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ON THE CHOW GROUPS OF HYPERSURFACES IN SYMPLECTIC GRASSMANNIANS

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Abstract. Let Y be a Plücker hypersurface in a symplectic Grassmannian $I_1 \operatorname{Gr}(3, n)$ or a bisymplectic Grassmannian $I_2 \operatorname{Gr}(3, n)$. We show that many Chow groups of Y inject into cohomology.

1. Introduction. Given a smooth projective variety Y over $\mathbb{C}$, let $A_i(Y) := CH_i(Y)_{\mathbb{Q}}$ denote the Chow groups of Y (i.e. the groups of i-dimensional algebraic cycles on Y with $\mathbb{Q}$-coefficients, modulo rational equivalence). Let $A_i^{hom}(Y) \subset A_i(Y)$ denote the subgroup of homologically trivial cycles.

The famous Bloch–Beilinson conjectures [8], [25] predict that the Hodge level of the cohomology of Y should have an influence on the size of the Chow groups of Y. For surfaces, this is the notorious Bloch conjecture, which is still an open problem. For hypersurfaces in projective space, the precise prediction is as follows:

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Conjecture 1.1. *Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Then*

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le \frac{n}{d} - 1.$$

Conjecture 1.1 is still open; partial results have been obtained in [16], [23], [19], [5], [6].

In [14], I considered a version of Conjecture 1.1 for Plücker hyperplane sections of Grassmannians. In this note, we look at the case of Plücker hyperplane sections of *symplectic Grassmannians*. Recall that inside the Grassmannian $\mathrm{Gr}(3, n)$ (of 3-dimensional subspaces of an n-dimensional vector space), the symplectic Grassmannian $I_1 \mathrm{Gr}(3, n) \subset \mathrm{Gr}(3, n)$ parametrizes subspaces that are isotropic with respect to some fixed skew-symmetric 2-form. The precise prediction (cf. Subsection 3.2 below) is as follows:

Conjecture 1.2. *Let*

$$Y := I_1 \mathrm{Gr}(3, n) \cap H \quad \subset \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n - 4.$$

The main result of this note is a partial verification of Conjecture 1.2:

Theorem (= Theorem 3.4)**.** *Let*

$$Y := I_1 \mathrm{Gr}(3, n) \cap H \quad \subset \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n - 5.$$

Moreover, in case $n \le 10$ or $n = 12$ we have

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n - 4.$$

To prove Theorem 3.4, we rely on the recent notion of *projections* among (symplectic) Grassmannians [2]. Combined with the Chow-theoretic Cayley trick [9], this reduces Theorem 3.4 to understanding the Chow groups of a hyperplane section in an ordinary Grassmannian $\mathrm{Gr}(3, n+1)$. This last problem was handled in [14].

As a consequence of Theorem 3.4, some instances of the generalized Hodge conjecture are verified:

Corollary (= Corollary 4.1). *Let Y be as in Theorem 3.4, and assume $n \le 10$ or $n = 12$. Then $H^{\dim Y}(Y,\mathbb{Q})$ is supported on a subvariety of codimension $n-3$.*

Other consequences are as follows:

Corollary (= Corollary 4.2). *Let*

$$Y := I_1 \operatorname{Gr}(3,n) \cap H \quad \subset \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding).
(i) If $n \le 8$, then Y has finite-dimensional motive (in the sense of [11]).
(ii) If $n \le 9$, then Y has trivial Griffiths groups (and so Voevodsky's smash conjecture is true for Y).
(iii) If $n \le 10$, the Hodge conjecture is true for Y.

Applying the same method, we can also say something about hypersurfaces in *bisymplectic Grassmannians*. (Recall that the bisymplectic Grassmannian $I_2 \operatorname{Gr}(3,n) \subset \operatorname{Gr}(3,n)$ is the locus of 3-spaces that are isotropic with respect to two fixed generic skew-forms.)

Theorem (= Theorem 3.5). *Assume n is even, and let*

$$Y := I_2 \operatorname{Gr}(3,n) \cap H \quad \subset \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n-7.$$

Moreover, in case $n \le 10$ we have

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n-6.$$

The hyperplane sections $I_1 \operatorname{Gr}(3,9) \cap H$ and $I_2 \operatorname{Gr}(3,8) \cap H$ are of particular interest: they are Fano varieties of K3 type, and they are related to hyperplane sections $\operatorname{Gr}(3,10) \cap H$ and hence to Debarre–Voisin hyperkähler fourfolds, cf. [2, Section 4].

Conventions 1. *In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we denote by $A_j(Y) := CH_j(Y)_{\mathbb{Q}}$ the Chow group of j-dimensional cycles on Y with $\mathbb{Q}$-coefficients; for Y smooth of dimension n the notations $A_j(Y)$ and $A^{n-j}(Y)$ are* *used interchangeably. The notations* $A^j_{hom}(Y)$ *and* $A^j_{AJ}(X)$ *will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.*

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [20], [17]) will be denoted $\mathcal{M}_{\rm rat}$.

2. Preliminaries.

2.1. Cayley's trick and Chow groups.

Theorem 2.1 (Jiang [9]). *Let* $E \to U$ *be a vector bundle of rank* $r \geq 2$ *over a projective variety* U, *and let* $S := s^{-1}(0) \subset U$ *be the zero locus of a regular section* $s \in H^0(U,E)$ *such that* S *is smooth of dimension* $\dim U - \operatorname{rank} E$. *Let* $X := w^{-1}(0) \subset \mathbb{P}(E)$ *be the zero locus of the regular section* $w \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *that corresponds to* s *under the natural isomorphism* $H^0(U,E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$. *There is an isomorphism of integral Chow motives*

$$h(X) \cong h(S)(1-r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \quad \text{in } \mathcal{M}^{\mathbb{Z}}_{\rm rat}.$$

Proof. This is [9, Theorem 3.1]. Both the isomorphism and its inverse are explicitly described. □

Remark 2.2. In the set-up of Theorem 2.1, a cohomological relation between X, S and U was established in [12, Prop. 4.3] (cf. also [7, section 3.7], as well as [2, Proposition 46] for a generalization). A relation on the level of derived categories was established in [18, Theorem 2.10] (cf. also [10, Theorem 2.4] and [2, Proposition 47]).

2.2. Linear sections of $\mathrm{Gr}(2,n)$.

Proposition 2.3. *Let*

$$Y := \mathrm{Gr}(2,n) \cap H_1 \cap \cdots \cap H_s \quad \subset \ \mathbb{P}^{\binom{n}{2}-1}$$

be a smooth dimensionally transverse intersection with s *hyperplanes (with respect to the Plücker embedding). Assume* $s \leq 2$. *Then*

$$A_i^{hom}(Y) = 0 \quad \forall\ i.$$

Proof. This uses a geometric construction that can be found in [4].

Let $P \subset \mathbb{P}(V_n)$ be a fixed hyperplane, and consider (as in [4, Section 2.3]) the rational map

$$\mathrm{Gr}(2, V_n) \dashrightarrow P$$

sending a line in $\mathbb{P}(V_n)$ to its intersection with P. This map is resolved by blowing up a subvariety $\sigma_{11}(P) \cong \mathrm{Gr}(2, n-1)$, resulting in a morphism

$$\Gamma\colon\ \ \widetilde{\mathrm{Gr}}\ \to\ P$$

(where $\widetilde{\mathrm{Gr}} \to \mathrm{Gr}(2, V_n)$ denotes the blow-up with center $\sigma_{11}(P)$).

Let $\widetilde{Y} \to Y$ be the blow-up of Y with center $\sigma_{11}(P) \cap Y$, and let us consider the morphism

$$\Gamma_Y\colon\ \ \widetilde{Y}\ \to\ P,$$

obtained by restricting Γ.

In case $s = 1$ and P is generic with respect to Y, the morphism Γ_Y is a $\mathbb{P}^{n-3}$-fibration over P. It follows that $\widetilde{Y}$, and hence Y, has trivial Chow groups.

In case $s = 2$, and P chosen generically with respect to Y, the morphism Γ_Y is generically a $\mathbb{P}^{n-4}$-fibration over P, and there are finitely many points in P where the fiber is $\mathbb{P}^{n-3}$. Applying Theorem 2.1, this implies that $\widetilde{Y}$, and hence also Y, has trivial Chow groups. □

2.3. Hyperplane sections of $\mathbf{Gr(3,n)}$.

Theorem 2.4. *Let*

$$Y := \mathrm{Gr}(3, n) \cap H \quad\ \subset\ \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\ i \le n-3.$$

Moreover, in case $n \le 11$ or $n = 13$ we have

$$A_i^{hom}(Y) = 0 \quad \forall\ i \le n-2.$$

P r o o f. This is [14, Theorems 3.1 and 3.2], which uses the notion of *jumps* between Grassmannians as developed in [2]. □

3. Main results.

3.1. Projections. As in [2], let $I_r \,\mathrm{Gr}(k, n) \subset \mathrm{Gr}(k, n)$ parametrize linear subspaces that are isotropic with respect to r fixed generic skew-forms. One has

$$\dim I_r \,\mathrm{Gr}(k, n) = k(n-k) - r\binom{k}{2}.$$

For example, $I_r \,\mathrm{Gr}(2, n)$ is just the intersection of $\mathrm{Gr}(2, n)$ with r Plücker hyperplanes. The case $I_2 \,\mathrm{Gr}(k, n)$ is studied in detail in [1].

To relate hyperplane sections of different symplectic Grassmannians, Bernardara–Fatighenti–Manivel [2] have developed a theory of *projections*. The starting point is a rational map

$$\pi\colon\ \mathrm{Gr}(k,n+1)\ \dashrightarrow\ \mathrm{Gr}(k,n),$$

determined by the choice of a line in the $n+1$-dimensional vector space. If Y' is a hyperplane section of $I_r\,\mathrm{Gr}(k,n+1)$, one can restrict π to Y'. A detailed analysis of the case $k=3$ yields the following:

Theorem 3.1 ([2]). *Assume n is even and $r\le 1$, or n is odd and $r=0$. Let*

$$Y' := I_r\,\mathrm{Gr}(3,n+1)\cap H$$

be a smooth hyperplane section. There exists a commutative diagram

$$\begin{array}{ccccccc}
E & \hookrightarrow & \widetilde{Y}' & \hookleftarrow & & F & \\
\downarrow & & \downarrow{\scriptstyle\sigma} & \searrow{\scriptstyle\tau} & & & \searrow{\scriptstyle q} \\
Z' & \hookrightarrow & Y' & \stackrel{\pi'}{\dashrightarrow} & I_r\,\mathrm{Gr}(3,n) & \hookleftarrow & Y
\end{array}$$

where $Y := I_{r+1}\,\mathrm{Gr}(3,n)\cap H$ is a smooth hyperplane section. The morphism σ is the blow-up with center $Z'\cong I_{r+1}\,\mathrm{Gr}(2,n)$. The morphism q is a $\mathbb{P}^3$-fibration, while τ is a $\mathbb{P}^2$-fibration over the complement of Y.

Proof. This is contained in [2, Section 3.2] (NB: note that our n is $n-1$ in loc. cit.). As explained in loc. cit., the assumptions on n and r guarantee that the target $I_r\,\mathrm{Gr}(3,n)$ and the hyperplane section Y are generic and hence smooth. □

3.2. Motivating the conjecture. As a consequence of Theorem 3.1, one can compute the Hodge level of hyperplane sections Y of symplectic Grassmannians: surprisingly, it turns out that (at least for $n>8$) Y is of "Calabi–Yau type":

Theorem 3.2 ([2]). *Let*

$$Y := I_1\,\mathrm{Gr}(3,n)\cap H \quad \subset\ \mathbb{P}^{\binom{n}{k}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Assume $n>8$. Then Y has Hodge coniveau $n-3$. More precisely, the Hodge numbers verify

$$h^{p,\dim Y-p}(Y)=\begin{cases}1 & \text{if}\quad p=n-3,\\ 0 & \text{if}\quad p<n-3.\end{cases}$$

Proof. This is implicit in [2], as we now explain. Let the set-up be as in Theorem 3.1. Then [2, Proposition 6] relates Y and Y' on the level of cohomology: one has an isomorphism of Hodge structures

$$(1)\quad H^{j-6}(Y,\mathbb{Q})(-3)\oplus\bigoplus_{i=0}^{2} H^{j-2i}(I_r\,\mathrm{Gr}(3,n),\mathbb{Q})(-i) \xrightarrow{\cong} H^{j}(Y',\mathbb{Q})\oplus\bigoplus_{i=1}^{c-1} H^{j-2i}(I_{r+1}\,\mathrm{Gr}(2,n),\mathbb{Q})(-i)$$

(where c denotes the codimension of Z' in Y'). Setting $r=0$ and combining with [2, Theorem 3] (which gives the Hodge numbers of Y'), plus the fact that $\mathrm{Gr}(3,n)$ and $I_1\,\mathrm{Gr}(2,n)$ have algebraic cohomology, this gives the required Hodge numbers of Y. □

Theorem 3.2 motivates Conjecture 1.2. Indeed, the *generalized Bloch conjecture* [25, Conjecture 1.10] predicts that any variety Y with Hodge coniveau $\geq c$ has

$$A_i^{hom}(Y)=0\quad\forall\, i<c.$$

Note that at least for $n>8$, the bound of Conjecture 1.2 is optimal: assuming $A_i^{hom}(Y)=0$ for $j\leq n-3$ and applying the Bloch–Srinivas argument [3], one would get the vanishing $h^{n-3,\dim Y-n+3}(Y)=0$, contradicting Theorem 3.2.

3.3. A relation of motives. The cohomological relation (1) between Y and Y' also exists as (and actually is implied by) a relation on the level of the Grothendieck ring of varieties [2, Proposition 4], and on the level of derived categories [2, Proposition 5]. To complete the picture, we now lift the relation (1) to the level of Chow motives:

Proposition 3.3. *Let notation and assumptions be as in Theorem 3.1. There is an isomorphism of integral Chow motives*

$$h(Y)(-3)\oplus\bigoplus_{i=0}^{2} h(I_r\,\mathrm{Gr}(3,n))(-i) \xrightarrow{\cong} h(Y')\oplus\bigoplus_{i=1}^{c-1} h(I_{r+1}\,\mathrm{Gr}(2,n))(-i)\quad \text{in } \mathcal{M}_{\mathrm{rat}}^{\mathbb{Z}}\,.$$

(Here c denotes the codimension of Z' in Y'.)

Proof. The idea is to express the motive of $\widetilde{Y}'$ in two different ways:

The blow-up formula expresses $h(\widetilde{Y}')$ in terms of $h(Y')$; this gives the right-hand side of the relation.

Looking at [2, Section 3.2], one finds that $\widetilde{Y}'$ is the total space of a projectivization $\mathbb{P}(E)$ where E is the vector bundle $E := \mathcal{O} \oplus \mathcal{U}^*$ on $I_r\,\mathrm{Gr}(3,n)$ (in the notation of loc. cit.), and Y is given by a section of E. That is, we are in the setting of Cayley's trick, and so Theorem 2.1 expresses $h(\widetilde{Y}')$ in terms of $h(Y)$; this gives the left-hand side of the relation. □

3.4. Hyperplane sections of $I_1\,\mathrm{Gr}(3,n)$.

Theorem 3.4. *Let*

$$Y := I_1\,\mathrm{Gr}(3,n) \cap H \quad \subset \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n-5.$$

Moreover, in case $n \le 10$ or $n = 12$ we have

$$A_i^{hom}(Y) = 0 \quad \forall\, i \le n-4.$$

P r o o f. A generic hyperplane section Y is attained by the construction of Theorem 3.1 with $r = 0$, i.e. there is a smooth hyperplane section

$$Y' := \mathrm{Gr}(3, n+1) \cap H,$$

related to Y via the projection of Theorem 3.1. In this case, Proposition 3.3 implies that there is an injection of Chow groups

$$A_i^{hom}(Y) \hookrightarrow A_{i+3}^{hom}(Y') \oplus \bigoplus A_*^{hom}(I_1\,\mathrm{Gr}(2,n)).$$

The symplectic Grassmannian $I_1\,\mathrm{Gr}(2,n)$ is nothing but a Plücker hyperplane section of $\mathrm{Gr}(2,n)$, and so Proposition 2.3 gives the vanishing

$$A_*^{hom}(I_1\,\mathrm{Gr}(2,n)) = 0.$$

The variety Y' is a hyperplane section of $\mathrm{Gr}(3,n+1)$, and so Theorem 2.4 gives the vanishing

$$A_{i+3}^{hom}(Y') = 0 \quad \forall\, i \le n-5,$$

with the additional vanishing for $i = n-4$ for small n. This proves the theorem for generic sections Y.

A standard spread argument allows to extend to *all* smooth hyperplane sections: Let $\mathcal{Y} \to B$ denote the universal family of all smooth hyperplane sections of $\mathrm{Gr}(k,n)$, and let $B^\circ \subset B$ denote the Zariski open subset parametrizing smooth Y verifying the set-up of Theorem 3.1. Doing the Bloch–Srinivas argument [3] (cf. also [13]), the above implies that for each $b\in B^\circ$ one has a decomposition of the diagonal

$$\Delta_{Y_b} = \gamma_b + \delta_b \quad \text{in } A^{\dim Y_b}(Y_b\times Y_b) \tag{2}$$

where γ_b is completely decomposed (i.e. $\gamma_b\in A^*(Y_b)\otimes A^*(Y_b)$) and δ_b is supported on $Y_b\times W_b$ with $\operatorname{codim} W_b = n-2$ (and $\operatorname{codim} W_b = n-1$ for small n). Using the Hilbert schemes argument of [24, Proposition 3.7] (cf. also [15, Proposition A.1] for the precise form used here), the γ_b, δ_b, W_b exist relatively, i.e. one can find a cycle $\gamma\in (p_1)^*A^*(\mathcal{Y})\cdot (p_2)^*A^*(\mathcal{Y})$, a subvariety $\mathcal{W}\subset\mathcal{Y}$ of codimension $n-2$, and a cycle δ supported on $\mathcal{Y}\times_{B^\circ}\mathcal{W}$ such that

$$\Delta_{\mathcal{Y}}|_b = \gamma|_b + \delta|_b \quad \text{in } A^{\dim Y_b}(Y_b\times Y_b) \quad \forall\, b\in B^\circ .$$

Let $\bar\gamma,\bar\delta\in A^{\dim Y_b}(\mathcal{Y}\times_B\mathcal{Y})$ be cycles that restrict to γ resp. δ. The spread lemma [25, Lemma 3.2] implies that

$$\Delta_{\mathcal{Y}}|_b = \bar\gamma|_b + \bar\delta|_b \quad \text{in } A^{\dim Y_b}(Y_b\times Y_b) \quad \forall\, b\in B.$$

Given any $b_1\in B\setminus B^\circ$, using the moving lemma, one can find representatives for $\bar\gamma$ and $\bar\delta$ in general position with respect to the fiber $Y_{b_1}\times Y_{b_1}$. Restricting to the fiber, this implies that the diagonal of Y_{b_1} has a decomposition as in (2). Letting the decomposition (2) act on Chow groups, this shows that

$$A_i^{hom}(Y_b) = 0 \quad \forall\, i\le n-5\ , \quad \forall\, b\in B$$

(with the additional vanishing for $i=n-4$ for small n). □

3.5. Hyperplane sections of $I_2\,\mathrm{Gr}(3,n)$.

Theorem 3.5. *Assume n is even, and let*

$$Y := I_2\,\mathrm{Gr}(3,n)\cap H \quad \subset\ \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall\, i\le n-7.$$

Moreover, in case $n\le 10$ we have

$$A_i^{hom}(Y) = 0 \quad \forall\, i\le n-6.$$

P r o o f. For n even, a generic hyperplane section Y is attained by the construction of Theorem 3.1 with $r = 1$, i.e. there is a smooth hyperplane section

$$Y' := I_1 \operatorname{Gr}(3, n+1) \cap H,$$

related to Y via the projection of Theorem 3.1. In this case, Proposition 3.3 implies that there is an injection of Chow groups

$$A_i^{hom}(Y) \hookrightarrow A_{i+3}^{hom}(Y') \oplus \bigoplus A_*^{hom}(I_2 \operatorname{Gr}(2, n)).$$

The bisymplectic Grassmannian $I_2 \operatorname{Gr}(2, n)$ is nothing but an intersection $\operatorname{Gr}(2, n) \cap H_1 \cap H_2$ (where the H_j are Plücker hyperplanes), and so Proposition 2.3 gives the vanishing

$$A_*^{hom}(I_2 \operatorname{Gr}(2, n)) = 0.$$

The variety Y' is a hyperplane section of $I_1 \operatorname{Gr}(3, n+1)$, and so Theorem 3.4 gives the vanishing

$$A_{i+3}^{hom}(Y') = 0 \quad \forall\, i \leq n - 7,$$

with the additional vanishing for $i = n - 6$ for small n. This proves the theorem for generic sections Y.

The extension to *all* smooth hyperplane sections Y is done just as in the proof of Theorem 3.4. □

4. Some consequences.

Corollary 4.1. (i) *Let Y be as in Theorem* 3.4 *and $n \leq 10$ or $n = 12$, or as in Theorem* 3.5 *and $n \leq 10$. Then $H^{\dim Y}(Y, \mathbb{Q})$ is supported on a subvariety of codimension $n - 3$.*
(ii) *Let Y be as in Theorem* 3.5 *and $n \leq 10$. Then $H^{\dim Y}(Y, \mathbb{Q})$ is supported on a subvariety of codimension $n - 5$.*

P r o o f. This follows in standard fashion from the Bloch–Srinivas argument [3]. Let us treat (i) (the argument for (ii) is the same). The vanishing

$$A_i^{hom}(Y) = 0 \quad \forall\, i \leq n - 4$$

(Theorem 3.4) is equivalent to the decomposition

$$\Delta_Y = \gamma + \delta \quad \text{in } A^{\dim Y}(Y \times Y),$$

where γ is a completely decomposed cycle (i.e. $\gamma \in A^*(Y) \otimes A^*(Y)$), and δ has support on $Y \times W$ with $W \subset Y$ of codimension $n - 3$ (to see this equivalence, one can look for instance at [13, Theorem 1.7]). Let $H^{\dim Y}_{tr}(Y,\mathbb{Q})$ denote the transcendental cohomology (i.e. the complement of the algebraic part under the cup product pairing). The cycle γ does not act on $H^{\dim Y}_{tr}(Y,\mathbb{Q})$. The action of δ on $H^{\dim Y}_{tr}(Y,\mathbb{Q})$ factors over W, and so

$$H^{\dim Y}_{tr}(Y,\mathbb{Q}) \ \subset\ H^{\dim Y}_{W}(Y,\mathbb{Q}).$$

Since the algebraic part of $H^{\dim Y}(Y,\mathbb{Q})$ is (by definition) supported in codimension $\dim Y/2$, this settles the corollary. □

Corollary 4.2. *Let*

$$Y := I_1\,\mathrm{Gr}(3,n)\cap H \ \subset\ \mathbb{P}^{\binom{n}{3}-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding).
(i) If $n\le 8$, then Y has finite-dimensional motive (in the sense of [11]*).*
(ii) If $n\le 9$, then Y has trivial Griffiths groups (and so Voevodsky's smash conjecture [22] *is true for Y, i.e. numerical equivalence and smash-equivalence coincide on Y).*
(iii) If $n\le 10$, the Hodge conjecture is true for Y.

P r o o f. This is similar to the argument of Corollary 4.1.
(i) The vanishing

$$A^{hom}_i(Y)=0 \quad \forall\, i\le n-4$$

(Theorem 3.4) is equivalent to the decomposition of the diagonal

$$\Delta_Y=\gamma+\delta \quad \text{in } A^{\dim Y}(Y\times Y),$$

where γ is a completely decomposed cycle, and δ has support on $Y\times W$ with $W\subset Y$ of codimension $n-3$ (cf. [3] or [13]). The dimension of Y is $3n-13$, and so (looking at the action of the diagonal) one finds that

$$A^*_{AJ}(Y)=0$$

as long as $n\le 8$. This implies Kimura finite-dimensionality of Y [21, Theorem 4].
(ii) The vanishing of Theorem 3.4 implies that

$$\mathrm{Niveau}(A_*(Y))\le 2$$

(in the sense of [13]), i.e. the motive of Y factors over a surface. Since surfaces have trivial Griffiths groups, the conclusion follows.

(*iii*) The vanishing of Theorem 3.4 implies that

$$\text{Niveau}(A_*(Y)) \leq 3$$

(in the sense of [13]), i.e. the motive of Y factors over a threefold. Since threefolds verify the Hodge conjecture, the conclusion follows. □

We leave it to the zealous reader to formulate and prove a version of Corollary 4.2 for bisymplectic Grassmannians.

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