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# STRONG CONVERSE INEQUALITIES FOR THE WEIGHTED SIMULTANEOUS APPROXIMATION BY THE SZÁSZ-MIRAKJAN OPERATOR*

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Abstract. We establish two-term strong converse estimates of the rate of weighted simultaneous approximation by the Szász-Mirakjan operator for smooth functions in the supremum norm on the non-negative semi-axis. We consider Jacobi-type weights. The estimates are stated in terms of appropriate moduli of smoothness or $K$-functionals.

**1. Main results.** The Szász-Mirakjan operator for a function $f(x)$ defined on $[0,\infty)$ is given by

$$S_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) s_{n,k}(x), \quad s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \quad n \ge 1, \quad x \ge 0,$$

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as $n$ is not necessarily an integer.

Let $C[0,\infty)$ denote the space of the continuous, not necessarily bounded, functions on $[0,\infty)$, and $L_\infty[0,\infty)$ be the space of the essentially bounded Lebesgue measurable function on $[0,\infty)$, equipped with the essential supremum norm $\|\circ\|$.

We will consider simultaneous approximation by the Szász-Mirakjan operator in the essential supremum norm on $[0,\infty)$ with weights of the form

$$w(x) = w(\gamma_0,\gamma_\infty;x) = \left(\frac{x}{1+x}\right)^{\gamma_0}(1+x)^{\gamma_\infty}. \tag{1.1}$$

Let $r \in \mathbb{N}_+$ and $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. We denote by $\mathbb{N}_+$ the set of the positive integers. In [8, Theorem 1.2] we proved the direct estimate

$$\|w(S_n f - f)^{(r)}\| \le c\,\widetilde{K}_r(f^{(r)}, n^{-1})_w$$

for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$. Here and henceforward $c$ stands for a positive constant (not necessarily the same at each occurrence), which is independent of the approximated function $f$ and the degree of the operator $n$. The $K$-functional $\widetilde{K}_r(f^{(r)},t)_w$ is defined by

$$\widetilde{K}_r(f^{(r)},t)_w := \inf\Big\{\|w(f^{(r)}-g^{(r)})\| + t\|w\big(\widetilde{D}g\big)^{(r)}\| \\ : g \in AC^{r+1}[0,\infty),\ wg^{(r)}, w\big(\widetilde{D}g\big)^{(r)} \in L_\infty[0,\infty)\Big\},$$

where $\widetilde{D}g(x) := xg''(x)$, $AC^m[0,\infty)$ is the set of the functions which along with their derivatives up to order $m$ are absolutely continuous on $[a,b]$ for every $[a,b] \subset [0,\infty)$.

In the present paper, we will establish the following converse inequality.

**Theorem 1.1.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0,\gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exists $R \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $k,n \ge 1$ with $k \ge Rn$ there holds*

$$\widetilde{K}_r(f^{(r)},n^{-1})_w \le c\frac{k}{n}\Big(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\|\Big).$$

*In particular,*

$$\widetilde{K}_r(f^{(r)},n^{-1})_w \le c\Big(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\Big).$$

*The constant $c > 0$ is independent of $f$, $k$ and $n$.*

The rate of the simultaneous approximation by the Szász-Mirakjan operator can be estimated by simpler function characteristics—moduli of smoothness.

We will use the weighted Ditzian-Totik modulus of smoothness $\omega_\varphi^2(f,t)_w$ defined in [5, p. 56] with $\varphi(x) := \sqrt{x}$ and the weighted modulus of continuity

$$\omega(f,t)_w := \sup_{0<h\le t} \|w\overrightarrow{\Delta}_h f\|,$$

where

$$\overrightarrow{\Delta}_h f(x) := f(x+h) - f(x), \quad x \ge 0.$$

In [8, Theorem 1.1] it was established that

$$\|w(S_n f - f)^{(r)}\| \le c\left(\omega_\varphi^2(f^{(r)}, n^{-1/2})_w + \omega(f^{(r)}, n^{-1})_w\right), \quad n \ge n_0, \tag{1.2}$$

with some $n_0 \ge 1$ for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$ provided that $0 \le \gamma_0 < r$, whereas $\gamma_\infty$ is arbitrary. Also, there was shown that the second term on the right above is redundant if $0 < \gamma_0 < r$ and $\gamma_\infty > 0$.

Here we will derive from Theorem 1.1 the following converse estimate.

**Theorem 1.2.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0,\gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exist $R, n_0 \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$ there hold*

$$\omega_\varphi^2(f^{(r)}, n^{-1/2})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge n_0,$$

*and*

$$\omega(f^{(r)}, n^{-1})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge 1.$$

*The constant $c > 0$ is independent of $f$ and $n$.*

We say that the real-valued functions $A(f,n)$ and $B(f,n)$ are equivalent and write $A(f,n) \sim B(f,n)$ for $f$ and $n$ in specified domains iff there exists a positive constant $c$ such that $c^{-1}B(f,n) \le A(f,n) \le c\,B(f,n)$ for all $f$ and $n$ in the specified domains.

Theorems 1.1 and 1.2, [8, Theorems 1.1 and 1.2], and properties of the $K$-functionals and moduli (see [5, Theorem 6.1.1]) imply the following equivalences.

**Corollary 1.3.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0,\gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exist $R, n_0 \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge n_0$ there hold*

$$\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\| \sim \widetilde{K}_r(f^{(r)}, n^{-1})_w$$

$$\sim \omega_\varphi^2(f^{(r)}, n^{-1/2})_w + \omega(f^{(r)}, n^{-1})_w.$$

In particular, the direct inequality (1.2) and Theorem 1.2 (or Corollary 1.3) readily imply a big $O$-characterization of the rate of the simultaneous approximation by the Szász-Mirakjan operator.

**Corollary 1.4.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Let also $f \in C[0,\infty)$ be such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and $0 < \alpha \le 1$. Then*

$$\|w(S_n f - f)^{(r)}\| = O(n^{-\alpha}) \iff \omega_\varphi^2(f^{(r)}, t)_w = O(t^{2\alpha}) \quad \text{and} \quad \omega(f^{(r)}, t)_w = O(t^\alpha).$$

The approximation of $f'$ with $(S_n f)'$ is closely related to the approximation by means of the Szász-Mirakjan-Kantorovich operator. This operator is defined for functions $f(x)$, which are summable on every compact subinterval of $[0,\infty)$, by

$$\widetilde{S}_n f(x) := \sum_{k=0}^{\infty} s_{n,k}(x)\, n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\, du, \quad x \ge 0.$$

We set

$$F(x) := \int_0^x f(u)\, du, \quad x \ge 0.$$

Then, by virtue of (2.8) below,

$$\widetilde{S}_n f(x) = (S_n F)'(x).$$

Now, Theorems 1.1 and 1.2 yield the following converse inequalities for the simultaneous approximation by the Szász-Mirakjan-Kantorovich operator in weighted $L_\infty$-spaces.

**Theorem 1.5.** *Let $r \in \mathbb{N}_0$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r+1$ and $\gamma_\infty \ne r+1$. Then there exists $R \ge 1$ such that for all $f(x)$, which are summable on every compact subinterval of $[0,\infty)$, $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there holds*

$$\widetilde{K}_{r+1}(f^{(r)}, n^{-1})_w \le c\left(\|w(\widetilde{S}_n f - f)^{(r)}\| + \|w(\widetilde{S}_{Rn} f - f)^{(r)}\|\right).$$

**Theorem 1.6.** *Let $r \in \mathbb{N}_0$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1), as $0 \le \gamma_0 < r+1$ and $\gamma_\infty \ne r+1$. Then there exist $R, n_0 \ge 1$ such that for all $f(x)$,* *which are summable on every compact subinterval of* $[0,\infty)$, $f \in AC_{loc}^{r-1}(0,\infty)$ *and* $wf^{(r)} \in L_\infty[0,\infty)$ *there hold*

$$\omega_\varphi^2(f^{(r)}, n^{-1/2})_w \le c\left(\|w(\widetilde{S}_n f - f)^{(r)}\| + \|w(\widetilde{S}_{Rn} f - f)^{(r)}\|\right), \quad n \ge n_0,$$

*and*

$$\omega(f^{(r)}, n^{-1})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge 1.$$

*The constant* $c > 0$ *is independent of* $f$ *and* $n$.

Here the assumption $f \in AC_{loc}^{r-1}(0,\infty)$ is to be ignored for $r = 0$. The unweighted case, that is $w = 1$, for $r = 0$ was considered in [10] in $L_p[0,\infty)$, $1 < p \le \infty$. Weaker converse results for $r = 0$, but for more general operators in some instances, were obtained earlier in [5, Theorems 9.3.2 and 10.1.3] and [14, 15].

The contents of the paper are organized as follows. In the next section we establish a Voronovskaya-type estimate and several Bernstein-type inequalities for the simultaneous approximation by the Szász-Mirakjan operator in weighted $L_\infty$-norm. Then, in the last section, we apply them to verify Theorem 1.1 and by means of the method for proving converse inequalities, described in [4]. There we also give a proof of Theorem 1.2.

## 2. Basic assertions.

We begin with several notations and known auxiliary results.

Let $AC_{loc}^{m}(0,\infty)$ denote the set of the functions which along with their derivatives up to order $m$ are absolutely continuous on $[a,b]$ for every $[a,b] \subset (0,\infty)$.

We set $s_{n,k} := 0$ for $k < 0$. Direct computations yield the following two formulas for the derivatives of $s_{n,k}(x)$, $k \in \mathbb{N}_0$:

$$s'_{n,k}(x) = n(s_{n,k-1}(x) - s_{n,k}(x)) \tag{2.1}$$

and

$$s'_{n,k}(x) = \frac{1}{x}(k - nx)\, s_{n,k}(x). \tag{2.2}$$

For a sequence $\{a_k\}_{k\in\mathbb{Z}}$ we define $\Delta a_k := a_k - a_{k-1}$ and $\Delta^r a_k := \Delta(\Delta^{r-1} a_k)$. Set $s_k(n,x) := s_{n,k}(x)$. Then iterating (2.1), we get

$$s_{n,k}^{(r)}(x) = (-1)^r n^r \Delta^r s_k(n,x). \tag{2.3}$$

Likewise, using (2.2), we get by induction on $r$ the formula (cf. [5, (9.4.9)])

$$s_{n,k}^{(r)}(x) = x^{-r} s_{n,k}(x) \sum_{0\le i\le r/2} (nx)^i \sum_{j=0}^{r-2i} d_{r,i,j}(k-nx)^j, \tag{2.4}$$

where $d_{r,i,j}$ are constants, whose value is independent of $n$ and $k$.

For $\ell \in \mathbb{N}_0$ we set

$$T_{n,\ell}(x) := n^\ell S_n\left((\circ - x)^\ell\right)(x) = \sum_{k=0}^{\infty} (k-nx)^\ell s_{n,k}(x). \tag{2.5}$$

As is known (see [5, Lemma 9.5.5]), we have for $\ell \ge 1$

$$T_{n,\ell}(x) = \sum_{1\le\rho\le\ell/2} d_{\ell,\rho}(nx)^\rho,$$

where $d_{\ell,\rho}$ are constants, whose value is independent of $n$. We follow the convention that an empty sum is identically 0. In particular, we have (see e.g. [12, p. 94])

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = T_{n,3}(x) = nx, \\ &T_{n,4}(x) = 3(nx)^2 + nx. \end{aligned} \tag{2.6}$$

Identity (2.5) yields for $m \ge 1$

$$0 \le T_{n,2\ell}(x) \le c \begin{cases} nx, & nx \le 1, \\ (nx)^\ell, & nx \ge 1. \end{cases}$$

Then, by means of Cauchy's inequality and the identity $\sum_{k=0}^{\infty} s_{n,k}(x) \equiv 1$, we get

$$0 \le \sum_{k=0}^{\infty} |k-nx|^\ell s_{n,k}(x) \le \sqrt{T_{n,2\ell}(x)} \le c \begin{cases} 1, & nx \le 1, \\ (nx)^{\ell/2}, & nx \ge 1. \end{cases} \tag{2.7}$$

We will also use the quantities

$$T_{r,n,\ell}(x) := \sum_{k=0}^{\infty} (k-nx)^\ell s_{n,k}^{(r)}(x).$$

To recall, the forward finite difference of $f : [0,\infty) \to \mathbb{R}$ with step $h > 0$ is defined by $\overrightarrow{\Delta}_h f(x) := f(x+h) - f(x)$, $x \ge 0$. We have the following formula for its $r$th iterate, $\overrightarrow{\Delta}_h^r := \overrightarrow{\Delta}_h(\overrightarrow{\Delta}_h^{r-1})$,

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x + (r-i)h), \quad x \geq 0.$$

As is known (see [13] or [5, (9.4.3)])

$$(S_n f)^{(r)}(x) = n^r \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^r f\left(\frac{k}{n}\right) s_{n,k}(x), \quad x \geq 0. \tag{2.8}$$

In [8, Proposition 3.1] it was shown that if $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ is given by (1.1) with $0 \leq \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$, then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \geq 1$ there holds

$$\|w(S_n f)^{(r)}\| \leq c\,\|wf^{(r)}\|. \tag{2.9}$$

Next, we will establish a Voronovskaya-type inequality. A basic tool in its proof is the following formula.

**Lemma 2.1.** *Let $r \in \mathbb{N}_+$, $\gamma \in \mathbb{R}$ and $n \geq 1$. Let also $f \in C[0,\infty)$ be such that $\varphi^\gamma f \in L_\infty[1,\infty)$, $f \in AC_{loc}^{r+3}(0,\infty)$ and $\varphi^{2r+6} f^{(r+4)} \in L[0,1]$. Then*

$$\begin{aligned}
&\left(S_n f(x) - f(x) - \frac{1}{2n}\,\widetilde{D} f(x)\right)^{(r)} \\
&\quad = \frac{S(r+2,r)}{(r+1)(r+2)n^2}\, f^{(r+2)}(x) \\
&\quad + \left(\frac{(3r+2)x}{12n^2} + \frac{S(r+3,r)}{(r+1)(r+2)(r+3)n^3}\right) f^{(r+3)}(x) \\
&\quad + \frac{1}{(r+3)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+3} f^{(r+4)}(u)\, du, \quad x > 0.
\end{aligned}$$

Here $S(m,r) := \dfrac{1}{r!} \displaystyle\sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-i)^m$ are the Stirling numbers of the second kind.

Proof. By [7, Proposition 2.1] with $p = 1$, $g = f$, $j = r+2, r+3$, $m = r+4$, $w_1 = \varphi^{2j-2}$ and $w_2 = \varphi^{2r+6}$ we get

$$\varphi^{2j-2} f^{(j)} \in L[0,1], \quad j = r+2, r+3. \tag{2.10}$$

Then (see e.g. [7, p. 106, (3.11)]) we have

$$\lim_{u \to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u) = 0, \quad \sigma = r+1, r+2. \tag{2.11}$$

By [8, Lemma 2.2] (the lemma is applicable by virtue of (2.10) with $j = r+2$), we have

$$\big(S_n f(x) - f(x)\big)^{(r)} = \frac{r}{2n}\, f^{(r+1)}(x) + \frac{1}{(r+1)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+1} f^{(r+2)}(u)\, du, \quad x > 0.$$

Next, we integrate by parts the integrals twice, as for the term with $k = 0$ we take into consideration (2.10) with $j = r+3$ and (2.11). Thus we arrive at

$$\begin{aligned}\big(S_n f(x) - f(x)\big)^{(r)} &= \frac{r}{2n}\, f^{(r+1)}(x) + \frac{1}{(r+2)!\, n^{r+2}}\, T_{r,n,r+2}(x) f^{(r+2)}(x) \\ &+ \frac{1}{(r+3)!\, n^{r+3}}\, T_{r,n,r+3}(x) f^{(r+3)}(x) \\ &+ \frac{1}{(r+3)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+3} f^{(r+4)}(u)\, du, \quad x > 0.\end{aligned}$$

We will show that

$$\begin{aligned} T_{r,n,r+2}(x) &= n^r \left( r!\, S(r+2, r) + \frac{(r+2)!}{2}\, nx \right), \\ T_{r,n,r+3}(x) &= n^r \left( r!\, S(r+3, r) + \frac{(r+3)!\,(3r+2)}{12}\, nx \right). \end{aligned} \tag{2.12}$$

Then, since $(\widetilde{D} f)^{(r)}(x) = r f^{(r+1)}(x) + x f^{(r+2)}(x)$, we get the assertion of the lemma.

By virtue of [8, Lemma 2.1] with $\ell = r+2, r+3$, we have

$$T_{r,n,r+2}(x) = n^r (d_1 + d_2 nx)$$

and

$$T_{r,n,r+3}(x) = n^r (d_3 + d_4 nx),$$

where $d_i$, $i = 1, \ldots, 4$ are constants whose value is independent of $n$ (and $x$).

Clearly, $s_{n,k}^{(r)}(0) = (-1)^{r-k} n^r \binom{r}{k}$ for $0 \le k \le r$, and $s_{n,k}^{(r)}(0) = 0$ for $k > r$. Therefore,

$$d_1 = n^{-r} T_{r,n,r+2}(0) = \sum_{k=0}^{\infty} k^{r+2} s_{n,k}^{(r)}(0)$$

$$= \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^{r+2}$$
$$= r!\, S(r+2, r).$$

Just similarly, we get

$$d_3 = r!\, S(r+3, r).$$

To calculate $d_2$ we use analogous considerations and also $T_{r,n,r+1}(x) \equiv n^r (r+1)!\, r/2$ (see [8, Lemma 2.1]) to obtain

$$\begin{aligned} d_2 &= n^{-r-1} T'_{r,n,r+2}(x) \\ &= -n^{-r}(r+2) T_{r,n,r+1}(x) + n^{-r-1} T_{r+1,n,r+2}(x) \\ &= \frac{(r+2)!}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_4 &= n^{-r-1} T'_{r,n,r+3}(x) \\ &= -n^{-r}(r+3) T_{r,n,r+2}(x) + n^{-r-1} T_{r+1,n,r+3}(x) \\ &= r!\big[(r+1) S(r+3, r+1) - (r+3) S(r+2, r)\big] \\ &= \frac{(r+3)!\,(3r+2)}{12}. \end{aligned}$$

Above we have used that (see [11, Section 3.4])

$$\begin{aligned} S(r+2, r) &= \binom{r+2}{3} + 3\binom{r+2}{4} \\ &= \frac{r(r+1)(r+2)(3r+1)}{24}. \end{aligned} \tag{2.13}$$

This completes the proof of (2.12). □

**Proposition 2.2.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $0 \le \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0, \infty)$ such that $f \in AC_{loc}^{r+3}(0, \infty)$ and $wf^{(r+2)}, wf^{(r+3)}, w\varphi^4 f^{(r+4)} \in L_\infty[0, \infty)$ and all $n \ge 1$ there holds*

$$\begin{aligned} &\left\| w\left( S_n f - f - \frac{1}{2n}\, \widetilde{D} f \right)^{(r)} \right\| \\ &\qquad \le \frac{c}{n^2} \left( \|wf^{(r+2)}\| + \|w\varphi^2 f^{(r+3)}\| + \|w\varphi^4 f^{(r+4)}\| \right) + \frac{c}{n^3}\, \|wf^{(r+3)}\|. \end{aligned}$$

*The constant $c > 0$ is independent of $f$ and $n$.*

**Remark 2.3.** Let us note that $wf^{(r+2)}, wf^{(r+3)}, w\varphi^4 f^{(r+4)} \in L_\infty[0,\infty)$ implies $w\varphi^2 f^{(r+3)} \in L_\infty[0,\infty)$. This can be shown by e.g. [9, Proposition 4.1] with $p=\infty$, $k=1$, $r$ fixed to be equal to 2, $g = f^{(r+2)}$ and $a = 1/2$ (or see [6, Lemma 1]), which yields

$$\|w\varphi^2 f^{(r+3)}\|_{[1/2,\infty)} \le c\left(\|wf^{(r+2)}\|_{[1/2,\infty)} + \|w\varphi^4 f^{(r+4)}\|_{[1/2,\infty)}\right). \tag{2.14}$$

Here $\|\circ\|_{[1/2,\infty)}$ stands for the essential supremum norm on the interval $[1/2,\infty)$.

Proof of Proposition 2.2. Note that $\varphi^{2r+6} f^{(r+4)} \in L[0,1]$. We set

$$\widetilde{R}_{r,n}(x) := \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x)\, \tilde{\rho}_{r,x}\left(\frac{k}{n}\right),$$

where

$$\tilde{\rho}_{r,x}(t) := \int_x^t (t-u)^{r+3} f^{(r+4)}(u)\, du. \tag{2.15}$$

In view of Lemma 2.1, we have

$$\left\| w\left(S_n f - f - \frac{1}{2n}\widetilde{D} f\right)^{(r)} \right\| \le \frac{c}{n^2}\left(\|wf^{(r+2)}\| + \|w\varphi^2 f^{(r+3)}\|\right) + \frac{c}{n^3}\|wf^{(r+3)}\| + \|w\widetilde{R}_{r,n}\|.$$

To complete the proof of the proposition, we will show that

$$\|w\widetilde{R}_{r,n}\| \le \frac{c}{n^3}\|wf^{(r+3)}\| + \frac{c}{n^2}\|w\varphi^4 f^{(r+4)}\|. \tag{2.16}$$

We use that

$$|\tilde{\rho}_{r,x}(t)| \le \left| \int_x^t \frac{|t-u|^{r+3}}{u^{\gamma_0+2}(1+u)^{\gamma_\infty-\gamma_0}}\, du \right| \|w\varphi^4 f^{(r+4)}\|. \tag{2.17}$$

By Hölder's inequality we arrive at

$$\left| \int_x^t \frac{|t-u|^{r+3}}{u^{\gamma_0+2}(1+u)^{\gamma_\infty-\gamma_0}}\, du \right| \le \left| \int_x^t \frac{|t-u|^{r+3}}{u^{p(\gamma_0+2)}}\, du \right|^{1/p} \left| \int_x^t \frac{|t-u|^{r+3}}{(1+u)^{q(\gamma_\infty-\gamma_0)}}\, du \right|^{1/q}, \tag{2.18}$$

where we have set $p := (r+3)/(\gamma_0+2)$ and $q$ is its conjugate exponent.

It is quite straightforward to verify that

$$\frac{|t-u|}{u} \le \frac{|t-x|}{x}$$

for $u$ between $x$ and $t$. Therefore,

$$\left|\int_x^t \frac{|t-u|^{r+3}}{u^{p(\gamma_0+2)}}\,du\right|^{1/p} \le \frac{|t-x|^{(r+4)/p}}{x^{\gamma_0+2}}. \tag{2.19}$$

Clearly, if $u$ is between $x$ and $t$, then

$$(1+u)^\gamma \le (1+x)^\gamma + (1+t)^\gamma$$

for any $\gamma \in \mathbb{R}$. Consequently,

$$\left|\int_x^t \frac{|t-u|^{r+3}}{(1+u)^{q(\gamma_\infty-\gamma_0)}}\,du\right|^{1/q} \le \frac{|t-x|^{(r+4)/q}}{(1+x)^{\gamma_\infty-\gamma_0}} + \frac{|t-x|^{(r+4)/q}}{(1+t)^{\gamma_\infty-\gamma_0}}. \tag{2.20}$$

Combining (2.17)-(2.20), we arrive at the estimate

$$\begin{aligned} &|w(x)\tilde{\rho}_{r,x}(t)| \\ &\quad\le \left(1+\frac{(1+x)^{\gamma_\infty-\gamma_0}}{(1+t)^{\gamma_\infty-\gamma_0}}\right)\frac{|t-x|^{r+4}}{x^2}\,\|w\varphi^4 f^{(r+4)}\|, \quad x>0,\ t\ge 0. \end{aligned} \tag{2.21}$$

We consider two cases.

Case 1: $nx \ge 1$. Inequality (2.21) implies

$$\begin{aligned} |w(x)R_{r,n}(x)| &\le \frac{1}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\|w\varphi^4 f^{(r+4)}\| \\ &+ \frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}\|w\varphi^4 f^{(r+4)}\|. \end{aligned} \tag{2.22}$$

To estimate the first sum above, we apply (2.4) and (2.7) to deduce

$$\begin{aligned} &\frac{1}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4} \\ &\quad\le \frac{c}{n^2}\sum_{0\le i\le r/2}(nx)^{i-r-2}\sum_{j=0}^{r-2i}\sum_{k=0}^{\infty}|k-nx|^{r+j+4}s_{n,k}(x) \\ &\quad\le \frac{c}{n^2}\sum_{0\le i\le r/2}\sum_{j=0}^{r-2i}(nx)^{(2i-r+j)/2} \le \frac{c}{n^2}, \end{aligned} \tag{2.23}$$

where at the last inequality we have taken into consideration that $2i - r + j \le 0$ for all $i$ and $j$ in the specified range.

We estimate the other sum in (2.22) in a similar way, as we also use Cauchy's inequality on the sum on $k$ in order to split $|k - nx|^{r+j+4}$ and $(1 + k/n)^{\gamma_0 - \gamma_\infty}$. We have

(2.24)

$$\begin{aligned}
&\frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}\\
&\le \frac{c\,(1+x)^{\gamma_\infty-\gamma_0}}{n^2}\sum_{0\le i\le r/2}(nx)^{i-r-2}\sum_{j=0}^{r-2i}\sum_{k=0}^{\infty}|k-nx|^{r+j+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}s_{n,k}(x)\\
&\le \frac{c\,(1+x)^{\gamma_\infty-\gamma_0}}{n^2}\sum_{0\le i\le r/2}(nx)^{i-r-2}\sum_{j=0}^{r-2i}\sqrt{\sum_{k=0}^{\infty}|k-nx|^{2(r+j+4)}s_{n,k}(x)}\\
&\qquad\times\sqrt{\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{2(\gamma_0-\gamma_\infty)}s_{n,k}(x)}.
\end{aligned}$$

By (2.7), we have

$$\sum_{k=0}^{\infty}|k-nx|^{2(r+j+4)}s_{n,k}(x)\le c\,(nx)^{r+j+4},\quad nx\ge 1. \tag{2.25}$$

It was shown in [5, p. 163] that

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{m}s_{n,k}(x)\le c\,(1+x)^{m},\quad x\ge 0,\quad m\in\mathbb{Z}. \tag{2.26}$$

Then by means of Hölder's inequality and the identity $\sum_{k=0}^{\infty}s_{n,k}(x)\equiv 1$ we derive (see [5, p. 162–163])

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{2(\gamma_0-\gamma_\infty)}s_{n,k}(x)\le c\,(1+x)^{2(\gamma_0-\gamma_\infty)},\quad x\ge 0. \tag{2.27}$$

Combining (2.24), (2.25) and (2.27), we arrive at

$$\frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}\le\frac{c}{n^2}.$$

Now, (2.22), (2.23) and the last estimate above yield

$$|w(x)\widetilde{R}_{r,n}(x)| \le \frac{c}{n^2}\,\|w\varphi^4 f^{(r+4)}\|, \quad nx \ge 1. \tag{2.28}$$

Case 2: $nx \le 1$. By means of (2.3) and summation by parts we derive for $n \ge 1$ the relation (cf. (2.8))

$$\widetilde{R}_{r,n}(x) = n^r \sum_{k=0}^{\infty} \overrightarrow{\Delta}^r_{1/n}\tilde{\rho}_{r,x}\left(\frac{k}{n}\right) s_{n,k}(x).$$

Consequently,

$$|w(x)\widetilde{R}_{r,n}(x)| \le c\,n^r \max_{i=0,\dots,r} \sum_{k=0}^{\infty} \left| w(x)\,\tilde{\rho}_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x). \tag{2.29}$$

We will estimate the terms for $k=0$ and $k=1$ separately. For the sum on $k \ge 2$, we apply (2.21) and Cauchy's inequality to arrive at

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left| w(x)\,\tilde{\rho}_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x)\\
&\quad\le \frac{1}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\|\\
&\qquad + \frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4}\left(1+\frac{k+i}{n}\right)^{\gamma_0-\gamma_\infty} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\|\\
&\quad\le \frac{1}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\|\\
&\qquad + \frac{c}{x^2}\sqrt{\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{2(r+4)} s_{n,k}(x)}\\
&\qquad\quad \times \sqrt{\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{2(\gamma_0-\gamma_\infty)} s_{n,k}(x)}\,\|w\varphi^4 f^{(r+4)}\|.
\end{aligned}$$

We will show that

$$\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x) \le \frac{c\,x^2}{n^{l-2}}, \quad l \in \mathbb{N}_+,\ l \ge 2, \tag{2.30}$$

and

$$\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x) \le c\,(nx)^2, \quad \gamma\in\mathbb{R}, \tag{2.31}$$

for $nx \le 1$ and $i = 0, \dots, r$.

Then we will get

$$\sum_{k=2}^{\infty}\left|w(x)\,\tilde\rho_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x) \le \frac{c}{n^{r+2}}\,\|w\varphi^4 f^{(r+4)}\|, \quad i=0,\dots,r. \tag{2.32}$$

To verify (2.30)–(2.31), we apply [8, (3.16) and (3.17)] to the right-hand side of the trivial inequalities

$$\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x) \le nx\sum_{k=1}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x)$$

and

$$\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x) \le nx\sum_{k=1}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x),$$

where $0 \le x \le 1/n$, $l\in\mathbb{N}_+$ and $\gamma\in\mathbb{R}$.

Now, let us consider the terms for $k=0,1$ in (2.29). For $k=0$ and $i=0$ we again use (2.21) to get directly

$$\begin{aligned} |w(x)\,\tilde\rho_{r,x}(0)| &\le c\,x^{r+2}\|w\varphi^4 f^{(r+4)}\| \\ &\le \frac{c}{n^{r+2}}\,\|w\varphi^4 f^{(r+4)}\|. \end{aligned} \tag{2.33}$$

It remains to estimate $\tilde\rho_{r,x}(i/n)$, defined in (2.15), for $i=1,\dots,r+1$. To this end, we expand $(i/n-u)^{r+3}$ by the binomial formula to get

$$\left|w(x)\tilde\rho_{r,x}\left(\frac{i}{n}\right)\right| \le c\,x^{\gamma_0}\sum_{j=0}^{r+3}\frac{1}{n^{r-j+3}}\left|\int_x^{i/n} u^j f^{(r+4)}(u)\,du\right|. \tag{2.34}$$

Clearly, for $j=2,\dots,r+3$ we have

$$\begin{aligned} x^{\gamma_0}\left|\int_x^{i/n} u^j f^{(r+4)}(u)\,du\right| &\le c\,x^{\gamma_0}\int_x^{i/n} u^{j-\gamma_0-2}du\,\|w\varphi^4 f^{(r+4)}\| \\ &\le \frac{c\,x^{\gamma_0}}{n}\left(\frac{1}{n^{j-\gamma_0-2}}+x^{j-\gamma_0-2}\right)\|w\varphi^4 f^{(r+4)}\| \end{aligned}$$

$$\leq \frac{c}{n^{j-1}}\,\|w\varphi^4 f^{(r+4)}\|, \quad x \in (0, 1/n].$$

For the integral in (2.34) with $j = 0$ we have

$$\begin{aligned} x^{\gamma_0}\left|\int_x^{i/n} f^{(r+4)}(u)\,du\right| &= x^{\gamma_0}\left|f^{(r+3)}\left(\frac{i}{n}\right) - f^{(r+3)}(x)\right| \\ &\leq \left(\frac{i}{n}\right)^{\gamma_0}\left|f^{(r+3)}\left(\frac{i}{n}\right)\right| + x^{\gamma_0}|f^{(r+3)}(x)| \\ &\leq c\,\|wf^{(r+3)}\|, \quad x \in (0, 1/n]. \end{aligned}$$

Similarly, for the integral with $j = 1$, we have, after integrating by parts,

$$\begin{aligned} &x^{\gamma_0}\left|\int_x^{i/n} uf^{(r+4)}(u)\,du\right| = x^{\gamma_0}\left|\int_x^{i/n} u\,d\,f^{(r+3)}(u)\right| \\ &\leq \frac{1}{n}\left[\left(\frac{i}{n}\right)^{\gamma_0}\left|f^{(r+3)}\left(\frac{i}{n}\right)\right| + x^{\gamma_0}|f^{(r+3)}(x)|\right] + x^{\gamma_0}\int_x^{i/n} |f^{(r+3)}(u)|\,du \\ &\leq \frac{c}{n}\,\|wf^{(r+3)}\|, \quad x \in (0, 1/n]. \end{aligned}$$

Thus we have established for $nx \leq 1$ and $i = 1, \dots, r+1$

$$\left|w(x)\tilde{\rho}_{r,x}\left(\frac{i}{n}\right)\right| \leq \frac{c}{n^{r+3}}\,\|wf^{(r+3)}\| + \frac{c}{n^{r+2}}\,\|w\varphi^4 f^{(r+4)}\|. \tag{2.35}$$

Inequalities (2.29), (2.32), (2.33) and (2.35) yield

$$|w(x)\widetilde{R}_{r,n}(x)| \leq \frac{c}{n^3}\,\|wf^{(r+3)}\| + \frac{c}{n^2}\,\|w\varphi^4 f^{(r+4)}\|, \quad nx \leq 1.$$

This along with (2.28) completes the proof of (2.16). □

Similar point-wise Voronovskaya-type estimates were established in [1, Theorem 2] for any $r \in \mathbb{N}_0$ and $w(x) := (1+x)^{-2}$, and also in [2] for general linear positive operators, which in particular include $S_n$, for the first and second derivative and weights $w(x) := (1+x)^{-m}$, where $m \in \mathbb{N}_+$.

We proceed to several Bernstein-type inequalities.

**Proposition 2.4.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \leq \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \geq 1$ there hold:*

(a) $\|w(S_n f)^{(r+1)}\| \leq cn\|wf^{(r)}\|$;

(b) $\|w\varphi^2(S_nf)^{(r+2)}\| \le cn\|wf^{(r)}\|$.

Proof. (a) By virtue of (2.8) with $r+1$ in place of $r$, we have

$$\begin{aligned}|(S_nf)^{(r+1)}(x)| &= n^{r+1}\left|\sum_{k=0}^{\infty}\overrightarrow{\Delta}_{1/n}^{r+1}f\left(\frac{k}{n}\right)s_{n,k}(x)\right|\\ &\le 2n^{r+1}\max_{j=0,1}\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{r}f\left(\frac{k+j}{n}\right)\right|s_{n,k}(x),\quad x\ge 0.\end{aligned}$$

Let us recall that (see e.g. [3, p. 45])

$$\overrightarrow{\Delta}_{h}^{r}f(x) = h^r\int_0^r M_r(u)f^{(r)}(x+hu)\,du,\quad x\ge 0,$$

where $M_r$ is the $r$-fold convolution of the characteristic function of $[0,1]$ with itself and

$$0\le M_r(u)\le c\,u^{r-1},\quad u\in[0,r].$$

Therefore,

$$\left|\overrightarrow{\Delta}_{1/n}^{r}f\left(\frac{k}{n}\right)\right| \le \frac{c}{n^r}\int_0^r \frac{u^{r-1}}{w\left(\frac{k+u}{n}\right)}\,du\,\|wf^{(r)}\|,\quad k\in\mathbb{N}_0. \tag{2.36}$$

Consequently,

$$\begin{aligned}&|w(x)(S_nf)^{(r+1)}(x)|\\ &\qquad\le cnw(x)\max_{j=0,1}\sum_{k=0}^{\infty}\int_0^r\frac{u^{r-1}}{w\left(\frac{k+j+u}{n}\right)}\,du\,s_{n,k}(x)\,\|wf^{(r)}\|,\quad x\ge 0.\end{aligned} \tag{2.37}$$

It is quite straightforward to obtain (see [8, Proposition 3.1]) that

$$\int_0^r\frac{u^{r-1}}{w\left(\frac{k+u}{n}\right)}\,du \le c\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0},\quad k\ge 0; \tag{2.38}$$

hence,

$$\int_0^r\frac{u^{r-1}}{w\left(\frac{k+u+1}{n}\right)}\,du \le c\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0},\quad k\ge 0, \tag{2.39}$$

as well.

It was shown in [5, (10.2.4)] that

$$\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{l} s_{n,k}(x)\le c\,x^{-l},\quad x>0,\quad l\in\mathbb{N}_0,$$

This along with (2.26), the identity $\sum_{k=0}^{\infty} s_{n,k}(x)\equiv 1$ and Hölder's inequality yields (see [5, p. 162–163])

$$\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0} s_{n,k}(x)\le \frac{c}{w(x)},\quad x>0, \tag{2.40}$$

for all $\gamma_0\ge 0$ and $\gamma_\infty\in\mathbb{R}$.

Estimates (2.37)-(2.40) imply (a).

(b) As in the proof of Proposition 2.2 we consider the cases $nx\ge 1$ and $nx\le 1$ separately.

Case 1: $nx\ge 1$. We differentiate identity (2.8) twice to get

$$(S_nf)^{(r+2)}(x)=n^r\sum_{k=0}^{\infty}\overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k}{n}\right) s''_{n,k}(x).$$

We note that the series on the right-hand side of (2.8) can be differentiated term-by-term any number of times because, under the assumptions on $f$, the resulting series are uniformly convergent on any finite closed subinterval of $[0,\infty)$, as can be shown by means of the Weierstrass M-test.

Using (2.2) (cf. (2.4) with $r=2$), we compute that

$$s''_{n,k}(x)=\frac{s_{n,k}(x)}{x^2}\left(-(k-nx)+(k-nx)^2-nx\right),\quad k\in\mathbb{N}_0.$$

Therefore,

$$\begin{aligned}&|w(x)\varphi^2(x)(S_nf)^{(r+2)}(x)|\\ &\quad\le n^r\frac{w(x)}{x}\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k}{n}\right)\right|\left(|k-nx|+(k-nx)^2+nx\right) s_{n,k}(x),\quad x>0.\end{aligned}$$

Then we combine (2.36) and (2.38) to estimate $|\overrightarrow{\Delta}_{1/n}^{r} f(k/n)|$ and derive the inequality

(2.41)
$$
\begin{aligned}
&|w(x)\varphi^2(x)(S_nf)^{(r+2)}(x)| \\
&\qquad \le c\frac{w(x)}{x}\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}|k-nx|s_{n,k}(x)\,\|wf^{(r)}\| \\
&\qquad\quad + c\frac{w(x)}{x}\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}(k-nx)^2s_{n,k}(x)\,\|wf^{(r)}\| \\
&\qquad\quad + cnw(x)\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}s_{n,k}(x)\,\|wf^{(r)}\|.
\end{aligned}
$$

We further estimate the first two sums above, using Cauchy's inequality (2.40) with $2\gamma_0$ in place of $\gamma_0$ and $2\gamma_\infty$ in place of $\gamma_\infty$, and (2.6), to arrive at

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}|k-nx|s_{n,k}(x) \\
&\qquad \le \sqrt{\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{2\gamma_0}\left(\frac{n}{n+k}\right)^{2(\gamma_\infty-\gamma_0)}s_{n,k}(x)}\sqrt{T_{n,2}(x)} \\
&\qquad \le c\sqrt{w^{-2}(x)}\sqrt{nx}\le c\frac{nx}{w(x)}
\end{aligned}
\tag{2.42}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}(k-nx)^2s_{n,k}(x) \\
&\qquad \le \sqrt{\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{2\gamma_0}\left(\frac{n}{n+k}\right)^{2(\gamma_\infty-\gamma_0)}s_{n,k}(x)}\sqrt{T_{n,4}(x)} \\
&\qquad \le c\sqrt{w^{-2}(x)}nx = c\frac{nx}{w(x)}.
\end{aligned}
\tag{2.43}
$$

Now, combining (2.41) with (2.42), (2.43) and (2.40), we get

$$
|w(x)\varphi^2(x)(S_nf)^{(r+2)}(x)| \le cn\,\|wf^{(r)}\|, \quad nx\ge 1. \tag{2.44}
$$

Case 2: $nx\le 1$. We differentiate identity (2.8) with $r+1$ in place of $r$

and thus get

$$(S_n f)^{(r+2)}(x) = n^{r+1} \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^{r+1} f\left(\frac{k}{n}\right) s'_{n,k}(x).$$

Then we use (2.2), (2.36), (2.38), (2.39) and (2.42) to get

$$\begin{aligned}
&|w(x)\varphi^2(x)(S_n f)^{(r+2)}(x)| \\
&\quad\le 2n^{r+1} w(x) \max_{j=0,1} \sum_{k=0}^{\infty} \left| \overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k+j}{n}\right)\right| |k-nx| s_{n,k}(x) \\
&\quad\le cnw(x) \sum_{k=0}^{\infty} \left(\frac{n}{k+1}\right)^{\gamma_0} \left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0} |k-nx| s_{n,k}(x)\, \|wf^{(r)}\| \\
&\quad\le cnw(x) \frac{nx}{w(x)}\, \|wf^{(r)}\| \\
&\quad\le cn\, \|wf^{(r)}\|, \quad x \in (0, 1/n].
\end{aligned}$$

where at the last estimate we have taken into consideration that $nx \le 1$.

Thus we have established

$$|w(x)\varphi^2(x)(S_n f)^{(r+2)}(x)| \le cn\, \|wf^{(r)}\|, \quad nx \le 1. \tag{2.45}$$

Estimates (2.44) and (2.45) verify assertion (b). □

Since $\big(\widetilde{D}g\big)^{(r)} = rg^{(r+1)} + \varphi^2 g^{(r+2)}$, Proposition 2.4 immediately yields the following inequality.

**Corollary 2.5.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there holds*

$$\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\| \le cn\|wf^{(r)}\|.$$

We will also use the following inequalities, which follow from Proposition 2.4 and the embedding inequalities [8, Proposition 2.4].

**Corollary 2.6.** *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then for all $f \in AC^{r+1}[0,\infty)$ such that $wf^{(r)} \in L_\infty[0,\infty)$ and $w\big(\widetilde{D}f\big)^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there hold:*

(a) $\|w(S_n f)^{(r+2)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(b) $\|w(S_n^2 f)^{(r+3)}\| \le cn^2\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(c) $\|w\varphi^2(S_nf)^{(r+3)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(d) $\|w\varphi^4(S_nf)^{(r+4)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$.

P r o o f. (a) By virtue of [8, (2.15)], we have

$$\|wf^{(r+1)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|. \tag{2.46}$$

This shows, in the first place, that $wf^{(r+1)} \in L_\infty[0,\infty)$. Then we apply Proposition 2.4(a) with $r+1$ in place of $r$ to get

$$\|w(S_nf)^{(r+2)}\| \le cn\|wf^{(r+1)}\|,$$

which combined with (2.46) yields (a).

(b) The assertion follows from Proposition 2.4(a) with $r+2$ in place of $r$ and $S_nf$ in place of $f$ and (a).

(c) Similarly to (a), we apply Proposition 2.4(b) with $r+1$ in place of $r$ and (2.46) to derive

$$\begin{aligned}\|w\varphi^2(S_nf)^{(r+3)}\| &\le cn\|wf^{(r+1)}\|\\ &\le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.\end{aligned}$$

(d) We apply Proposition 2.4(b) with $r+2$ in place of $r$ and $w\varphi^2$ in place of $w$. Thus we get

$$\|w\varphi^4(S_nf)^{(r+4)}\| \le cn\|w\varphi^2f^{(r+2)}\|. \tag{2.47}$$

Let us note that the assumption in Proposition 2.4(b) on the weight exponent at 0 now is $0 \le \gamma_0 + 1 < r+2$, which is satisfied. As for the assumptions on the function, it remains only to observe that $w\varphi^2f^{(r+2)} \in L_\infty[0,\infty)$. It follows from [8, (2.16)], by virtue of which we have

$$\|w\varphi^2f^{(r+2)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.$$

The last estimate and (2.47) yield (d). □

## 3. Proofs of Theorems 1.1 and 1.2.

P r o o f o f T h e o r e m 1.1. We apply the method to establish converse inequalities given in [4, Theorem 3.2]. This theorem is not directly applicable because the Voronovskaya-type estimate has a different form—compare [4, (3.4)] and Proposition 2.2. However, the same idea still works.

We set $g_n := S_n^3 f$. First, we will show that $g_n$ is in the domain on which the infimum in the definition of the $K$-functional $\widetilde{K}_r(f^{(r)},t)_w$ is taken and hence

$$\widetilde{K}_r(f^{(r)},n^{-1})_w \le \|w(f^{(r)}-g_n^{(r)})\| + \frac{1}{n}\|w\big(\widetilde{D}g_n\big)^{(r)}\|. \tag{3.1}$$

Indeed, clearly, $g_n \in AC^{r+1}[0,\infty)$. Next, iterating (2.9), we see that $wg_n^{(r)} \in L_\infty[0,\infty)$, whereas $w\big(\widetilde{D}g_n\big)^{(r)} \in L_\infty[0,\infty)$ follows from Corollary 2.5 and (2.9), which imply

$$\begin{aligned}\|w\big(\widetilde{D}g_n\big)^{(r)}\| &= \|w\big(\widetilde{D}S_n^3 f\big)^{(r)}\|\\ &\le cn\|wS_n^2 f^{(r)}\|\\ &\le cn\|wf^{(r)}\|.\end{aligned}$$

Let $I$ stand for the identity map in the $L_\infty$-space with the weight $w$ on $[0,\infty)$. We have, by virtue of (2.9),

$$\begin{aligned}\|w(f^{(r)}-g_n^{(r)})\| &= \|w\big[(I+S_n+S_n^2)(f-S_nf)\big]^{(r)}\|\\ &\le c\,\|w(f-S_nf)^{(r)}\|.\end{aligned} \tag{3.2}$$

To complete the proof of the theorem, we will show that there exists $R \ge 1$ such that for all $n,k \ge 1$ such that $k \ge Rn$ there holds

$$\frac{1}{n}\|w\big(\widetilde{D}g_n\big)^{(r)}\| \le c\frac{k}{n}\left(\|w(S_nf-f)^{(r)}\| + \|w(S_kf-f)^{(r)}\|\right). \tag{3.3}$$

Then the first assertion of Theorem 1.1 follows from (3.1)-(3.3).

Let $k \ge n \ge 1$. We want to apply Proposition 2.2 with $g_n$ in place of $f$. To this end, we first verify that $wg_n^{(r+2)}, wg_n^{(r+3)}, w\varphi^4 g_n^{(r+4)} \in L_\infty[0,\infty)$. To show it and, moreover, get estimates of their weighted $L_\infty$-norms, we apply Corollary 2.6, (a), (b) and (d) with $S_nf$ in place of $f$ (note that $w\big(\widetilde{D}S_nf\big)^{(r)} \in L_\infty[0,\infty)$ by Corollary 2.5). Thus we get

$$\|w(S_n^2f)^{(r+2)}\| \le cn\|w\big(\widetilde{D}S_nf\big)^{(r)}\|, \tag{3.4}$$

$$\|w(S_n^3f)^{(r+3)}\| \le cn^2\|w\big(\widetilde{D}S_nf\big)^{(r)}\|, \tag{3.5}$$

and

$$\|w\varphi^4(S_n^2f)^{(r+4)}\| \le cn\|w\big(\widetilde{D}S_nf\big)^{(r)}\|. \tag{3.6}$$

Further, by means of (2.9) with $S_n^2 f$ in place of $f$, we get from (3.4) and (3.6)

$$\|w(S_n^3f)^{(r+2)}\| \le c\,\|w(S_n^2f)^{(r+2)}\| \le cn\|w\big(\widetilde{D}S_nf\big)^{(r)}\|, \tag{3.7}$$

and

$$\|w\varphi^4(S_n^3 f)^{(r+4)}\| \le c\,\|w\varphi^4(S_n^2 f)^{(r+4)}\| \le cn\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|. \tag{3.8}$$

For the application of (2.9) in the latter case, we observe that the assumption on the weight exponent at 0 is $0 \le \gamma_0 + 2 < r + 4$, which is satisfied.

Having verified that $wg_n^{(r+2)}, wg_n^{(r+3)}, w\varphi^4 g_n^{(r+4)} \in L_\infty[0,\infty)$, we next apply Proposition 2.2 with $k$ in place of $n$ and $g_n$ in place of $f$ to arrive at

$$\begin{aligned}
\frac{1}{n}\big\|w\big(\widetilde{D}g_n\big)^{(r)}\big\| &\le \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f - \frac{1}{2k}\,\widetilde{D}(S_n^3 f)\right)^{(r)}\right\| \\
&\qquad + \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f\right)^{(r)}\right\| \\
&\le \frac{c}{nk}\left(\|w(S_n^3 f)^{(r+2)}\| + \|w\varphi^2(S_n^3 f)^{(r+3)}\| + \|w\varphi^4(S_n^3 f)^{(r+4)}\|\right) \\
&\qquad + \frac{c}{nk^2}\,\|w(S_n^3 f)^{(r+3)}\| + \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f\right)^{(r)}\right\|.
\end{aligned} \tag{3.9}$$

We will estimate the terms on the right.

Similarly as above, we use (2.9) with $w\varphi^2$ in place of $w$ and $S_n^2 f$ in place of $f$, and Corollary 2.6(c) with $S_n f$ in place of $f$ to get

$$\begin{aligned}
\|w\varphi^2(S_n^3 f)^{(r+3)}\| &\le c\,\|w\varphi^2(S_n^2 f)^{(r+3)}\| \\
&\le cn\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|.
\end{aligned} \tag{3.10}$$

Here the application of (2.9) is justified since the assumption on the weight exponent at 0 is $0 \le \gamma_0 + 1 < r + 3$, which is clearly satisfied.

By virtue of (3.7), (3.10) and (3.8), we have

$$\begin{aligned}
\frac{1}{nk}&\left(\|w(S_n^3 f)^{(r+2)}\| + \|w\varphi^2(S_n^3 f)^{(r+3)}\| + \|w\varphi^4(S_n^3 f)^{(r+4)}\|\right) \\
&\le \frac{c}{k}\,\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|.
\end{aligned} \tag{3.11}$$

Also, by (3.5), we get

$$\frac{1}{nk^2}\,\|w(S_n^3 f)^{(r+3)}\| \le \frac{c}{k}\,\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|, \tag{3.12}$$

where we have also taken into account that $n \le k$.

To estimate the last term on the right of (3.9) we use the representation

$$S_k(S_n^3 f) - S_n^3 f = S_k(S_n^3 f - f) + (S_k f - f) + (f - S_n^3 f).$$

Therefore, using also (2.9) and (3.2), we arrive at

$$\left\|w\left(S_k(S_n^3 f)-S_n^3 f\right)^{(r)}\right\| \le c\left(\|w(S_n f-f)^{(r)}\|+\|w(S_k f-f)^{(r)}\|\right). \tag{3.13}$$

We combine (3.9) with (3.11)-(3.13) to derive

$$\begin{aligned}\frac{1}{n}\|w\big(\widetilde{D}g_n\big)^{(r)}\| \\ \le \frac{c}{k}\left\|w\big(\widetilde{D}S_n f\big)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f-f)^{(r)}\|+\|w(S_k f-f)^{(r)}\|\right).\end{aligned} \tag{3.14}$$

Next, we will relate $\left\|w\big(\widetilde{D}S_n f\big)^{(r)}\right\|$ to $\left\|w\big(\widetilde{D}g_n\big)^{(r)}\right\|$. Using Corollary 2.5 and (2.9), we get

$$\begin{aligned}\left\|w\big(\widetilde{D}S_n f\big)^{(r)}\right\| &\le \left\|w\big(\widetilde{D}S_n^3 f\big)^{(r)}\right\| + \left\|w\big[\widetilde{D}S_n(f-S_n^2 f)\big]^{(r)}\right\| \\ &\le \left\|w\big(\widetilde{D}g_n\big)^{(r)}\right\|| + cn\left\|w(f-S_n^2 f)^{(r)}\right\| \\ &\le \left\|w\big(\widetilde{D}g_n\big)^{(r)}\right\| + cn\left\|w\big[(I+S_n)(f-S_n f)\big]^{(r)}\right\| \\ &\le \left\|w\big(\widetilde{D}g_n\big)^{(r)}\right\| + cn\left\|w(S_n f-f)^{(r)}\right\|.\end{aligned}$$

Hence (3.14) yields

$$\begin{aligned}\frac{1}{n}\|w\big(\widetilde{D}g_n\big)^{(r)}\| \\ \le \frac{c}{k}\left\|w\big(\widetilde{D}g_n\big)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f-f)^{(r)}\|+\|w(S_k f-f)^{(r)}\|\right)\end{aligned} \tag{3.15}$$

for all $k \ge n \ge 1$.

Let $R \ge 1$ and $k \ge Rn$. Then

$$\frac{c}{k} \le \frac{c}{Rn},$$

where $c$ is the constant in (3.15). We fix $R$ so large that $c/R \le 1/2$. Then (3.15) implies

$$\begin{aligned}\frac{1}{n}\|w\big(\widetilde{D}g_n\big)^{(r)}\| \\ \le \frac{1}{2n}\left\|w\big(\widetilde{D}S_n f\big)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f-f)^{(r)}\|+\|w(S_k f-f)^{(r)}\|\right)\end{aligned}$$

for all $n, k \ge 1$ such that $k \ge Rn$; hence the first assertion of the theorem follows. □

In the proof of Theorem 1.2 we will make use of the $K$-functionals

$$K_{2,\varphi}(f,t)_w := \inf\left\{\|w(f-g)\| + t\|w\varphi^2 g''\|\right.$$

$$: g \in AC^1_{loc}(0,\infty),\ wg, w\varphi^2 g'' \in L_\infty[0,\infty)\big\}$$

and

$$K_1(f,t)_w := \inf\big\{\|w(f-g)\| + t\|wg'\| \\ : g \in AC_{loc}(0,\infty),\ wg, wg' \in L_\infty[0,\infty)\big\},$$

where $wf \in L_\infty[0,\infty)$ and $t > 0$.

Ditzian and Totik [5, Theorem 6.1.1] showed that there exist positive constants $c$ and $t_0$ such that for all $f$ with $wf \in L_\infty[0,\infty)$ and all $t \in (0,t_0]$ there holds

$$c^{-1}\omega^2_\varphi(f,t)_w \le K_{2,\varphi}(f,t^2)_w \le c\,\omega^2_\varphi(f,t)_w. \tag{3.16}$$

Analogously to the unweighted case (see e.g. [3, Chapter 6, Theorem 2.4]), we have

$$c^{-1}\omega(f,t)_w \le K_1(f,t)_w \le c\,\omega(f,t)_w, \quad t > 0. \tag{3.17}$$

Proof of Theorem 1.2. In view of Theorem 1.1 and the left inequalities in (3.16)–(3.17), it is sufficient to show that

$$K_{2,\varphi}(f,t)_w \le c\,\widetilde{K}_r(f,t)_w \tag{3.18}$$

and

$$K_1(f,t)_w \le c\,\widetilde{K}_r(f,t)_w, \tag{3.19}$$

where $wf \in L_\infty[0,\infty)$ and $t > 0$.

Let $g \in AC^{r+1}[0,\infty)$ with $wg^{(r)}, w\big(\widetilde{D}g\big)^{(r)} \in L_\infty[0,\infty)$ be arbitrarily fixed. Then, clearly, $g^{(r)} \in AC^1_{loc}(0,\infty)$. By virtue of [8, (2.16)], we have

$$\|w\varphi^2 g^{(r+2)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.$$

This implies that $w\varphi^2\big(g^{(r)}\big)'' \in L_\infty[0,\infty)$ and

$$K_{2,\varphi}(f,t)_w \le \|f - g^{(r)}\| + t\,\|w\varphi^2\big(g^{(r)}\big)''\| \\ \le c\left(\|f - g^{(r)}\| + t\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|\right).$$

Taking the infimum on $g$, we straightforwardly arrive at (3.18).

Relation (3.19) is established just similarly by means of [8, (2.15)]. □

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