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CONSTRUCTION OF TWO INFINITE CLASSES OF STRONGLY REGULAR GRAPHS USING MAGIC SQUARES

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Dedicated to French mathematician Philippe de La Hire
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Abstract. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. Using a method for constructing the magic and semi-magic squares of order $2k+1$, we have created two infinite classes of strongly regular graphs (i) strongly regular graph of order $n = (2k+1)^2$ and degree $r = 8k$ with $\tau = 2k+5$ and $\theta = 12$ and (ii) strongly regular graph of order $n = (2k+1)^2$ and degree $r = 6k$ with $\tau = 2k+1$ and $\theta = 6$ for $k \geq 2$.

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1. Introduction. Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \dots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex k. We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [1]. Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of a connected strongly regular graph G. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2 and λ_3. Further, let $\overline{r} = (n-1)-r$, $\overline{\lambda}_2 = -\lambda_3 - 1$ and $\overline{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of G. Then $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$, where $\overline{\tau} = \tau(\overline{G})$ and $\overline{\theta} = \theta(\overline{G})$.

Remark 1. If G is a disconnected strongly regular graph of degree r then $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H.

Remark 2. We also know that a strongly regular graph $G = \overline{mK_{r+1}}$ if and only if $\theta = r$. Since $\lambda_2\lambda_3 = -(r-\theta)$ it follows that $G = \overline{mK_{r+1}}$ if and only if $\lambda_2 = 0$.

Remark 3. (i) A strongly regular graph G of order $n = 4k+1$ and degree $r = 2k$ with $\tau = k-1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then G is an integral graph.

Remark 4. The line graph of the complete bipartite graph $K_{n,n}$ is called a lattice graph and is denoted by $L(n)$. It is a strongly regular graph of order n^2 and degree $2(n-1)$ with $\tau = n-2$ and $\theta = 2$.

Let $X = X[x_{ij}]$ be a square matrix of order n with all distinct x_{ij} which belong to the set $\{1, 2, \dots, n^2\}$. Let $G[X]$ be a graph obtained from the matrix $X[x_{ij}]$ in the following way: (i) the vertex set of the graph $G[X]$ is $V(G[X]) = \{x_{ij} \,|\, i, j = 1, 2, \dots, n\}$ and (ii) the neighborhood of the vertex x_{ij} is $S_{x_{ij}} = S_{x_{i,-j}} \cup S_{x_{-i,j}}$ where

$$S_{x_{i,-j}} = \{x_{i1}, x_{i2}, \dots, x_{i,j-1}, x_{i,j+1}, \dots, x_{in}\}, \quad (1)$$

$$S_{x_{-i,j}} = \{x_{1j}, x_{2j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{nj}\} \quad (2)$$

for[1] any $i, j = 1, 2, \ldots, n$. Since $|S_{x_{ij}}| = |S_{x_{i,-j}}| + |S_{x_{-i,j}}| = (n-1) + (n-1)$ we note that $G[X]$ is a regular graph of order n^2 and degree $r = 2(n-1)$. Let x_{st} be adjacent to x_{ij}. Then x_{st} belongs to the i-th row or to the j-th column. Without loss of generality we may assume that x_{st} belongs to the i-th row. In this case we have $s = i$ and $t \neq j$. So we obtain

$$|S_{x_{ij}} \cap S_{x_{it}}| = |S_{x_{i,-j}} \cap S_{x_{i,-t}}| + |S_{x_{i,-j}} \cap S_{x_{-i,t}}| + |S_{x_{-i,j}} \cap S_{x_{i,-t}}| + |S_{x_{-i,j}} \cap S_{x_{-i,t}}|\,.$$

We note $|S_{x_{i,-j}} \cap S_{x_{-i,t}}| = 0$ because $x_{it} \notin S_{x_{it}}$ and $|S_{x_{-i,j}} \cap S_{x_{i,-t}}| = 0$ because $x_{ij} \notin S_{x_{ij}}$. Next, we have $|S_{x_{-i,j}} \cap S_{x_{-i,t}}| = 0$ because $t \neq j$. In the view of this we get $|S_{x_{ij}} \cap S_{x_{it}}| = |S_{x_{i,-j}} \cap S_{x_{i,-t}}|$. Since $x_{ij} \notin S_{x_{ij}}$ and $x_{it} \notin S_{x_{it}}$ we find that $|S_{x_{ij}} \cap S_{x_{it}}| = n - 2$ for any two adjacent vertices x_{ij} and x_{st}.

Further, let us assume that x_{ij} and x_{st} are two distinct non-adjacent vertices of the graph $G[X]$. In this case x_{st} neither belongs to the i-th row of the matrix X nor belongs to the j-th column of the matrix X, which provides that $s \neq i$ and $t \neq j$. So we obtain

$$|S_{x_{ij}} \cap S_{x_{st}}| = |S_{x_{i,-j}} \cap S_{x_{s,-t}}| + |S_{x_{i,-j}} \cap S_{x_{-s,t}}| + |S_{x_{-i,j}} \cap S_{x_{s,-t}}| + |S_{x_{-i,j}} \cap S_{x_{-s,t}}|\,.$$

We note $|S_{x_{i,-j}} \cap S_{x_{s,-t}}| = 0$ because $s \neq i$ and $|S_{x_{-i,j}} \cap S_{x_{-s,t}}| = 0$ because $t \neq j$. Since $x_{it} \in S_{x_{i,-j}}$ and $x_{it} \in S_{x_{-s,t}}$ we find that $|S_{x_{i,-j}} \cap S_{x_{-s,t}}| = 1$. Since $x_{sj} \in S_{x_{-i,j}}$ and $x_{sj} \in S_{x_{s,-t}}$ we find that $|S_{x_{-i,j}} \cap S_{x_{s,-t}}| = 1$. Finally, we arrive at

$$|S_{x_{ij}} \cap S_{x_{st}}| = |S_{x_{i,-j}} \cap S_{x_{-s,t}}| + |S_{x_{-i,j}} \cap S_{x_{s,-t}}| = 1 + 1\,,$$

which provides[2] that $G[X]$ is a strongly regular graph of order n^2 and degree $r = 2(n-1)$ with $\tau = n - 2$ and $\theta = 2$. Therefore, according to Remark 4 it follows that $G[X] = L(n)$ for $n \geq 2$.

2. Magic squares of order $2k+1$. Let $M_n = M_n[m_{ij}]$ be a square matrix of order n with all distinct m_{ij} which belong to the set $\{1, 2, \ldots, n^2\}$. The matrix M_n is called the magic square of order n if the sum of all elements in any row and column and both diagonals is the same. The matrix M_n is called the semi-magic square of order n if the sum of all elements in any row and

[1]In other words, the adjacent vertices of the vertex x_{ij} are obtained from the matrix $X[x_{ij}]$ by crossing out its i-th row and by crossing out its j-th column. We note that $S_{x_{i,-j}} \cap S_{x_{-i,j}} = \emptyset$ for $i, j = 1, 2, \ldots, n$.

[2]Using this trivial result which is related to construction the infinite class of strongly regular graphs $L(n)$, we are motivated to investigate strongly regular graphs using magic squares.

column is the same. We shall now demonstrate how to construct a magic square of order 5 created by "the method of cyclic permutations" established by French mathematician Philippe de La Hire, as follows. Let $(\pi(1),\pi(2),\pi(3),\pi(4),\pi(5)) = (2,5,4,1,3)$ be a fixed permutation of the numbers $1,2,3,4,5$ and let $(\pi(0),\pi(5),\pi(10),\pi(15),\pi(20)) = (20,0,10,5,15)$ be a fixed permutation of the numbers $0,5,10,15,20$. Using the method of cyclic permutations we obtain the following two matrices

2	5	4	1	3
4	1	3	2	5
3	2	5	4	1
5	4	1	3	2
1	3	2	5	4

20	0	10	5	15
5	15	20	0	10
0	10	5	15	20
15	20	0	10	5
10	5	15	20	0

`K[5][5] and L[5][5]`

Then the matrix $M_5[m_{ij}] = K_5[k_{ij}] + L_5[\ell_{ij}]$ is a magic square of order 5, where $K_5[k_{ij}] = \mathtt{K}[5][5]$ and $L_5[\ell_{ij}] = \mathtt{L}[5][5]$.

We now proceed to obtain a new method for creating the semi-magic squares of order $2k+1$ for $k \geq 1$, which is based on "the method of cyclic permutations", as follows. First, let us assume that $(\pi(1),\pi(2),\dots,\pi(2k+1))$ is a fixed permutation of the numbers $1,2,\dots,2k+1$. Let

$\pi(1)$	$\pi(2)$	$\cdots$	$\pi(k)$	$\pi(k{+}1)$	$\pi(k{+}2)$	$\cdots$	$\pi(2k)$	$\pi(2k{+}1)$
$\pi(k{+}1)$	$\pi(k{+}2)$	$\cdots$	$\pi(2k)$	$\pi(2k{+}1)$	$\pi(1)$	$\cdots$	$\pi(k{-}1)$	$\pi(k)$
$\pi(2k{+}1)$	$\pi(1)$	$\cdots$	$\pi(k{-}1)$	$\pi(k)$	$\pi(k{+}1)$	$\cdots$	$\pi(2k{-}1)$	$\pi(2k)$
$\pi(k)$	$\pi(k{+}1)$	$\cdots$	$\pi(2k{-}1)$	$\pi(2k)$	$\pi(2k{+}1)$	$\cdots$	$\pi(k{-}2)$	$\pi(k{-}1)$
$\pi(2k)$	$\pi(2k{+}1)$	$\cdots$	$\pi(k{-}2)$	$\pi(k{-}1)$	$\pi(k)$	$\cdots$	$\pi(2k{-}2)$	$\pi(2k{-}1)$
$\pi(k{-}1)$	$\pi(k)$	$\cdots$	$\pi(2k{-}2)$	$\pi(2k{-}1)$	$\pi(2k)$	$\cdots$	$\pi(k{-}3)$	$\pi(k{-}2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi(3)$	$\pi(4)$	$\cdots$	$\pi(k{+}2)$	$\pi(k{+}3)$	$\pi(k{+}4)$	$\cdots$	$\pi(1)$	$\pi(2)$
$\pi(k{+}3)$	$\pi(k{+}4)$	$\cdots$	$\pi(1)$	$\pi(2)$	$\pi(3)$	$\cdots$	$\pi(k{+}1)$	$\pi(k{+}2)$
$\pi(2)$	$\pi(3)$	$\cdots$	$\pi(k{+}1)$	$\pi(k{+}2)$	$\pi(k{+}3)$	$\cdots$	$\pi(2k{+}1)$	$\pi(1)$
$\pi(k{+}2)$	$\pi(k{+}3)$	$\cdots$	$\pi(2k{+}1)$	$\pi(1)$	$\pi(2)$	$\cdots$	$\pi(k)$	$\pi(k{+}1)$

$K[2k+1][2k+1]$

Second, let us assume that $(\pi(0), \pi(2k+1), \ldots, \pi(2k(2k+1)))$ is a fixed permutation of the numbers $0, 2k+1, \ldots, 2k(2k+1)$. Let $\overline{k} = 2k+1$ and let

$\pi(0)$	$\pi(\overline{k})$	$\cdots$	$\pi((k-1)\overline{k})$	$\pi(k\overline{k})$	$\pi((k+1)\overline{k})$	$\cdots$	$\pi((2k-1)\overline{k})$	$\pi(2k\overline{k})$
$\pi((k+1)\overline{k})$	$\pi((k+2)\overline{k})$	$\cdots$	$\pi(2k\overline{k})$	$\pi(0)$	$\pi(\overline{k})$	$\cdots$	$\pi((k-1)\overline{k})$	$\pi(k\overline{k})$
$\pi(\overline{k})$	$\pi(2\overline{k})$	$\cdots$	$\pi(k\overline{k})$	$\pi((k+1)\overline{k})$	$\pi((k+2)\overline{k})$	$\cdots$	$\pi(2k\overline{k})$	$\pi(0)$
$\pi((k+2)\overline{k})$	$\pi((k+3)\overline{k})$	$\cdots$	$\pi(0)$	$\pi(\overline{k})$	$\pi(2\overline{k})$	$\cdots$	$\pi(k\overline{k})$	$\pi((k+1)\overline{k})$
$\pi(2\overline{k})$	$\pi(3\overline{k})$	$\cdots$	$\pi((k+1)\overline{k})$	$\pi((k+2)\overline{k})$	$\pi((k+3)\overline{k})$	$\cdots$	$\pi(0)$	$\pi(\overline{k})$
$\pi((k+3)\overline{k})$	$\pi((k+4)\overline{k})$	$\cdots$	$\pi(\overline{k})$	$\pi(2\overline{k})$	$\pi(3\overline{k})$	$\cdots$	$\pi((k+1)\overline{k})$	$\pi((k+2)\overline{k})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi((2k-1)\overline{k})$	$\pi(2k\overline{k})$	$\cdots$	$\pi((k-3)\overline{k})$	$\pi((k-2)\overline{k})$	$\pi((k-1)\overline{k})$	$\cdots$	$\pi((2k-3)\overline{k})$	$\pi((2k-2)\overline{k})$
$\pi((k-1)\overline{k})$	$\pi(k\overline{k})$	$\cdots$	$\pi((2k-2)\overline{k})$	$\pi((2k-1)\overline{k})$	$\pi(2k\overline{k})$	$\cdots$	$\pi((k-3)\overline{k})$	$\pi((k-2)\overline{k})$
$\pi(2k\overline{k})$	$\pi(0)$	$\cdots$	$\pi((k-2)\overline{k})$	$\pi((k-1)\overline{k})$	$\pi(k\overline{k})$	$\cdots$	$\pi((2k-2)\overline{k})$	$\pi((2k-1)\overline{k})$
$\pi(k\overline{k})$	$\pi((k+1)\overline{k})$	$\cdots$	$\pi((2k-1)\overline{k})$	$\pi(2k\overline{k})$	$\pi(0)$	$\cdots$	$\pi((k-2)\overline{k})$	$\pi((k-1)\overline{k})$

$L[2k+1][2k+1]$

understanding that $0 = 0\cdot(2k+1)$ and $2k+1 = 1\cdot(2k+1)$. Then we can see that the matrix $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ is a semi-magic square of order $2k+1$, where $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ and $L_{2k+1}[\ell_{ij}] = L[2k+1][2k+1]$. Indeed, since $(k_{11}, k_{12}, \ldots, k_{1,2k+1}) = (\pi(1), \pi(2), \ldots, \pi(2k+1))$ and since the i-th row of the matrix K_{2k+1} is a cyclic permutation of its first row, we get

$$\sum_{j=1}^{2k+1} k_{ij} = \sum_{j=1}^{2k+1} k_{1j} = \sum_{j=1}^{2k+1} \pi(j) = \sum_{j=1}^{2k+1} j = (k+1)(2k+1)$$

for $i = 1, 2, \ldots, 2k+1$. According to $K[2k+1][2k+1]$, we have that

$$k_{i1} = \begin{cases} \pi(k+2-t), & \text{if} \quad i = 2t \\ \pi(2k+2-t), & \text{if} \quad i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Since $k+2-t \le k+1 < k+2 \le 2k+2-t$ it follows that $\pi(k+2-t) \ne \pi(2k+2-t)$ for $t = 1, 2, \ldots, k$. So we find that $(k_{11}, k_{21}, \ldots, k_{2k+1,1})^\top$ is a permutation $(\pi_c(1), \pi_c(2), \ldots, \pi_c(2k+1))$ of the numbers $1, 2, \ldots, 2k+1$. Next, the j-th column of the matrix $K[2k+1][2k+1]$ is a cyclic permutation of its first column. In the view of this, we get

$$\sum_{i=1}^{2k+1} k_{ij} = \sum_{i=1}^{2k+1} k_{i1} = \sum_{i=1}^{2k+1} \pi_c(i) = \sum_{i=1}^{2k+1} i = (k+1)(2k+1)$$

for $j = 1, 2, \ldots, 2k+1$. We note (i) since the i-th row of the matrix K_{2k+1} is a cyclic permutation of its first row then[3] any fixed number $p \in \{1, 2, \ldots, 2k+1\}$ is presented in the i-th row of the matrix K_{2k+1} only one time and (ii) since the j-th column of the matrix K_{2k+1} is a cyclic permutation of its first column then any fixed number $p \in \{1, 2, \ldots, 2k+1\}$ is presented in the j-th column of the matrix K_{2k+1} only one time.

Further, since $(\ell_{11}, \ell_{12}, \ldots, \ell_{1,2k+1}) = (\pi(0), \pi(2k+1), \ldots, \pi(2k(2k+1)))$ and since the i-th row of the matrix L_{2k+1} is a cyclic permutation of its first row, we get

$$\sum_{j=1}^{2k+1} \ell_{ij} = \sum_{j=1}^{2k+1} \ell_{1j} = \sum_{j=1}^{2k+1} \pi((j-1)(2k+1)) = (2k+1) \sum_{j=1}^{2k+1} (j-1) = k(2k+1)^2$$

for $i = 1, 2, \ldots, 2k+1$. According to $L[2k+1][2k+1]$, we have that

$$\ell_{i1} = \begin{cases} \pi((k+t)(2k+1)), & \text{if} \quad i = 2t \\ \pi(t(2k+1)), & \text{if} \quad i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Next, since $t \le k < k+1 \le k+t$ it follows that $\pi(t(2k+1)) \ne \pi((k+t)(2k+1))$ for $t = 1, 2, \ldots, k$. So we find that $(\ell_{11}, \ell_{21}, \ldots, \ell_{2k+1,1})^\top$ is a permutation $(\pi_c(0), \pi_c(2k+1), \ldots, \pi_c(2k(2k+1)))$ of the numbers $0, 2k+1, \ldots, 2k(2k+1)$. Since the j-th column of the matrix $L[2k+1][2k+1]$ is a cyclic permutation of its first column, we obtain

$$\sum_{i=1}^{2k+1} \ell_{ij} = \sum_{i=1}^{2k+1} \ell_{i1} = \sum_{i=1}^{2k+1} \pi_c((i-1)(2k+1)) = (2k+1) \sum_{i=1}^{2k+1} (i-1) = k(2k+1)^2$$

for $j = 1, 2, \ldots, 2k+1$. We note (i) since the i-th row of the matrix L_{2k+1} is a cyclic permutation of its first row then[4] any fixed number $q \in \{0, 2k+1, \ldots, 2k(2k+1)\}$ is presented in the i-th row of the matrix L_{2k+1} only one time and (ii) since the j-th column of the matrix L_{2k+1} is a cyclic permutation of its first column then any fixed number $q \in \{0, 2k+1, \ldots, 2k(2k+1)\}$ is presented in the j-th column of the matrix L_{2k+1} only one time. Since $m_{ij} = k_{ij} + \ell_{ij}$ we get

$$\sum_{i=1}^{2k+1} m_{ij} = \sum_{j=1}^{2k+1} m_{ij} = (k+1)(2k+1) + k(2k+1)^2 = (2k+1)\left(\frac{(2k+1)^2+1}{2}\right)$$

[3] Namely, according to $K[2k+1][2k+1]$ the all cyclic permutations related to rows and columns of the matrix K_{2k+1} are mutually different, which provides that $k_{sj} \ne k_{tj}$ and $k_{is} \ne k_{it}$ for $j = 1, 2, \ldots, 2k+1$ and $i = 1, 2, \ldots, 2k+1$ if $s \ne t$.

[4] Namely, according to $L[2k+1][2k+1]$ the all cyclic permutations related to rows and columns of the matrix L_{2k+1} are mutually different, which provides that $\ell_{sj} \ne \ell_{tj}$ and $\ell_{is} \ne \ell_{it}$ for $j = 1, 2, \ldots, 2k+1$ and $i = 1, 2, \ldots, 2k+1$ if $s \ne t$.

for $i, j = 1, 2, \dots, 2k+1$.

Theorem 1. *Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ where $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ and $L_{2k+1}[\ell_{ij}] = L[2k+1][2k+1]$. Then $M_{2k+1}[m_{ij}]$ is a semi-magic square of order $2k+1$ for $k \geq 1$.*

Proof. In order to prove that M_{2k+1} is a semi-magic square it is sufficient to show that $m_{ij} \in \{1, 2, \dots, (2k+1)^2\}$ and m_{ij} are mutually different for $i, j = 1, 2, \dots, 2k+1$. Indeed, since $k_{ij} \in \{1, 2, \dots, 2k+1\}$ and $\ell_{ij} \in \{0, 2k+1, \dots, 2k(2k+1)\}$ we have $m_{ij} \in \{1, 2, \dots, (2k+1)^2\}$ for $i, j = 1, 2, \dots, 2k+1$. Next, according to $K[2k+1][2k+1]$ we have that

(3)
$$k_{ij} = \begin{cases} \pi(k+1-t+j), & \text{if} \quad i = 2t \wedge k+1-t+j \leq 2k+1 \\ \pi(k+1-t+j-(2k+1)), & \text{if} \quad i = 2t \wedge k+1-t+j > 2k+1 \\ \pi(2k+1-t+j), & \text{if} \quad i = 2t+1 \wedge 2k+1-t+j \leq 2k+1 \\ \pi(2k+1-t+j-(2k+1)), & \text{if} \quad i = 2t+1 \wedge 2k+1-t+j > 2k+1 \end{cases}$$

for $t = 1, 2, \dots, k$ and $j = 1, 2, \dots, 2k+1$. Next, according to $L[2k+1][2k+1]$ we have that

(4)
$$\ell_{ij} = \begin{cases} \pi((k+t+j-1)(2k+1)), & \text{if} \quad i = 2t \wedge k+t+j-1 \leq 2k \\ \pi((k+t+j-1-(2k+1))(2k+1)), & \text{if} \quad i = 2t \wedge k+t+j-1 > 2k \\ \pi((t+j-1)(2k+1)), & \text{if} \quad i = 2t+1 \wedge t+j-1 \leq 2k \\ \pi((t+j-1-(2k+1))(2k+1)), & \text{if} \quad i = 2t+1 \wedge t+j-1 > 2k \end{cases}$$

for $t = 1, 2, \dots, k$ and $j = 1, 2, \dots, 2k+1$. Since $m_{ij} = k_{ij} + \ell_{ij} = \pi(p) + \pi(q(2k+1))$ and since the numbers $\pi(p) \in \{1, 2, \dots, 2k+1\}$ and $\pi(q(2k+1)) \in \{0, 2k+1, \dots, 2k(2k+1)\}$, it follows that k_{ij} and ℓ_{ij} are uniquely determined. In other words, if $m_{ij} = k_{ij} + \ell_{ij}$, $m_{st} = k_{st} + \ell_{st}$ and $m_{ij} = m_{st}$ then $k_{ij} = k_{st}$ and $\ell_{ij} = \ell_{st}$. We now proceed to show that m_{ij} are mutually different for $i, j = 1, 2, \dots, 2k+1$. On the contrary, assume that $m_{ij} = m_{\mu\nu}$ for some $(i, j) \neq (\mu, \nu)$. Then $m_{ij} = \pi(p_0) + \pi(q_0(2k+1)) = m_{\mu\nu}$ for some $\pi(p_0) \in \{1, 2, \dots, 2k+1\}$ and $\pi(q_0(2k+1)) \in \{0, 2k+1, \dots, 2k(2k+1)\}$, which provides that $k_{ij} = k_{\mu\nu}$ and $\ell_{ij} = \ell_{\mu\nu}$. Without loss of generality we may assume that $i \neq \mu$. Since $\pi(q_0(2k+1))$ is presented in the j-th column of the matrix L_{2k+1} only one time, we find that $j \neq \nu$. Since[5] the i-th row and the μ-th row of the matrix K_{2k+1} is

[5]Of course, since the j-th column and the ν-th column of the matrix K_{2k+1} is a cyclic permutation of its first column and since the j-th column and the ν-th column of the matrix L_{2k+1} is a cyclic permutation of its first column, we can easily see that any m_{sj} in the j-th

a cyclic permutation of its first row and since the i-th row and the μ-th row of the matrix L_{2k+1} is a cyclic permutation of its first row, we can easily see that any m_{is} in the i-th row is also presented in the μ-th row. Indeed, we have

$$m_{i,j+1} = \pi(p_0+1) + \pi((q_0+1)(2k+1)) = m_{\mu,\nu+1}\,,$$

(i) understanding that $\pi(p_0+1) = \pi(1)$ if $p_0+1 = 2k+2$ and $\pi((q_0+1)(2k+1)) = \pi(0)$ if $q_0+1 = 2k+1$ and (ii) understanding that $m_{i,j+1} = m_{i1}$ if $j+1 = 2k+2$ and $m_{\mu,\nu+1} = m_{\mu 1}$ if $\nu+1 = 2k+2$. In the view of this, we can assume that $j = 1$. Since $j \neq \nu$ we have that $\nu \in \{2, 3, \ldots, 2k+1\}$. Finally, in order to prove that M_{2k+1} is a semi-magic square we shall consider the following four cases:

Case 1. ($i = 2t$ and $\mu = 2s$). Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(k+1-s+\nu)$ and $\pi((k+t)(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (i) $k+2-t = k+1-s+\nu$ and (ii) $k+t = k+s+\nu-1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(k+1-s+\nu)$ and $\pi((k+t)(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $k+2-t = k+1-s+\nu$ and (iv) $k+t = k+s+\nu-1-(2k+1)$. Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (v) $k+2-t = k+1-s+\nu-(2k+1)$ and (vi) $k+t = k+s+\nu-1$. Using (v) and (vi) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $k+2-t = k+1-s+\nu-(2k+1)$ and (viii) $k+t = k+s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2, 3, \ldots, 2k+1\}$.

Case 2. ($i = 2t$ and $\mu = 2s+1$). Consider the case when $2k+1-s+\nu \leq 2k+1$ and $s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu)$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (i) $k+2-t = 2k+1-s+\nu$ and (ii) $k+t = s+\nu-1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $2k+1-s+\nu \leq 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu)$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $k+2-t = 2k+1-s+\nu$ and (iv) $k+t = s+\nu-1-(2k+1)$. Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case

column is also presented in the ν-th column.

when $2k+1-s+\nu > 2k+1$ and $s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (v) $k+2-t = 2k+1-s+\nu-(2k+1)$ and (vi) $k+t = s+\nu-1$. Using (v) and (vi) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $2k+1-s+\nu > 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $k+2-t = 2k+1-s+\nu-(2k+1)$ and (viii) $k+t = s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2,3,\ldots,2k+1\}$.

Case 3. ($i = 2t+1$ and $\mu = 2s$). Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (i) $2k+2-t = k+1-s+\nu$ and (ii) $t = k+s+\nu-1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $2k+2-t = k+1-s+\nu$ and (iv) $t = k+s+\nu-1-(2k+1)$. Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (v) $2k+2-t = k+1-s+\nu-(2k+1)$ and (vi) $t = k+s+\nu-1$. Using (v) and (vi) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $2k+2-t = k+1-s+\nu-(2k+1)$ and (viii) $t = k+s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2,3,\ldots,2k+1\}$.

Case 4. ($i = 2t+1$ and $\mu = 2s+1$). Consider the case when $2k+1-s+\nu \leq 2k+1$ and $s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (i) $2k+2-t = 2k+1-s+\nu$ and (ii) $t = s+\nu-1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $2k+1-s+\nu \leq 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $2k+2-t = 2k+1-s+\nu$ and (iv) $t = s+\nu-1-(2k+1)$. Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $2k+1-s+\nu > 2k+1$ and $s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (v) $2k+2-t = 2k+1-s+\nu-(2k+1)$ and (vi) $t = s+\nu-1$. Using (v) and (vi) we obtain $2\nu = 2k+3$, a contradiction because $2 \nmid 2k+3$. Consider the case when $2k+1-s+\nu > 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $2k+2-t = 2k+1-s+\nu-(2k+1)$ and (viii) $t = s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2, 3, \dots, 2k+1\}$. This completes the proof. □

Theorem 2. *Let* $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ *where* $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ *and* $L_{2k+1}[\ell_{ij}] = L[2k+1][2k+1]$. *Then*[6] $M_{2k+1}[m_{ij}]$ *is a magic square of order* $2k+1$ *if* $3 \nmid 2k+1$.

P r o o f. In order to prove that M_{2k+1} is a magic square it is sufficient to show that the all elements in both diagonals of the matrix K_{2k+1} and the matrix L_{2k+1} are mutually different. First, according to $K[2k+1][2k+1]$ we have that

$$k_{ii} = \begin{cases} \pi(k+t+1), & \text{if} \quad i = 2t \\ \pi(t+1), & \text{if} \quad i = 2t+1 \end{cases}$$

for $t = 1, 2, \dots, k$. Since $t+1 < k+t+1$ it follows that k_{ii} are mutually different for $i = 1, 2, \dots, 2k+1$. Next, according to $K[2k+1][2k+1]$ we have that

$$k_{i,2k+2-i} = \begin{cases} \pi(k+2-3t), & \text{if} \quad i = 2t \wedge k+2-3t \geq 0 \\ \pi(k+2-3t+2k+1), & \text{if} \quad i = 2t \wedge k+2-3t < 0 \\ \pi(2k+1-3t), & \text{if} \quad i = 2t+1 \wedge 2k+1-3t \geq 0 \\ \pi(2(2k+1)-3t), & \text{if} \quad i = 2t+1 \wedge 2k+1-3t < 0 \end{cases}$$

for $t = 1, 2, \dots, k$. Since $3 \nmid 2k+1$ and $k+2 = 3(k+1)-(2k+1)$ it follows that $k+2-3t \neq 0$ and $2k+1-3t \neq 0$. Let $2k+1 \equiv \varepsilon \mod 3$ where $\varepsilon \in \{-1, 1\}$. Then we have (i) $k+2-3t \equiv -\varepsilon \mod 3$ and (ii) $k+2-3t+2k+1 \equiv 0 \mod 3$, which provides that $k_{2t,2k+2-2t}$ are mutually different for $t = 1, 2, \dots, k$. Since (iii) $2k+1-3t \equiv \varepsilon \mod 3$ and (iv) $2(2k+1)-3t \equiv -\varepsilon \mod 3$, we find that $k_{2t+1,2k+2-(2t+1)}$ are mutually different for $t = 1, 2, \dots, k$. Of course, since $3 \nmid 2k+1$ we have $k_{1,2k+1} = \pi(2k+1) \neq k_{2t+1,2k+2-(2t+1)}$ for $t = 1, 2, \dots, k$. On the contrary, assume that $k_{i,2k+2-i} = k_{j,2k+2-j}$ for some $i \neq j$. Then according to

[6] The first row of the matrix $K_{2k+1}[k_{ij}]$ is a permutation of the numbers $1, 2, \dots, 2k+1$ and it can be selected in $(2k+1)!$ different ways, and the first row of the matrix $L_{2k+1}[\ell_{ij}]$ is a permutation of the numbers $0, 2k+1, \dots, 2k(2k+1)$ and it can be selected in $(2k+1)!$ different ways, which provides that the matrix $M_{2k+1}[m_{ij}]$ can be formed in $(2k+1)! \times (2k+1)!$ different ways.

(i), (ii), (iii) and (iv) it must be $\pi(k+2-3t) = \pi(2(2k+1)-3s)$, which provides that $k+2-3t = 2(2k+1)-3s$ for some $t = 1,2,\dots,k$ and $s = 1,2,\dots,k$. Then $k+2-3t \le k-1 < k+1 < 2(2k+1)-3s$, a contradiction.

Next, we shall now demonstrate that the all elements in both diagonals of the matrix $L[2k+1][2k+1]$ are mutually different. Indeed, according to $L[2k+1][2k+1]$ we have that

$$\ell_{ii} = \begin{cases} \pi((k-1+3t)(2k+1)), & \text{if} \quad i = 2t \wedge k-1+3t \le 2k+1 \\ \pi((k-1+3t-(2k+1))(2k+1)), & \text{if} \quad i = 2t \wedge k-1+3t > 2k+1 \\ \pi(3t(2k+1)), & \text{if} \quad i = 2t+1 \wedge 3t \le 2k+1 \\ \pi(3t-(2k+1))(2k+1)), & \text{if} \quad i = 2t+1 \wedge 3t > 2k+1 \end{cases}$$

for $t = 1,2,\dots,k$. Since $3 \nmid 2k+1$ and $k-1 = 3k-(2k+1)$ it follows that $k-1+3t \ne 2k+1$ and $3t \ne 2k+1$. Let $2k+1 \equiv \varepsilon \bmod 3$ where $\varepsilon \in \{-1,1\}$. Then we have (i) $k-1+3t \equiv -\varepsilon \bmod 3$ and (ii) $k-1+3t-(2k+1) \equiv \varepsilon \bmod 3$, which provides that $\ell_{2t,2t}$ are mutually different for $t = 1,2,\dots,k$. Since (iii) $3t \equiv 0 \bmod 3$ and (iv) $3t-(2k+1) \equiv -\varepsilon \bmod 3$, we find that $\ell_{2t+1,2t+1}$ are mutually different for $t = 1,2,\dots,k$. Of course, since $3 \nmid 2k+1$ we have $\ell_{11} = \pi(0) \ne \ell_{2t+1,2t+1}$ for $t = 1,2,\dots,k$. On the contrary, assume that $\ell_{ii} = \ell_{jj}$ for some $i \ne j$. Then according to (i), (ii), (iii) and (iv) it must be $\pi((k-1+3t)(2k+1)) = \pi((3s-(2k+1))(2k+1))$, which provides that $k-1+3t = 3s-(2k+1)$ for some $t = 1,2,\dots,k$ and $s = 1,2,\dots,k$. Then $k-1+3t \ge k+2 > k > 3s-(2k+1)$, a contradiction. Next, according to $L[2k+1][2k+1]$ we have that

$$\ell_{i,2k+2-i} = \begin{cases} \pi((k-t)(2k+1)), & \text{if} \quad i = 2t \\ \pi((2k-t)(2k+1)), & \text{if} \quad i = 2t+1 \end{cases}$$

for $t = 1,2,\dots,k$. Since $k-t < 2k-t$ it follows that ℓ_{ii} are mutually different for $i = 1,2,\dots,2k+1$. □

Corollary 1. *Let* $M_n[m_{ij}] = K_n[k_{ij}] + L_n[\ell_{ij}]$ *for* $n \in 2\mathbb{N}+1$, *where* $K_n[k_{ij}] = K[n][n]$ *and* $L_n[\ell_{ij}] = L[n][n]$. *Then*

$$M_n[m_{ij}] = \begin{cases} \text{the magic square,} & \text{if} \quad n = 6k-1 \\ \text{the magic square,} & \text{if} \quad n = 6k+1 \\ \text{the semi-magic square,} & \text{if} \quad n = 6k-3 \end{cases}$$

for $k \in \mathbb{N}$.

Remark 5. In case that $k = 2$ the applied method of cyclic permutations for creating the magic squares is reduced to the method of cyclic permutations for creating the magic squares of order 5 established by French mathematician Philippe de La Hire.

3. Two infinite classes of strongly regular graphs. Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ be a semi-magic square of order $2k+1$ for $k \geq 1$. Let $G[M_{2k+1}]$ be a graph obtained from the matrix $M_{2k+1}[m_{ij}]$ in the following way: (i) the vertex set of the graph $G[M_{2k+1}]$ is $V(G[M_{2k+1}]) = \{m_{ij} \,|\, i,j = 1,2,\dots,2k+1\}$ and (ii) the neighborhood of the vertex $m_{ij} = k_{ij} + \ell_{ij}$ is $S_{m_{ij}} = S_{m_{i,-j}} \cup S_{m_{-i,j}} \cup K_{ij} \cup L_{ij}$ where

$$K_{ij} = \{m_{st} \,|\, k_{st} = k_{ij} \text{ and } (s,t) \neq (i,j)\}, \tag{5}$$

$$L_{ij} = \{m_{st} \,|\, \ell_{st} = \ell_{ij} \text{ and } (s,t) \neq (i,j)\}, \tag{6}$$

for $s,t = 1,2,\dots,2k+1$. We note that $K_{ij} \cap L_{ij} = \emptyset$ for $i,j = 1,2,\dots,2k+1$. Indeed, on the contrary, assume that $m_{st} \in K_{ij} \cap L_{ij}$. Then $m_{st} = k_{st} + \ell_{st} = k_{ij} + \ell_{ij} = m_{ij}$, a contradiction. Namely, it is easy to see that $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}, L_{ij}$ are mutually disjoint. For the sake of an example, let us show that $S_{m_{i,-j}} \cap K_{ij} = \emptyset$. On the contrary, assume that $m_{st} \in S_{m_{i,-j}} \cap K_{ij}$. Using (1) it follows that $s = i$ and $t \neq j$. Since $m_{it} \in K_{ij}$ and $k_{it} = k_{ij}$ we find that k_{ij} is presented in the i-th row of the matrix $K_{2k+1}[k_{ij}]$ two times, a contradiction. Since $k_{ij} \in K_{2k+1} = K_{2k+1}[k_{ij}]$ is presented in the i-th row and the j-th column only one time and $m_{ij} \notin K_{ij}$, we obtain $|K_{ij}| = (2k+1)-1$. Similarly, since $\ell_{ij} \in L_{2k+1} = L_{2k+1}[\ell_{ij}]$ is presented in the i-th row and the j-th column only one time and $m_{ij} \notin L_{ij}$, we obtain $|L_{ij}| = (2k+1) - 1$. Therefore, we have

$$|S_{m_{ij}}| = |S_{m_{i,-j}}| + |S_{m_{-i,j}}| + |K_{ij}| + |L_{ij}| = 2k + 2k + 2k + 2k,$$

which provides that $G[M_{2k+1}]$ is a regular[7] graph of order $n = (2k+1)^2$ and degree $r = 8k$.

Theorem 3. *Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ be a semi-magic square of order $2k+1$ for $k \geq 2$. Then $G[M_{2k+1}]$ is a strongly regular graph of order $n = (2k+1)^2$ and degree $r = 8k$ with $\tau = 2k+5$ and $\theta = 12$.*

Proof. First, assume that m_{ij} and $m_{\mu\nu}$ are two distinct non-adjacent vertices of the graph $G[M_{2k+1}]$. In this case we have $\mu \neq i$ and $\nu \neq j$. On the contrary, assume that $\mu = i$ or $\nu = j$. Without loss of generality we can assume

[7] If $k = 1$ then the corresponding graph $G[M_{2k+1}]$ is reduced to the complete graph K_9, a case that will be excluded.

that $\mu = i$ and $\nu \neq j$. Then $m_{i\nu} \in S_{m_{i,-j}}$, which means that $m_{i\nu}$ and m_{ij} are adjacent, a contradiction. Since $m_{\mu\nu} = k_{\mu\nu} + \ell_{\mu\nu}$ it is easy to see $k_{\mu\nu} \neq k_{ij}$ and $\ell_{\mu\nu} \neq \ell_{ij}$. Indeed, if we assume $k_{\mu\nu} = k_{ij}$ then $m_{\mu\nu} \in K_{ij}$, which means that $m_{\mu\nu}$ and m_{ij} are adjacent, a contradiction. We shall now (1^0) prove that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| = 3$. Since k_{ij} is presented in the μ-th row of the matrix K_{2k+1} it follows that there exist $s \neq \nu$ so that $k_{\mu s} = k_{ij}$, which provides that $m_{\mu s} \in S_{m_{\mu,-\nu}}$ and $m_{\mu s} \in K_{ij} \subseteq S_{m_{ij}}$. Similarly, since ℓ_{ij} is presented in the μ-th row of the matrix L_{2k+1} it follows that there exist $t \neq \nu$ so that $\ell_{\mu t} = \ell_{ij}$, which provides that $m_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} \in L_{ij} \subseteq S_{m_{ij}}$. Since $S_{m_{\mu,-\nu}} \cap S_{m_{i,-j}} = \emptyset$ and since $S_{m_{\mu,-\nu}} \cap S_{m_{-i,j}} = \{m_{\mu j}\} \subseteq S_{m_{ij}}$, we obtain[8] that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| \geq 3$. Next, let $m_{\mu x} \in S_{m_{\mu,-\nu}}$ and let $m_{\mu x} \notin \{m_{\mu j}, m_{\mu s}, m_{\mu t}\}$, which provides that $x \notin \{j, s, t\}$. It remains to demonstrate that $m_{\mu x} \notin S_{m_{ij}}$. On the contrary, assume that $m_{\mu x} = k_{\mu x} + \ell_{\mu x} \in S_{m_{ij}}$. Then according to (1), (2), (5) and (6) we find that $m_{\mu x} \in K_{ij}$ or $m_{\mu x} \in L_{ij}$. Without loss of generality we may assume $m_{\mu x} \in K_{ij}$. In this case we have $k_{\mu x} = k_{ij}$. Since $k_{\mu s} = k_{ij}$ we find that k_{ij} is presented in the μ-th row of the matrix K_{2k+1} two times, a contradiction. This completes the assertion (1^0). Using the same arguments as in the proof of (1^0), we can (2^0) prove that $|S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| = 3$. We shall now ($3^0$) prove that $|S_{m_{ij}} \cap K_{\mu\nu}| = 3$. Since $k_{\mu\nu}$ is presented in the i-th row of the matrix K_{2k+1} it follows that there exist $t \neq j$ so that $k_{it} = k_{\mu\nu}$, which provides that $m_{it} \in K_{\mu\nu}$ and $m_{it} \in S_{m_{i,-j}} \subseteq S_{m_{ij}}$. Since $k_{\mu\nu}$ is presented in the j-th column of the matrix K_{2k+1} it follows that there exist $s \neq i$ so that $k_{sj} = k_{\mu\nu}$, which provides that $m_{sj} \in K_{\mu\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap K_{\mu\nu} = \emptyset$. On the contrary, assume that $m_{xy} \in K_{ij} \cap K_{\mu\nu}$. Then $k_{xy} = k_{ij}$ and $k_{xy} = k_{\mu\nu}$, which provides that $k_{\mu\nu} = k_{ij}$, a contradiction. Further, let $P_{ij} = \{p + \ell_{ij} \,|\, p \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}\}$ and let $Q_{ij} = \{k_{ij} + q \,|\, q \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{ij}\}\}$ for $i, j = 1, 2, \ldots, 2k+1$. Due to the fact that k_{ij} is presented in the i-th row and the j-th column of the matrix K_{2k+1} only one time, we easily see $P_{ij} = L_{ij}$ for $i, j = 1, 2, \ldots, 2k+1$. Due to the fact that ℓ_{ij} is presented in the i-th row and the j-th column of the matrix L_{2k+1} only one time, we easily see $Q_{ij} = K_{ij}$ for $i, j = 1, 2, \ldots, 2k+1$. Let $p_0 \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}$ such that $p_0 = k_{\mu\nu}$ and let $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{\mu\nu}\}$ such that $q_0 = \ell_{ij}$. Then $p_0 + \ell_{ij} \in L_{ij} \subseteq S_{m_{ij}}$ and $k_{\mu\nu} + q_0 \in K_{\mu\nu}$. So we obtain $p_0 + \ell_{ij} = p_0 + q_0 = k_{\mu\nu} + q_0$, which provides that $|L_{ij} \cap K_{\mu\nu}| \geq 1$ and $|S_{m_{ij}} \cap K_{\mu\nu}| \geq 3$. Since $p_0 \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}$ and $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{\mu\nu}\}$ are uniquely determined we obtain $|L_{ij} \cap K_{\mu\nu}| = 1$, which completes the assertion (3^0). Using the same arguments

[8]Since $m_{\mu j} \in S_{m_{-i,j}}$, $m_{\mu s} \in K_{ij}$, $m_{\mu t} \in L_{ij}$ and $S_{m_{-i,j}}, K_{ij}, L_{ij}$ are mutually disjoint it follows that $m_{\mu j}, m_{\mu s}$ and $m_{\mu t}$ are three distinct vertices.

as in the proof of (3^0), we can (4^0) prove that $|S_{m_{ij}} \cap L_{\mu\nu}| = 3$. Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = |S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| + |S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| + |S_{m_{ij}} \cap K_{\mu\nu}| + |S_{m_{ij}} \cap L_{\mu\nu}|,$$

from which we obtain $|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = 12$ for any two distinct non-adjacent vertices m_{ij} and $m_{\mu\nu}$. Next, let m_{ij} and $m_{\mu\nu}$ be two adjacent vertices of the graph $G[M_{2k+1}]$. We shall now consider the following two cases:

Case 1. ($m_{\mu\nu} \in S_{m_{i,-j}}$ or $m_{\mu\nu} \in S_{m_{-i,j}}$). Without loss of generality we can assume that $m_{\mu\nu} \in S_{m_{i,-j}}$. In this case we have $\mu = i$ and $\nu \neq j$. We shall now (1^0) prove that $|S_{m_{ij}} \cap S_{m_{i,-\nu}}| = 2k-1$. Since $m_{ij} \notin S_{m_{i,-j}}$ and $m_{i\nu} \notin S_{m_{i,-\nu}}$ we have $|S_{m_{i,-j}} \cap S_{m_{i,-\nu}}| = (2k+1)-2$, which provides that $|S_{m_{ij}} \cap S_{m_{i,-\nu}}| \geq 2k-1$. Since $m_{ij} \notin S_{m_{ij}}$ and $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}$ and L_{ij} are mutually disjoint it follows that $S_{m_{i,-\nu}}, S_{m_{-i,j}}, K_{ij}$ and L_{ij} are also mutually disjoint, which completes the assertion (1^0). We shall now (2^0) prove that $|S_{m_{ij}} \cap S_{m_{-i,\nu}}| = 2$. Since $m_{i\nu} \notin S_{m_{-i,\nu}}$ we have that $S_{m_{i,-j}} \cap S_{m_{-i,\nu}} = \emptyset$ and $S_{m_{-i,j}} \cap S_{m_{-i,\nu}} = \emptyset$. Since k_{ij} is presented in the ν-th column of the matrix K_{2k+1} it follows that there exist $s \neq \mu$ so that $k_{s\nu} = k_{ij}$, which provides that $m_{s\nu} \in S_{m_{-i,\nu}}$ and $m_{s\nu} \in K_{ij} \subseteq S_{m_{ij}}$. Similarly, since ℓ_{ij} is presented in the ν-th column of the matrix L_{2k+1} it follows that there exist $t \neq \mu$ so that $\ell_{t\nu} = \ell_{ij}$, which provides that $m_{t\nu} \in S_{m_{-i,\nu}}$ and $m_{t\nu} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion (2^0). We shall now (3^0) prove that $|S_{m_{ij}} \cap K_{i\nu}| = 2$. Since $m_{i\nu} \notin K_{i\nu}$ and $S_{m_{i,-\nu}} \cap K_{i\nu} = \emptyset$ it follows that $S_{m_{i,-j}} \cap K_{i\nu} = \emptyset$. Since $k_{i\nu}$ is presented in the j-th column of the matrix K_{2k+1} it follows that there exist $s \neq i$ so that $k_{sj} = k_{i\nu}$, which provides that $m_{sj} \in K_{i\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap K_{i\nu} = \emptyset$. On the contrary, assume that $m_{st} \in K_{ij} \cap K_{i\nu}$. Then $k_{st} = k_{ij}$ and $k_{st} = k_{i\nu}$ which yields $k_{ij} = k_{i\nu}$, a contradiction. Next, since $K_{i\nu} = Q_{i\nu}$ and $Q_{i\nu} = \{k_{i\nu} + q \,|\, q \in \{0, 2k+1, \ldots, 2k(2k+1)\} \smallsetminus \{\ell_{i\nu}\}\}$ there exist $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \smallsetminus \{\ell_{i\nu}\}$ such that $q_0 = \ell_{ij}$. In the view of this, we have $k_{i\nu} + q_0 \in K_{i\nu}$ and $k_{i\nu} + q_0 \in L_{ij} \subseteq S_{m_{ij}}$, which completes the assertion (3^0). We shall now (4^0) prove that $|S_{m_{ij}} \cap L_{i\nu}| = 2$. Since $m_{i\nu} \notin L_{i\nu}$ and $S_{m_{i,-\nu}} \cap L_{i\nu} = \emptyset$ it follows that $S_{m_{i,-j}} \cap L_{i\nu} = \emptyset$. Since $\ell_{i\nu}$ is presented in the j-th column of the matrix L_{2k+1} it follows that there exist $s \neq i$ so that $\ell_{sj} = \ell_{i\nu}$, which provides that $m_{sj} \in L_{i\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $L_{ij} \cap L_{i\nu} = \emptyset$. On the contrary, assume that $m_{st} \in L_{ij} \cap L_{i\nu}$. Then $\ell_{st} = \ell_{ij}$ and $\ell_{st} = \ell_{i\nu}$ which yields $\ell_{ij} = \ell_{i\nu}$, a contradiction. Next, since $L_{i\nu} = P_{i\nu}$ and $P_{i\nu} = \{p + \ell_{i\nu} \,|\, p \in \{1, 2, \ldots, 2k+1\} \smallsetminus \{k_{i\nu}\}\}$ there exist $p_0 \in \{1, 2, \ldots, 2k+1\} \smallsetminus \{k_{i\nu}\}$ such that $p_0 = k_{ij}$. In the view of this, we have $p_0 + \ell_{i\nu} \in L_{i\nu}$ and $p_0 + \ell_{i\nu} \in K_{ij} \subseteq S_{m_{ij}}$, which completes the assertion (4^0).

Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}} \cap S_{m_{i\nu}}| = |S_{m_{ij}} \cap S_{m_{i,-\nu}}| + |S_{m_{ij}} \cap S_{m_{-i,\nu}}| + |S_{m_{ij}} \cap K_{i\nu}| + |S_{m_{ij}} \cap L_{i\nu}|,$$

from which we obtain $|S_{m_{ij}} \cap S_{m_{i\nu}}| = (2k-1)+2+2+2$ for any two adjacent vertices m_{ij} and $m_{i\nu}$.

CASE 2. ($m_{\mu\nu} \in K_{ij}$ or $m_{\mu\nu} \in L_{i,j}$). Without loss of generality we can assume that $m_{\mu\nu} \in K_{ij}$. Since $S_{m_{i,-j}}, S_{m_{-i,j}}$ and K_{ij} are mutually disjoint it follows that $\mu \neq i$ and $\nu \neq j$. Since $m_{\mu\nu} = k_{\mu\nu} + \ell_{\mu\nu}$ and $m_{\mu\nu} \in K_{ij}$ we obtain $m_{\mu\nu} = k_{ij} + \ell_{\mu\nu}$, from which we obtain $k_{\mu\nu} = k_{ij}$ and $\ell_{\mu\nu} \neq \ell_{ij}$. We shall now ($1^0$) prove that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| = 2$. Since $\mu \neq i$ and $\nu \neq j$ we have $S_{m_{i,-j}} \cap S_{m_{\mu,-\nu}} = \emptyset$ and $S_{m_{-i,j}} \cap S_{m_{\mu,-\nu}} = \{m_{\mu j}\} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap S_{m_{\mu,-\nu}} = \emptyset$. On the contrary, assume that $m_{\mu t} \in K_{ij} \cap S_{m_{\mu,-\nu}}$. Then $m_{\mu t} = k_{\mu t} + \ell_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} = k_{ij} + \ell_{\mu t} \in K_{ij}$ which yields $k_{\mu t} = k_{ij}$. Since $k_{\mu\nu} = k_{ij}$ and $k_{\mu t} = k_{ij}$ we have $k_{\mu\nu} = k_{\mu t}$. Finally, since $k_{\mu\nu}$ is presented in the μ-th row of the matrix K_{2k+1} only one time we obtain $t = \nu$. In the view of this, we find that $m_{\mu\nu} \in S_{m_{\mu,-\nu}}$, a contradiction. Next, since ℓ_{ij} is presented in the μ-th row of the matrix L_{2k+1} it follows that there exist $t \neq \nu$ so that $\ell_{\mu t} = \ell_{ij}$, which provides that $m_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion (1^0). We shall now (2^0) prove that $|S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| = 2$. Since $\mu \neq i$ and $\nu \neq j$ we have $S_{m_{i,-j}} \cap S_{m_{-\mu,\nu}} = \{m_{i\nu}\} \subseteq S_{m_{ij}}$ and $S_{m_{-i,j}} \cap S_{m_{-\mu,\nu}} = \emptyset$. We shall now demonstrate that $K_{ij} \cap S_{m_{-\mu,\nu}} = \emptyset$. On the contrary, assume that $m_{s\nu} \in K_{ij} \cap S_{m_{-\mu,\nu}}$. Then $m_{s\nu} = k_{s\nu} + \ell_{s\nu} \in S_{m_{-\mu,\nu}}$ and $m_{s\nu} = k_{ij} + \ell_{s\nu} \in K_{ij}$ which yields $k_{s\nu} = k_{ij}$. Since $k_{\mu\nu} = k_{ij}$ and $k_{s\nu} = k_{ij}$ we have $k_{\mu\nu} = k_{s\nu}$. Finally, since $k_{\mu\nu}$ is presented in the ν-th column of the matrix K_{2k+1} only one time we obtain $s = \mu$. In the view of this, we find that $m_{\mu\nu} \in S_{m_{-\mu,\nu}}$, a contradiction. Next, since ℓ_{ij} is presented in the ν-th column of the matrix L_{2k+1} it follows that there exist $t \neq \mu$ so that $\ell_{t\nu} = \ell_{ij}$, which provides that $m_{t\nu} \in S_{m_{-\mu,\nu}}$ and $m_{t\nu} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion (2^0). We shall now (3^0) prove that $|S_{m_{ij}} \cap K_{\mu\nu}| = 2k-1$. Since $k_{\mu\nu} = k_{ij}$ we have that $K_{ij} = \{m_{st} \,|\, k_{st} = k_{ij} \text{ and } (s,t) \neq (i,j)\} \subseteq S_{m_{ij}}$ and $K_{\mu\nu} = \{m_{st} \,|\, k_{st} = k_{ij} \text{ and } (s,t) \neq (\mu,\nu)\}$. Since $m_{ij} \notin K_{ij}$ and $m_{\mu\nu} \notin K_{\mu\nu}$ we find that $|K_{ij} \cap K_{\mu\nu}| = (2k+1)-2$. Since $m_{ij} \notin S_{m_{ij}}$ and $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}, L_{ij} \subseteq S_{m_{ij}}$ are mutually disjoint it follows that $S_{m_{i,-j}}, S_{m_{-i,j}}, L_{ij}$ and $K_{\mu\nu}$ are also mutually disjoint, which completes the assertion (3^0). We shall now (4^0) prove that $|S_{m_{ij}} \cap L_{\mu\nu}| = 2$. Since $\ell_{\mu\nu}$ is presented in the i-th row of the matrix L_{2k+1} it follows that there exist $t \neq j$ so that $\ell_{it} = \ell_{\mu\nu}$, which provides that $m_{it} = k_{it} + \ell_{\mu\nu} \in L_{\mu\nu}$ and $m_{it} = k_{it} + \ell_{it} \in S_{m_{i,-j}} \subseteq S_{m_{ij}}$. Since $\ell_{\mu\nu}$ is presented in the j-th column of the matrix L_{2k+1} it follows that there exist $s \neq i$ so that $\ell_{sj} = \ell_{\mu\nu}$, which provides that $m_{sj} =$

$k_{sj}+\ell_{\mu\nu}\in L_{\mu\nu}$ and $m_{sj}=k_{sj}+\ell_{sj}\in S_{m_{-i},j}\subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij}\cap L_{\mu\nu}=\emptyset$. On the contrary, assume that $m_{st}=k_{st}+\ell_{st}\in K_{ij}\cap L_{\mu\nu}$. Then $k_{st}=k_{ij}$ and $\ell_{st}=\ell_{\mu\nu}$. Since $k_{\mu\nu}=k_{ij}$ we obtain $k_{st}=k_{\mu\nu}$, which provides that $m_{\mu\nu}=k_{\mu\nu}+\ell_{\mu\nu}\in L_{\mu\nu}$, a contradiction. We shall now demonstrate that $L_{ij}\cap L_{\mu\nu}=\emptyset$. On the contrary, assume that $m_{st}=k_{st}+\ell_{st}\in L_{ij}\cap L_{\mu\nu}$. Then $\ell_{st}=\ell_{ij}$ and $\ell_{st}=\ell_{\mu\nu}$, which provides that $\ell_{\mu\nu}=\ell_{ij}$, a contradiction. This completes the assertion (4^0). Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}}\cap S_{m_{\mu\nu}}|=|S_{m_{ij}}\cap S_{m_{\mu,-\nu}}|+|S_{m_{ij}}\cap S_{m_{-\mu,\nu}}|+|S_{m_{ij}}\cap K_{\mu\nu}|+|S_{m_{ij}}\cap L_{\mu\nu}|,$$

from which we obtain $|S_{m_{ij}}\cap S_{m_{\mu\nu}}|=2+2+(2k-1)+2$ for any two adjacent vertices m_{ij} and $m_{\mu\nu}$, which completes the proof. □

Let $G^-[M_{2k+1}]$ be a graph obtained from the matrix $M_{2k+1}[m_{ij}]$ in the following way: (i) the vertex set of the graph $G^-[M_{2k+1}]$ is $V(G^-[M_{2k+1}])=\{m_{ij}\,|\,i,j=1,2,\ldots,2k+1\}$ and (ii) the neighborhood[9] of the vertex $m_{ij}=k_{ij}+\ell_{ij}$ is $S_{m_{ij}}=S_{m_{i,-j}}\cup S_{m_{-i},j}\cup K_{ij}$. Using the same arguments as in the proof of Theorem 3, we can prove the following result.

Theorem 4. *Let $M_{2k+1}[m_{ij}]=K_{2k+1}[k_{ij}]+L_{2k+1}[\ell_{ij}]$ be a semi-magic square of order $2k+1$ for $k\geq 2$. Then $G^-[M_{2k+1}]$ is a strongly regular graph of order $n=(2k+1)^2$ and degree $r=6k$ with $\tau=2k+1$ and $\theta=6$.*

Remark 6. If $k=1$ then the corresponding graph $G^-[M_{2k+1}]$ is reduced to the trivial strongly regular graph $\overline{3K_3}$, a case that is excluded in Theorem 4.

4. Magic squares of order $6k+3$. Using the applied method of cyclic permutations for creating the magic and semi-magic squares, in this section with a minor modification of "the first permutation" we create the magic squares of order $6k+3$ for $k\geq 0$. First, let us assume that $(\pi(1),\pi(2),\ldots,\pi(6k+3))$ is a fixed permutation of the numbers $1,2,\ldots,6k+3$. Second, let us assume that $(\pi(0),\pi(6k+1),\ldots,\pi((6k+2)(6k+3)))$ is a fixed permutation of the numbers $0,6k+3,\ldots,(6k+2)(6k+3)$.

Let us define $X=\{k+2,k+4,\ldots,k+2(2k+1)\}\subseteq\{1,2,\ldots,6k+3\}$ and let $Y=\{1,2,\ldots,6k+3\}\smallsetminus X$. Let us define $X_+=\{(k+1)\overline{k},(k+3)\overline{k},\ldots,(k+4k+1)\overline{k}\}\subseteq\{0,\overline{k},\ldots,(6k+2)\overline{k}\}$ and let $Y_+=\{0,\overline{k},\ldots,(6k+2)\overline{k}\}\smallsetminus X_+$, where $\overline{k}=6k+3$. Let $\pi(X)$ be the set of all permutations of the set X and let $\pi(Y)$ be the set of all permutations of the set Y. Of course, since $|X|=2k+1$ and

[9]Of course, if we define $S_{m_{ij}}=S_{m_{i,-j}}\cup S_{m_{-i},j}\cup L_{ij}$ we also obtain that $G^-[M_{2k+1}]$ is a strongly regular graph of order $n=(2k+1)^2$ and degree $r=6k$ with $\tau=2k+1$ and $\theta=6$.

$|Y| = 4k+2$ we have $|\pi(X)| = (2k+1)!$ and $|\pi(Y)| = (4k+2)!$. Similarly, let $\pi(X_+)$ be the set of all permutations of the set X_+ and let $\pi(Y_+)$ be the set of all permutations of the set Y_+. Of course, since $|X_+| = 2k+1$ and $|Y_+| = 4k+2$ we have $|\pi(X_+)| = (2k+1)!$ and $|\pi(Y_+)| = (4k+2)!$. Let $\mathbf{sum}\,\pi(x)$ be the sum of all elements in a fixed permutation $\pi(x) \in \pi(X)$. Then we have

$$\mathbf{sum}\,\pi(x) = \sum_{t=1}^{2k+1} (k+2t) = (2k+1)(3k+2)\,. \tag{7}$$

Let $\mathbf{sum}\,\pi_+(x)$ be the sum of all elements in a fixed permutation $\pi_+(x) \in \pi(X_+)$. Then we have

$$\mathbf{sum}\,\pi_+(x) = (6k+3)\sum_{t=1}^{2k+1} (k+(2t-1)) = (2k+1)(3k+1)(6k+3)\,. \tag{8}$$

The first row of the matrix K_{6k+3} contains the numbers of a fixed permutation $\pi(x) \in \pi(X)$ and the numbers of a fixed permutation $\pi(y) \in \pi(Y)$ obtained in the following way: (i) on the position $6k+3, 6k, \ldots, 3$ set up the numbers of $\pi(x)$ and (ii) on the position $t \notin \{6k+3, 6k, \ldots, 3\}$ set up the numbers of $\pi(y)$. According to $K[6k+3][6k+3]$ we note that the numbers of the permutation $\pi(x)$ are presented 3 times in the non-main diagonal of the matrix K_{6k+3}, understanding that $K_{6k+3} = K[6k+3][6k+3]$.

The first row of the matrix L_{6k+3} contains the numbers of a fixed permutation $\pi_+(x) \in \pi(X_+)$ and the numbers of a fixed permutation $\pi_+(y) \in \pi(Y_+)$ obtained in the following way: (i) on the position $1, 4, \ldots, 6k+1$ set up the numbers of $\pi_+(x)$ and (ii) on the position $t \notin \{1, 4, \ldots, 6k+1\}$ set up the numbers of $\pi_+(y)$. According to $L[6k+3][6k+3]$ we note that the numbers of the permutation $\pi_+(x)$ are presented 3 times in the main diagonal of the matrix L_{6k+3}, understanding that $L_{6k+3} = L[6k+3][6k+3]$. So we arrive at the following result:

Theorem 5. *Let $M_{6k+3}[m_{ij}] = K_{6k+3}[k_{ij}] + L_{6k+3}[\ell_{ij}]$ where $K_{6k+3}[k_{ij}] = K[6k+3][6k+3]$ and $L_{6k+3}[\ell_{ij}] = L[6k+3][6k+3]$. If the first row of the matrix $K_{6k+3}[k_{ij}]$ and the first row of the matrix $L_{6k+3}[\ell_{ij}]$ is created in a way described in this section then $M_{6k+3}[m_{ij}]$ is a magic square of order $6k+3$ for any $k \geq 0$.*

Proof. Keeping in mind that the numbers of the permutation $\pi(x)$ are presented 3 times in the non-main diagonal of the matrix K_{6k+3} and the numbers of the permutation $\pi_+(x)$ are presented 3 times in the main diagonal of the matrix

L_{6k+3}, using (7) and (8) we obtain[10]

$$3\,\mathbf{sum}\,\pi(x) + 3\,\mathbf{sum}\,\pi_+(x) = (6k+3)\left(\frac{(6k+3)^2+1}{2}\right),$$

which provides that $M_{6k+3}[m_{ij}] = K_{6k+3}[k_{ij}] + L_{6k+3}[\ell_{ij}]$ is a magic square[11] of order $6k+3$ for any $k \geq 0$. $\square$

Remark 7. We have written a source program in the programming language `Borland C++ Builder 5.5` for creating the magic squares of order $3, 5, \ldots, 999$. The source program can be sent to the interested people by request. Of course, the source programs for creating strongly regular graphs using magic squares can be also sent by request.

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[10](i) we note that $3\,\mathbf{sum}\,\pi(x)$ is equal to the sum of all elements in any row and any column of the matrix K_{6k+3} and (ii) we note that $3\,\mathbf{sum}\,\pi_+(x)$ is equal to the sum of all elements in any row and any column of the matrix L_{6k+3}.

[11]The first row of the matrix $K_{6k+3}[k_{ij}]$ is a permutation of the numbers that belong to the sets X, Y and it can be selected in $(2k+1)! \times (4k+2)!$ different ways, and the first row of the matrix $L_{6k+3}[\ell_{ij}]$ is a permutation of the numbers that belong to the sets X_+, Y_+ and it can be selected in $(2k+1)! \times (4k+2)!$ different ways, which provides that the matrix $M_{6k+3}[m_{ij}]$ can be formed in $((2k+1)!)^2 \times ((4k+2)!)^2$ different ways.

[12]The author is also very grateful to Marko Lepović for his suggestions on this paper.