mathcal{C}$ is extendable by Theorem 1.4, which contradicts the non-existence of a $[447, 5, 334]_4$ code. Hence $A_i = 0$ for $i \equiv -1 \pmod 4$, and we have $a_j = 0$ for all $j \notin \{47, 49, 63, 65, 79, 81, 83, 85, 95, 97,\ 103, 111, 113\}$.

Suppose $a_{85} > 0$ and let Δ_{85} be a 85-hyperplane. Then, the spectrum of Δ_{85} is $\tau_{21} = 85$ from Table 2. Setting $i = 85$, the maximum possible contribution of c_j's to the LHS of (4.3) is $(c_{111}, c_{113}) = (3, 1)$ for $t = 21$. Estimating the LHS of (4.3) for the spectrum of Δ_{85}, we get

$$\lambda_2 \le \frac{3\tau_{21} + 378 + 4291}{64} = \frac{4924}{64} = 76.9375\ldots.$$

which contradicts that $\lambda_2 \ge 106$ from (2.2). Hence $a_{85} = 0$.

Suppose $a_{83} > 0$ and let Δ_{83} be a 83-hyperplane. Then, the spectrum of Δ_{83} is $(\tau_{19}, \tau_{20}, \tau_{21}) = (5, 32, 48)$ from Table 2. Setting $i = 83$, the equation (4.4) has no solution for $t = 19$ since a 97-hyperplane and a 113-hyperplane contain no 19-plane by Lemmas 4.1 and 4.2, a contradiction. Hence $a_{83} = 0$.

Suppose $a_{103} > 0$ and let Δ_{103} be a 103-hyperplane. Let $H_1, \ldots, H_4$ be hyperplanes ($\neq \Delta_{103}$) through a plane π of Δ_{103} with $h_i = m_{\mathcal{C}}(H_i)$, $h_1 \leq \cdots \leq h_4$. The equation (3.1) with $s = k - 3$ gives

$$4m_{\mathcal{C}}(\pi) = m_{\mathcal{C}}(\Delta_{103}) + \sum_{i=1}^{4} h_i - 447. \tag{4.5}$$

Considering the spectra of 97-hyperplane and 113-hyperplane, we computed the possible values $h_1, \ldots, h_4$ of (4.5) and get Table 3. The first column shows the multiplicity of plane π of Δ_{103}, the second column shows $h_1, \ldots, h_4$ and the last column shows $n - h_i \pmod 4$, $i = 1, 2, 3, 4$. This implies that a 103-hyperplane has no plane π with even $m_{\mathcal{C}}(\pi) \leq 10$ as follows. Assume that Δ_{103} has a plane π_0 whose multiplicity is even. Let ℓ be a line in π_0. The planes $\pi_1, \pi_2, \pi_3, \pi_4$ through ℓ in Δ_{103} satisfy that

$$\begin{aligned} 4m_{\mathcal{C}}(\ell) &= m_{\mathcal{C}}(\pi_0) + \sum_{i=1}^{4} m_{\mathcal{C}}(\pi_i) - 103 \\ 0 &\equiv 0 + \sum_{i=1}^{4} m_{\mathcal{C}}(\pi_i) + 1 \pmod 2 \end{aligned}$$

This implies that $m_{\mathcal{C}}(\pi_1) \equiv 0$, that is, $m_{\mathcal{C}}(\pi_1) \leq 10$. Thus,

$$4m_{\mathcal{C}}(\ell) \leq 2 \cdot 10 + 3 \cdot 27 - 103 = -2,$$

a contradiction. Hence, there exists no plane with even multiplicity in Δ_{103}. The existence of 103-hyperplanes leads that $|M_2 \cap \{\pi_1, \pi_2, \pi_3, \pi_4\}| = 4$ where only $m_{\mathcal{C}}(\pi) = 27$. Moreover, it leads that $a_i = 0$ for $447 - i \equiv -2 \pmod{16}$ except for $i = 113$. Setting $i = 113$, the equation (3.1) with $s = k - 3$ is the following.

$$4m_{\mathcal{C}}(\pi) = 113 + \sum_{i=1}^{4} h_i - 447. \tag{4.6}$$

Now, the multiplicity of π equals $13, 17, 21, 25, 27$ or 29 and $a_j = 0$ for all $j \notin \{1, 31, 47, 49, 63, 65, 79, 81, 95, 97, 103, 111, 113\}$, i.e., $j \equiv 1, -1, 7 \pmod{16}$. If

Table 3. The possible solutions for (4.4)

$m_{\mathcal{C}}(\pi)$	h_1 h_2 h_3 h_4	M_0 or M_2
0	49 81 103 111	2 2 0 0
0	65 65 103 111	2 2 0 0
0	65 81 95 103	2 2 0 0
0	79 81 81 103	0 2 2 0
1	31 95 111 111	0 0 0 0
1	31 103 103 111	0 0 0 0
1	47 79 111 111	0 0 0 0
1	47 95 95 111	0 0 0 0
1	47 95 103 103	0 0 0 0
1	63 63 111 111	0 0 0 0
1	63 79 95 111	0 0 0 0
1	63 79 103 103	0 0 0 0
1	63 95 95 95	0 0 0 0
1	79 79 79 111	0 0 0 0
1	79 79 95 95	0 0 0 0
2	49 81 111 111	2 2 0 0
2	65 65 111 111	2 2 0 0
2	65 81 95 111	2 2 0 0
2	65 81 103 103	2 2 0 0
2	79 81 81 111	0 2 2 0
2	81 81 95 95	2 2 0 0
3	31 95 111 111	0 0 0 0
3	47 95 103 111	0 0 0 0
3	47 103 103 103	0 0 0 0
3	63 79 103 111	0 0 0 0
3	63 95 95 103	0 0 0 0
3	79 79 95 103	0 0 0 0
4	65 81 103 111	2 2 0 0
4	81 81 95 103	2 2 0 0
5	31 111 111 111	0 0 0 0
5	47 95 111 111	0 0 0 0
5	47 103 103 111	0 0 0 0
5	63 79 111 111	0 0 0 0
5	63 95 95 111	0 0 0 0
5	63 95 103 103	0 0 0 0
5	79 79 95 111	0 0 0 0
5	79 79 103 103	0 0 0 0
5	79 95 95 95	0 0 0 0
6	65 81 111 111	2 2 0 0
6	81 81 95 111	2 2 0 0
6	81 81 103 103	2 2 0 0
7	47 103 111 111	0 0 0 0
7	63 95 103 111	0 0 0 0
7	63 103 103 103	0 0 0 0
7	79 79 103 111	0 0 0 0
7	79 95 95 103	0 0 0 0
8	81 81 103 111	2 2 0 0
9	47 111 111 111	0 0 0 0
9	63 95 111 111	0 0 0 0
9	63 103 103 111	0 0 0 0
9	79 79 111 111	0 0 0 0
9	79 95 95 111	0 0 0 0
9	79 95 103 103	0 0 0 0
9	95 95 95 95	0 0 0 0
10	81 81 111 111	2 2 0 0
11	63 103 111 111	0 0 0 0
11	79 95 103 111	0 0 0 0
11	79 103 103 103	0 0 0 0
11	95 95 95 103	0 0 0 0
13	63 111 111 111	0 0 0 0
13	79 95 111 111	0 0 0 0
13	79 103 103 111	0 0 0 0
13	95 95 95 111	0 0 0 0
13	95 95 103 103	0 0 0 0
15	79 103 111 111	0 0 0 0
15	95 95 103 111	0 0 0 0
15	95 103 103 103	0 0 0 0
17	79 111 111 111	0 0 0 0
17	95 95 111 111	0 0 0 0
17	95 103 103 111	0 0 0 0
17	103 103 103 103	0 0 0 0
19	95 103 111 111	0 0 0 0
19	103 103 103 111	0 0 0 0
21	95 111 111 111	0 0 0 0
21	103 103 111 111	0 0 0 0
23	103 111 111 111	0 0 0 0
25	111 111 111 111	0 0 0 0
27	113 113 113 113	2 2 2 2

$m_{\mathcal{C}}(\pi) \equiv 1 \pmod 4$, the equation (4.6) modulo 16 is the following.

$$2 \equiv \sum_{i=1}^{4} h_i \pmod{16}.$$

From this congruence, there exists only one hyperplane H_i with $n - m_{\mathcal{C}}(H_i) \equiv 0 \pmod 4$. The condition $m_{\mathcal{C}}(\pi) \equiv 3 \pmod 4$ leads the same contributions, too. These imply that the set M_0 is a blocking set with respect to the lines of $\mathcal{P}^*$ and that $|M_0| = \theta_3$. Then M_0 is a hyperplane of $\mathcal{P}^*$, and $\mathcal{C}$ is extendable, a contradiction. Thus $a_{103} = 0$.

Now, $a_i = 0$ for all $i \notin \{1, 31, 47, 49, 63, 65, 75, 79, 95, 97, 111, 113\}$. Hence the weight distribution of $\mathcal{C}$ satisfies that $A_i = 0$ for all $i \not\equiv 0, -2 \pmod{16}$. Thus, $\mathcal{C}$ is extendable by Theorem 1.5, which contradicts the non-existence of a $[447, 5, 334]_4$ code. Thus Theorem 1.3 is proved. □

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