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GENERALIZED HELICOIDAL SURFACES IN MINKOWSKI 5-SPACE

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Abstract. In this paper, we study generalized helicoidal surfaces in Minkowski 5-space. We obtain the necessary and sufficient conditions for generalized helicoidal surfaces in Minkowski 5-space to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solve those equations and discuss the behavior of solutions.

1. Introduction. Helicoidal surfaces are a well-known type of surfaces in differential geometry. Helicoidal surfaces are a generalization of rotational surfaces. These surfaces are invariant by a subgroup of the group of isometries of the ambient space, called helicoidal group whose elements can be seen as a composition of a translation with a rotation for a given axis. In [4], the authors studied the space of all helicoidal surfaces in Euclidean 3-space which have constant mean curvatures or constant Gaussian curvatures. This space behaves as a circular cylinder, where a given generator corresponds to the rotational surfaces

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and each parallel corresponds to a periodic family of helicoidal surfaces. In [2], the cases with prescribed mean curvature or Gauss curvature have been studied.

Helicoidal surfaces were studied by many researchers in different spaces. In [6], authors constructed linear Weingarten helicoidal surfaces in Minkowski 3-space under the cubic screw motion. In [5], the authors constructed a helicoidal surface with a light-like axis with prescribed mean curvature or Gauss curvature given by smooth function in Minkowski 3-space and solved an open problem left in [3]. Also, in [7], the authors classify all helicoidal non-degenerate surfaces in Minkowski 3-space with constant mean curvature whose generating curve is the graph of a polynomial or a Lorentzian circle.

Besides, in [1], the authors studied rotational surfaces in higher dimensional Euclidean spaces. They obtained some results related with the curvature properties of these surfaces. Also they give examples of rotational surfaces in Euclidean 5-space.

Lastly, in [8], we studied generalized helicoidal surfaces in Euclidean 5-space. We obtained the necessary and sufficient conditions for generalized helicoidal surfaces in Euclidean 5-space to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solved those equations and discussed the completeness of the surfaces.

In this paper, we consider generalized helicoidal surfaces in 5-dimensional Minkowski space $\mathbb{R}^5_1$ with signature $(+,+,+,+,-)$, parametrized by

$$M: \quad F(t,u) = (\alpha(t)\cos u, \alpha(t)\sin u, \beta(t)\cos u, \beta(t)\sin u, u) \tag{1.1}$$

where α and β are smooth functions satisfying

$$\alpha^2+\beta^2>0, \quad (\alpha')^2+(\beta')^2>0 \quad \text{and} \quad \alpha^2+\beta^2-1\neq 0.$$

If $\alpha(t)=0$ or $\beta(t)=0$, then the surface M in (1.1) becomes a helicoidal surface in Minkowski 3-space. So the surface M as in (1.1) is called a generalized helicoidal surface.

We obtain the necessary and sufficient conditions for these surfaces to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solve those equations and discuss the behavior of solutions.

2. Preliminaries. Let $\mathbb{R}^n_q$ be the n-dimensional semi-Euclidean space of index q with inner product $\langle\ ,\ \rangle$ and flat connection D. Let M be a semi-Riemannian submanifold in $\mathbb{R}^n_q$. According to decomposition

$$\mathbb{R}^n_q|_M = TM \perp TM^{\perp},$$

we have

$$D_X Y = \nabla_X Y + h(X, Y),$$

and

$$D_X \xi = -A_\xi X + {}^\perp\nabla_X \, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^\perp\right)$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^\perp\nabla$ is the normal connection. We note that

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The normal curvature tensor ${}^\perp R$ is defined by

$${}^\perp R(X, Y)\xi = {}^\perp\nabla_X \, {}^\perp\nabla_Y \, \xi - {}^\perp\nabla_Y \, {}^\perp\nabla_X \, \xi - {}^\perp\nabla_{[X,Y]}\xi \,,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^\perp\right)$. Taking the normal part of the following equation

$$D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi = 0$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^\perp\right)$, we get the Ricci equation

$$\left\langle {}^\perp R(X, Y)\xi, \eta \right\rangle = \langle A_\eta X, A_\xi Y \rangle - \langle A_\xi X, A_\eta Y \rangle$$

where $\eta \in \Gamma\left(TM^\perp\right)$.

Let $\mathbb{R}^5_1$ be the 5-dimensional Minkowski space with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2.$$

In the following, we assume that M is a surface in $\mathbb{R}^5_1$. We say that M is spacelike if the induced metric is positive definite, and M is timelike if the induced metric is indefinite. We use the following convention on the ranges of indices:

$$1 \leq A, B \cdots \leq 5, \quad 1 \leq i, j \cdots \leq 2, \quad 3 \leq \alpha, \beta \cdots \leq 5.$$

Let $\{e_i\}$ be a local orthonormal frame field on M and $\{e_\alpha\}$ be a normal orthonormal frame field to M. Let $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Set

$$h_{ij}^\alpha = \varepsilon_\alpha \langle h(e_i, e_j), e_\alpha \rangle$$

and

$$R^{\alpha}_{\beta ij} = \varepsilon_{\alpha} \left\langle {}^{\perp}R\left(e_i, e_j\right) e_{\beta}, e_{\alpha} \right\rangle,$$

which are the components of the second fundamental form h and the normal curvature tensor ${}^{\perp}R$, respectively.

By the Ricci equation, the normal curvature tensor satisfies

$$R^{\alpha}_{\beta ij} = \varepsilon_{\alpha} \left(\left\langle A_{e_{\alpha}} e_i, A_{e_{\beta}} e_j \right\rangle - \left\langle A_{e_{\beta}} e_i, A_{e_{\alpha}} e_j \right\rangle \right).$$

Noting that

$$A_{e_{\alpha}} e_i = \varepsilon_{\alpha} \sum_k \varepsilon_k h^{\alpha}_{ik} e_k,$$

we obtain

$$R^{\alpha}_{\beta ij} = \varepsilon_{\beta} \sum_k \varepsilon_k \left(h^{\alpha}_{ik} h^{\beta}_{jk} - h^{\alpha}_{jk} h^{\beta}_{ik} \right).$$

The mean curvature vector H of M is given by

$$H = \frac{1}{2} \sum_{\alpha} \left(\varepsilon_1 h^{\alpha}_{11} + \varepsilon_2 h^{\alpha}_{22} \right) e_{\alpha}.$$

A surface M is called minimal if $H = 0$ identically.

The Gauss curvature K of M is given by

$$K = \varepsilon_1 \varepsilon_2 \sum_{\alpha} \varepsilon_{\alpha} \left(h^{\alpha}_{11} h^{\alpha}_{22} - \left(h^{\alpha}_{12}\right)^2 \right).$$

A surface M is called flat if $K = 0$ identically.

3. Generalized helicoidal surfaces in $\mathbb{R}^5_1$. In this section, we consider generalized helicoidal surface M parametrized by (1.1). Then we have

$$\begin{aligned} F_t &= \left(\alpha'(t) \cos u, \alpha'(t) \sin u, \beta'(t) \cos u, \beta'(t) \sin u, 0 \right), \\ F_u &= \left(-\alpha(t) \sin u, \alpha(t) \cos u, -\beta(t) \sin u, \beta(t) \cos u, 1 \right) \end{aligned}$$

and

$$\langle F_t, F_t \rangle = \left(\alpha'(t) \right)^2 + \left(\beta'(t) \right)^2, \quad \langle F_t, F_u \rangle = 0, \quad \langle F_u, F_u \rangle = \alpha^2(t) + \beta^2(t) - 1.$$

Then we can choose the followings:

$$e_1 = \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} F_t = \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} \left(\alpha' \cos u, \alpha' \sin u, \beta' \cos u, \beta' \sin u, 0 \right),$$

$$
\begin{aligned}
e_2 &= \frac{1}{\sqrt{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}}F_u \\
&= \frac{1}{\sqrt{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}}\left(-\alpha\sin u, \alpha\cos u, -\beta\sin u, \beta\cos u, 1\right), \\
e_3 &= \frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\beta\sin u, -\beta\cos u, -\alpha\sin u, \alpha\cos u, 0\right), \\
e_4 &= \frac{1}{\sqrt{(\alpha')^2+(\beta')^2}}\left(-\beta'\cos u, -\beta'\sin u, \alpha'\cos u, \alpha'\sin u, 0\right), \\
e_5 &= \frac{1}{\sqrt{\alpha^2+\beta^2}\sqrt{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}}\left(-\alpha\sin u, \alpha\cos u, -\beta\sin u, \beta\cos u, \alpha^2+\beta^2\right)
\end{aligned}
$$

where $\varepsilon_1 = \operatorname{sgn}\left(\alpha^2+\beta^2-1\right)$. Here $\{e_1, e_2\}$ is an orthonormal frame field on M with sign $(+, \varepsilon_1)$ and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame field to M with sign $(+, +, -\varepsilon_1)$.

Also we can easily obtain that

$$
\begin{aligned}
D_{e_1}e_1 &= \frac{(\beta'\alpha''-\alpha'\beta'')}{\left((\alpha')^2+(\beta')^2\right)^2}\left(\beta'\cos u, \beta'\sin u, -\alpha'\cos u, -\alpha'\sin u, 0\right), \\
D_{e_2}e_1 &= \frac{1}{\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}}\left(-\alpha'\sin u, \alpha'\cos u, -\beta'\sin u, \beta'\cos u, 0\right), \\
D_{e_2}e_2 &= \frac{1}{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}\left(-\alpha\cos u, -\alpha\sin u, -\beta\cos u, -\beta\sin u, 0\right).
\end{aligned}
$$

The components of the second fundamental form h are given as follows

$$
h^4_{11} = \frac{-\beta'\alpha''+\alpha'\beta''}{\left((\alpha')^2+(\beta')^2\right)^{3/2}}, \quad h^3_{12} = \frac{-\beta\alpha'+\alpha\beta'}{\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\varepsilon_1\left(\alpha^2+\beta^2-1\right)}\sqrt{\alpha^2+\beta^2}},
$$

$$
h^5_{12} = \frac{-\alpha\alpha'-\beta\beta'}{\left(\alpha^2+\beta^2-1\right)\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\alpha^2+\beta^2}},
$$

$$
h^4_{22} = \frac{-\beta\alpha'+\alpha\beta'}{\varepsilon_1\left(\alpha^2+\beta^2-1\right)\sqrt{(\alpha')^2+(\beta')^2}}, \quad h^3_{11} = h^5_{11} = h^4_{12} = h^3_{22} = h^5_{22} = 0.
$$

Then we get the following theorem and corollary.

Theorem 1. *Let M be a generalized helicoidal surface parametrized by* (1.1). *Then the mean curvature vector H of M is given by*

$$H = \frac{\left(\alpha'\beta'' - \beta'\alpha''\right)\left(\alpha^2+\beta^2-1\right) + \left(\alpha\beta'-\beta\alpha'\right)\left(\left(\alpha'\right)^2+\left(\beta'\right)^2\right)}{2\left(\alpha^2+\beta^2-1\right)\left(\left(\alpha'\right)^2+\left(\beta'\right)^2\right)^{3/2}} e_4.$$

Corollary 1. *Let M be a generalized helicoidal surface parametrized by* (1.1). *Then M is minimal if and only if*

$$\left(\alpha'\beta'' - \beta'\alpha''\right)\left(\alpha^2+\beta^2-1\right) + \left(\alpha\beta'-\beta\alpha'\right)\left(\left(\alpha'\right)^2+\left(\beta'\right)^2\right) = 0. \tag{3.1}$$

Let $\beta(t) = t$ in the equation (3.1). Then the minimal surface equation is

$$\left(\alpha^2+t^2-1\right)\alpha'' + \left(t\alpha'-\alpha\right)\left(\left(\alpha'\right)^2+1\right) = 0. \tag{3.2}$$

If $\alpha(t)$ is a linear function, that is, $\alpha(t) = pt+q$, then from the above equation, we have $q = 0$ and $\alpha(t) = pt$. Then the surface M is included in a 3-dimensional subspace of $\mathbb{R}^5_1$. So, in the following, we will consider the case where $\alpha(t)$ is a nonlinear function.

Multiplying (3.2) by $2\alpha' / \left(\left(\alpha'\right)^2+1\right)^2$, we can get

$$\left(t^2\frac{\left(\alpha'\right)^2}{\left(\alpha'\right)^2+1}\right)' - \left(\frac{\alpha^2-1}{\left(\alpha'\right)^2+1}\right)' = 0.$$

Thus we have

$$t^2\frac{\left(\alpha'\right)^2}{\left(\alpha'\right)^2+1} - \frac{\alpha^2-1}{\left(\alpha'\right)^2+1} = c_1$$

for a constant c_1. Then

$$\left(\alpha'\right)^2 = \frac{\alpha^2-1+c_1}{t^2-c_1}$$

So we have two cases (a) and (b) as

$$\begin{array}{lrcl} (a) & \alpha^2-1+c_1 & > & 0 \quad \text{and} \quad t^2-c_1>0, \\ (b) & \alpha^2-1+c_1 & < & 0 \quad \text{and} \quad t^2-c_1<0. \end{array}$$

In case (a), c_1 can be of any signature and $\alpha^2+t^2-1>0$, so that M is spacelike. In case (b), we have $0<c_1<1$ and $\alpha^2+t^2-1<0$, so that M is timelike.

(a) Firstly, we consider the case that M is spacelike. Then we have

$$\frac{\alpha'}{\sqrt{\alpha^2 - 1 + c_1}} = \pm\frac{1}{\sqrt{t^2 - c_1}}. \tag{3.3}$$

Changing t to $-t$, we may only consider the $(+)$ case if necessary.

When $c_1 = 0$, we have

$$\frac{\alpha'}{\sqrt{\alpha^2 - 1}} = \frac{1}{t}.$$

Integrating it we have

$$\log\left|\sqrt{\alpha^2 - 1} + \alpha\right| = \log|t| + c_2$$

for a constant c_2 and

$$\sqrt{\alpha^2 - 1} + \alpha = c_3 t$$

where $c_3 \neq 0$ is constant. Thus we get

$$\alpha = \frac{1}{2}\left(c_3 t + \frac{1}{c_3 t}\right)$$

and its graph is a hyperbola.

When $c_1 \neq 0$, integrating the equation (3.3) , we have

$$\log\left|\sqrt{\alpha^2 - 1 + c_1} + \alpha\right| = \pm\log|\sqrt{t^2 - c_1} + t| + c_2$$

for a constant c_2.

In the $(+)$ case,

$$\sqrt{\alpha^2 - 1 + c_1} + \alpha = c_3\left(\sqrt{t^2 - c_1} + t\right)$$

where $c_3 \neq 0$ is constant. In the $(-)$ case,

$$\sqrt{\alpha^2 - 1 + c_1} + \alpha = \frac{c_4}{\sqrt{t^2 - c_1} + t}$$

where $c_4 \neq 0$ is constant. Thus we get

$$\alpha_+(t) = \frac{1}{2c_1c_3}\left[\left(c_1c_3^2 + 1 - c_1\right)t + \left(c_1c_3^2 - 1 + c_1\right)\sqrt{t^2 - c_1}\right]$$

and

$$\alpha_{-}(t) = \frac{1}{2c_1c_4}\left[\left(c_1 - c_1^2 + c_4^2\right)t + \left(c_1 - c_1^2 - c_4^2\right)\sqrt{t^2 - c_1}\right].$$

Since $\alpha_{\pm}(t)$ is not a linear function, we have $c_1c_3^2 - 1 + c_1 \neq 0$ and $c_1 - c_1^2 - c_4^2 \neq 0$. The graph of $\alpha_{\pm}(t)$ is a hyperbola. When $c_1 < 0$, $\alpha_{\pm}(t)$ is defined for any $t \in \mathbb{R}$. When $c_1 > 0$, $\alpha_{\pm}(t)$ is defined for $|t| > \sqrt{c_1}$. In this case, we shall choose $c_4 = c_1c_3$. Then

$$\alpha_{-}(t) = \frac{1}{2c_1c_3}\left[\left(c_1c_3^2 + 1 - c_1\right)t - \left(c_1c_3^2 - 1 + c_1\right)\sqrt{t^2 - c_1}\right].$$

So the graph of $\alpha_{+}(t)$ and $\alpha_{-}(t)$ give the same hyperbola and can be connected smoothly and regularly.

(b) Now, we consider the case that M is timelike. Then we have

$$\frac{\alpha'}{\sqrt{1 - c_1 - \alpha^2}} = \pm\frac{1}{\sqrt{c_1 - t^2}}.$$

So we have

$$\begin{aligned}\alpha(t) &= \pm\sqrt{1 - c_1}\sin\left(\arcsin\left(\frac{t}{\sqrt{c_1}}\right) + c_2\right)\\ &= \pm\sqrt{\frac{1 - c_1}{c_1}}\left(t(\cos c_2) \pm \sqrt{c_1 - t^2}(\sin c_2)\right)\end{aligned}$$

for a constant c_2. Since $\alpha(t)$ is not a linear function, we have $\sin c_2 \neq 0$. Changing the signatures, we may assume that $\sin c_2 > 0$ and $\cos c_2 \geq 0$. The above $\alpha(t)$ gives a part of ellipse. The smooth and regular extension to a whole ellipse is possible when $\cos c_2 = 0$ and $\sin c_2 = 1$.

Thus we can give the following corollary.

Corollary 2. *The nonlinear solution of the minimal surface equation* (3.2) *is given by one of the followings*
(i) *for a constant* $c_3 \neq 0$,

$$\alpha(t) = \frac{1}{2}\left(c_3t + \frac{1}{c_3t}\right).$$

In this case, the surface M *is a spacelike minimal surface.*
(ii) *for nonzero constants* c_1, c_3 *and* c_4,

$$\alpha_{+}(t) = \frac{1}{2c_1c_3}\left[\left(c_1c_3^2 + 1 - c_1\right)t + \left(c_1c_3^2 - 1 + c_1\right)\sqrt{t^2 - c_1}\right]$$

and

$$\alpha_{-}(t) = \frac{1}{2c_1c_4}\left[\left(c_1 - c_1^2 + c_4^2\right)t + \left(c_1 - c_1^2 - c_4^2\right)\sqrt{t^2 - c_1}\right].$$

In this case, the surface M is a spacelike minimal surface.
(iii) for constants $c_1 \neq 0$ and c_2,

$$\alpha(t) = \pm\frac{\sqrt{1-c_1}}{\sqrt{c_1}}\left(t\cos c_2 \pm \sqrt{c_1 - t^2}\sin c_2\right).$$

In this case, the surface M is a timelike minimal surface.

In the following theorem, we give the Gauss curvature of the surface (1.1).

Theorem 2. *Let M be a generalized helicoidal surface parametrized by (1.1). Then the Gauss curvature K of M is given by*

$$K = \frac{(\alpha'\beta'' - \beta'\alpha'')(\alpha\beta' - \beta\alpha')}{(\alpha^2+\beta^2-1)\left((\alpha')^2+(\beta')^2\right)^2} + \frac{-(\alpha\beta'-\beta\alpha')^2\left(\alpha^2+\beta^2-1\right)+(\alpha\alpha'+\beta\beta')^2}{\left((\alpha')^2+(\beta')^2\right)(\alpha^2+\beta^2-1)^2(\alpha^2+\beta^2)}. \tag{3.4}$$

Corollary 3. *Let M be a generalized helicoidal surface parametrized by (1.1). Then M is flat if and only if*

$$\frac{(\alpha'\beta''-\beta'\alpha'')(\alpha\beta'-\beta\alpha')}{\left((\alpha')^2+(\beta')^2\right)} = \frac{(\alpha\beta'-\beta\alpha')^2\left(\alpha^2+\beta^2-1\right)-(\alpha\alpha'+\beta\beta')^2}{(\alpha^2+\beta^2-1)(\alpha^2+\beta^2)}.$$

To study flat surfaces, it is convenient to let

$$\alpha(t) = P(t)\cos(Q(t)) \quad \text{and} \quad \beta(t) = P(t)\sin(Q(t)) \tag{3.5}$$

where $P(t) > 0$ and $Q(t)$ are nonconstant smooth functions. From the equation (3.4), we have

$$K = \frac{(P')^4 + P^4(P')^2(Q')^2 - P^3\left(P^2-1\right)(Q')^2P'' + P^3\left(P^2-1\right)P'Q'Q''}{(P^2-1)^2\left((P')^2+P^2(Q')^2\right)^2}.$$

Then the surface M is flat if and only if

$$(P')^4 + P^4(P')^2(Q')^2 - P^3\left(P^2-1\right)(Q')^2P'' + P^3\left(P^2-1\right)P'Q'Q'' = 0. \tag{3.6}$$

Here for $P(t) = t$ where $t > 0, t \neq 1$, the equation (3.6) is rewritten as

$$t^3\left(t^2-1\right)Q'Q'' + t^4\left(Q'\right)^2 + 1 = 0. \tag{3.7}$$

Setting $Q'(t) = R(t)$, we get

$$t^3\left(t^2-1\right)RR' + t^4R^2 + 1 = 0$$

or

$$\frac{dR}{dt} + \frac{t}{t^2-1}R = -\frac{1}{t^3\left(t^2-1\right)}R^{-1}. \tag{3.8}$$

By multiplying (3.8) with $2\left(t^2-1\right)R$, we obtain

$$\left(\left(t^2-1\right)R^2\right)' = -\frac{2}{t^3}$$

which implies that

$$\left(t^2-1\right)\left(Q'\right)^2 = \frac{1}{t^2} - c_1$$

or

$$\left(Q'\right)^2 = \frac{1-c_1t^2}{t^2\left(t^2-1\right)}$$

for a constant c_1. So we have

$$Q' = \pm\frac{1}{t}\sqrt{\frac{1-c_1t^2}{t^2-1}}.$$

Let

$$I := \int \frac{1}{t}\sqrt{\frac{1-c_1t^2}{t^2-1}}dt.$$

(i) Assume that $c_1 = 0$, which implies that $t > 1$ and M is spacelike. Then we obtain

$$I = \int \frac{1}{t}\sqrt{\frac{1}{t^2-1}}dt = \arctan\left(\sqrt{t^2-1}\right) + c_2$$

and

$$Q(t) = \pm\left(\arctan\left(\sqrt{t^2-1}\right) + c_2\right)$$

for a constant c_2. Let

$$Q_+(t) = \arctan\left(\sqrt{t^2-1}\right) + c_3 \quad \text{and} \quad Q_-(t) = -\arctan\left(\sqrt{t^2-1}\right) + c_3$$

for a constant c_3. We denote by $t_+(Q)$ and $t_-(Q)$ the inverse functions of $Q_+(t)$ and $Q_-(t)$, respectively. Then

$$t_+(Q) = \frac{1}{\cos(Q - c_3)}, \quad c_3 < Q < \frac{\pi}{2} + c_3$$

and

$$t_-(Q) = \frac{1}{\cos(Q - c_3)}, \quad -\frac{\pi}{2} + c_3 < Q < c_3,$$

which can be connected smoothly and regularly at $Q = c_3$.

When $c_1 \neq 0$, set

$$\sqrt{\frac{1 - c_1 t^2}{t^2 - 1}} =: s.$$

Then

$$t^2 = \frac{s^2 + 1}{s^2 + c_1}$$

and

$$tdt = \frac{c_1 - 1}{(s^2 + c_1)^2} sds.$$

So we have

$$I = (c_1 - 1) \int \frac{s^2}{(1 + s^2)(s^2 + c_1)} ds = \int \left(\frac{c_1}{s^2 + c_1} - \frac{1}{1 + s^2} \right) ds.$$

(ii) Assume that c_1 is positive.

$(ii$-1) When $0 < c_1 < 1$, M is spacelike and $1 < t < 1/\sqrt{c_1}$.

$(ii$-2) When $c_1 > 1$, M is timelike and $1/\sqrt{c_1} < t < 1$.

Then we obtain

$$I = \sqrt{c_1} \arctan\left(\frac{s}{\sqrt{c_1}} \right) - \arctan s + c_2$$

and

$$Q(t) = \pm \left(\sqrt{c_1} \arctan\left(\sqrt{\frac{1 - c_1 t^2}{c_1 (t^2 - 1)}} \right) - \arctan\left(\sqrt{\frac{1 - c_1 t^2}{t^2 - 1}} \right) + c_2 \right)$$

where c_2 is a constant. Let

$$Q_+(t) = \sqrt{c_1} \arctan\left(\sqrt{\frac{1 - c_1 t^2}{c_1 (t^2 - 1)}} \right) - \arctan\left(\sqrt{\frac{1 - c_1 t^2}{t^2 - 1}} \right) + c_3$$

and

$$Q_-(t) = -\left(\sqrt{c_1}\arctan\left(\sqrt{\frac{1-c_1t^2}{c_1\left(t^2-1\right)}}\right) - \arctan\left(\sqrt{\frac{1-c_1t^2}{t^2-1}}\right)\right) + c_3$$

for a constant c_3. Then $Q_+(t)$ is increasing, $Q_-(t)$ is decreasing and

$$\lim_{t\to 1/\sqrt{c_1}} Q_\pm(t) = c_3, \quad \lim_{t\to 1} Q_\pm(t) = c_3 \pm \frac{\pi\left(\sqrt{c_1}-1\right)}{2},$$

$$\lim_{t\to 1/\sqrt{c_1}} Q'_\pm(t) = 0, \quad \lim_{t\to 1} Q'_\pm(t) = \pm\infty.$$

From the behavior as t tends to $1/\sqrt{c_1}$, the curves

$$\left(t\cos\left(Q_+(t)\right), t\sin\left(Q_+(t)\right)\right) \quad \text{and} \quad \left(t\cos\left(Q_-(t)\right), t\sin\left(Q_-(t)\right)\right)$$

cannot be connected as a regular curve.

(*iii*) Assume that c_1 is negative, which implies that $t > 1$ and M is spacelike. Then we can write $c_1 = -c_3^2$ for a positive c_3, which implies that

$$\begin{aligned} I &= -\int\left(\frac{c_3^2}{s^2-c_3^2} + \frac{1}{1+s^2}\right)ds = \frac{c_3}{2}\log\frac{s+c_3}{s-c_3} - \arctan s + c_4 \\ &= \frac{c_3}{2}\log\left(\frac{\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}}{\sqrt{1+c_3^2t^2}-c_3\sqrt{t^2-1}}\right) - \arctan\left(\sqrt{\frac{1+c_3^2t^2}{t^2-1}}\right) + c_4 \\ &= c_3\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right) - \arctan\left(\sqrt{\frac{1+c_3^2t^2}{t^2-1}}\right) - \frac{c_3}{2}\log\left(1+c_3^2\right) + c_4 \end{aligned}$$

where c_4 is a constant. Thus we have

$$\begin{aligned} Q(t) = \pm\Bigg(&c_3\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right) \\ &- \arctan\left(\sqrt{\frac{1+c_3^2t^2}{t^2-1}}\right) - \frac{c_3}{2}\log\left(1+c_3^2\right) + c_4\Bigg). \end{aligned}$$

Let

$$Q_+(t) = c_3\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right) - \arctan\left(\sqrt{\frac{1+c_3^2t^2}{t^2-1}}\right) + c_5$$

and

$$Q_-(t) = -\left(c_3 \log\left(\sqrt{1+c_3^2 t^2} + c_3\sqrt{t^2-1}\right) - \arctan\left(\sqrt{\frac{1+c_3^2 t^2}{t^2-1}}\right)\right) + c_6$$

for constants c_5, c_6. Then $Q_+(t)$ is an increasing function and

$$\lim_{t\to\infty} Q_+(t) = \infty, \quad \lim_{t\to 1} Q_+(t) = \frac{c_3}{2}\log\left(1+c_3^2\right) - \frac{\pi}{2} + c_5, \quad \lim_{t\to 1} Q'_+(t) = \infty.$$

Similarly, $Q_-(t)$ is a decreasing function and

$$\lim_{t\to\infty} Q_-(t) = -\infty, \quad \lim_{t\to 1} Q_-(t) = -\frac{c_3}{2}\log\left(1+c_3^2\right) + \frac{\pi}{2} + c_6, \quad \lim_{t\to 1} Q'_-(t) = -\infty.$$

We choose c_6 such that

$$\frac{c_3}{2}\log\left(1+c_3^2\right) - \frac{\pi}{2} + c_5 = -\frac{c_3}{2}\log\left(1+c_3^2\right) + \frac{\pi}{2} + c_6 =: Q_0.$$

Let $t_+(Q)$ denote the inverse function of $Q_+(t)$. It is an increasing function on (Q_0, ∞) and

$$\lim_{Q\to\infty} t_+(Q) = \infty, \quad \lim_{Q\to Q_0} t_+(Q) = 1, \quad \lim_{Q\to Q_0} t'_+(Q) = 0.$$

Let $t_-(Q)$ denote the inverse function of $Q_-(t)$. It is a decreasing function on $(-\infty, Q_0)$ and

$$\lim_{Q\to -\infty} t_-(Q) = \infty, \quad \lim_{Q\to Q_0} t_-(Q) = 1, \quad \lim_{Q\to Q_0} t'_-(Q) = 0.$$

Now we define a function $t(Q)$ on $\mathbb{R}$ such that $t(Q) = t_+(Q)$ for $Q > Q_0$, $t(Q_0) = 1$ and $t(Q) = t_-(Q)$ for $Q < Q_0$. Then $t(Q)$ is a C^1 function on $\mathbb{R}$ such that $t'(Q) = t'_+(Q)$ for $Q > Q_0$, $t'(Q_0) = 0$ and $t'(Q) = t'_-(Q)$ for $Q < Q_0$. For $Q \neq Q_0$, it satisfies the equation (3.6) where the parameter is Q and $P = t(Q)$. So

$$(t')^4 + t^4 (t')^2 - t^3\left(t^2-1\right) t'' = 0$$

and

$$t'' = \frac{(t')^4 + t^4 (t')^2}{t^3\left(t^2-1\right)}.$$

Noting that

$$(t')^2 = \left(\frac{dt}{dQ}\right)^2 = \frac{t^2\left(t^2-1\right)}{1-c_1 t^2},$$

we find that

$$\lim_{Q \to Q_0} t''(Q) = \frac{1}{1 - c_1}.$$

Thus we obtain that $t(Q)$ is a C^2 function on $\mathbb{R}$.

Thus we can give the following corollary.

Corollary 4. *The solution of the flat surface equation* (3.7) *is given by one of the followings*
(*i*) *for a constant* c_2,

$$Q(t) = \pm \left(\arctan \left(\sqrt{t^2 - 1} \right) + c_2 \right).$$

In this case, the surface M *is a spacelike flat surface.*
(*ii*) *for a positive constant* $c_1 \neq 1$ *and a constant* c_2,

$$Q(t) = \pm \left(\sqrt{c_1} \arctan \left(\sqrt{\frac{1 - c_1 t^2}{c_1 (t^2 - 1)}} \right) - \arctan \left(\sqrt{\frac{1 - c_1 t^2}{t^2 - 1}} \right) + c_2 \right).$$

When $0 < c_1 < 1$, *the surface* M *is a spacelike flat surface, and when* $c_1 > 1$, *the surface* M *is timelike flat surface.*
(*iii*) *for constants* $c_3 > 0$ *and* c_4,

$$Q(t) = \pm \left(c_3 \log \left(\sqrt{1 + c_3^2 t^2} + c_3 \sqrt{t^2 - 1} \right) \right.$$

$$\left. - \arctan \left(\sqrt{\frac{1 + c_3^2 t^2}{t^2 - 1}} \right) - \frac{c_3}{2} \log \left(1 + c_3^2 \right) + c_4 \right).$$

In this case, the surface M *is a spacelike flat surface.*

In the following theorem, we consider the case where the normal curvature tensor of M is identically zero.

Theorem 3. *Let* M *be a generalized helicoidal surface of parametrized by* (1.1). *Then the normal curvature tensor of* M *is identically zero if and only if*

$$\left((\alpha')^2 + (\beta')^2 \right) (\beta \alpha' - \alpha \beta') + (\alpha' \beta'' - \beta' \alpha'') (\alpha^2 + \beta^2 - 1) = 0. \tag{3.9}$$

P r o o f. We have

$$R^3_{412} = \frac{\varepsilon_1 (\beta\alpha' - \alpha\beta') \left[\left((\alpha')^2 + (\beta')^2 \right) (\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'') \left(\alpha^2 + \beta^2 - 1 \right) \right]}{\sqrt{\alpha^2 + \beta^2} \left(\varepsilon_1 (\alpha^2 + \beta^2 - 1) \right)^{3/2} \left((\alpha')^2 + (\beta')^2 \right)^2},$$

$$R^3_{512} = 0,$$

$$R^4_{512} = \frac{\varepsilon_1 (\alpha\alpha' + \beta\beta') \left[\left((\alpha')^2 + (\beta')^2 \right) (\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'') \left(\alpha^2 + \beta^2 - 1 \right) \right]}{\sqrt{\alpha^2 + \beta^2} \left(\alpha^2 + \beta^2 - 1 \right)^2 \left((\alpha')^2 + (\beta')^2 \right)^2}.$$

Thus ${}^{\perp}R = 0$ if and only if

$$\left((\alpha')^2 + (\beta')^2 \right) \left(\beta\alpha' - \alpha\beta' \right) + \left(\alpha'\beta'' - \beta'\alpha'' \right) \left(\alpha^2 + \beta^2 - 1 \right) = 0. \qquad \square$$

To study surfaces with zero normal curvature tensor, it is convenient to let

$$\alpha(t) = P(t)\cos(Q(t)) \quad \text{and} \quad \beta(t) = P(t)\sin(Q(t))$$

where $P(t) > 0$ and $Q(t)$ are nonconstant smooth functions. From the equation (3.9), we have

$$PQ' \left(P'' \left(1 - P^2 \right) - P \left(Q' \right)^2 \right) + \left(P^2 - 2 \right) \left(P' \right)^2 Q' + \left(P^2 - 1 \right) PP'Q'' = 0. \tag{3.10}$$

Here for $P(t) = t$ where $t > 0, t \neq 1$, the equation (3.10) is rewritten as

$$t\left(t^2 - 1 \right) Q'' - t^2 \left(Q' \right)^3 + \left(t^2 - 2 \right) Q' = 0. \tag{3.11}$$

Setting $Q'(t) = R(t)$, we get

$$\frac{dR}{dt} + \frac{t^2 - 2}{t\left(t^2 - 1\right)} R = \frac{t}{t^2 - 1} R^3. \tag{3.12}$$

By multiplying (3.12) with $-2\left(t^2 - 1\right)/t^4 R^3$, we obtain

$$\left(\frac{t^2 - 1}{t^4} \frac{1}{R^2} \right)' = -\frac{2}{t^3}$$

which implies that

$$\frac{t^2 - 1}{t^4} \frac{1}{\left(Q'\right)^2} = \frac{1}{t^2} - c_1$$

or

$$\left(Q'\right)^2 = \frac{t^2 - 1}{t^2\left(1 - c_1 t^2\right)}$$

for a constant c_1. So we have

$$Q' = \pm\frac{1}{t}\sqrt{\frac{t^2 - 1}{1 - c_1 t^2}}.$$

Let

$$I := \int \frac{1}{t}\sqrt{\frac{t^2 - 1}{1 - c_1 t^2}}dt.$$

(i) Assume that $c_1 = 0$, which implies that $t > 1$ and M is spacelike. Then we obtain

$$I = \int \frac{\sqrt{t^2 - 1}}{t}dt = \sqrt{t^2 - 1} - \arctan\left(\sqrt{t^2 - 1}\right) + c_2$$

and

$$Q\left(t\right) = \pm\left(\sqrt{t^2 - 1} - \arctan\left(\sqrt{t^2 - 1}\right) + c_2\right)$$

for a constant c_2. Let

$$Q_+\left(t\right) = \sqrt{t^2 - 1} - \arctan\left(\sqrt{t^2 - 1}\right) + c_3$$

and

$$Q_-\left(t\right) = -\left(\sqrt{t^2 - 1} - \arctan\left(\sqrt{t^2 - 1}\right)\right) + c_3$$

for a constant c_3.

When $c_1 \neq 0$, set

$$\sqrt{\frac{t^2 - 1}{1 - c_1 t^2}} =: s.$$

Then

$$t^2 = \frac{s^2 + 1}{1 + c_1 s^2}$$

and

$$tdt = \frac{1 - c_1}{\left(1 + c_1 s^2\right)^2}sds.$$

So we have

$$I = \left(1 - c_1\right)\int \frac{s^2}{\left(1 + s^2\right)\left(1 + c_1 s^2\right)}ds = \int \left(\frac{1}{1 + c_1 s^2} - \frac{1}{1 + s^2}\right)ds.$$

(ii) Assume that c_1 is positive.
(ii-1) When $0 < c_1 < 1$, M is spacelike and $1 < t < 1/\sqrt{c_1}$.
(ii-2) When $c_1 > 1$, M is timelike and $1/\sqrt{c_1} < t < 1$.
Then we obtain

$$\begin{aligned} I &= \int \left(\frac{1}{1+c_1 s^2} - \frac{1}{1+s^2} \right) ds = \frac{1}{\sqrt{c_1}} \arctan\left(\sqrt{c_1} s\right) - \arctan s + c_2 \\ &= \frac{1}{\sqrt{c_1}} \arctan\left(\sqrt{\frac{c_1\left(t^2-1\right)}{1-c_1 t^2}} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1-c_1 t^2}} \right) + c_2 \end{aligned}$$

and

$$Q\left(t\right) = \pm\left(\frac{1}{\sqrt{c_1}} \arctan\left(\sqrt{\frac{c_1\left(t^2-1\right)}{1-c_1 t^2}} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1-c_1 t^2}} \right) + c_2 \right)$$

where c_2 is a constant. Let

$$Q_+\left(t\right) = \frac{1}{\sqrt{c_1}} \arctan\left(\sqrt{\frac{c_1\left(t^2-1\right)}{1-c_1 t^2}} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1-c_1 t^2}} \right) + c_3$$

and

$$Q_-\left(t\right) = -\left(\frac{1}{\sqrt{c_1}} \arctan\left(\sqrt{\frac{c_1\left(t^2-1\right)}{1-c_1 t^2}} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1-c_1 t^2}} \right) \right) + c_3$$

for a constant c_3.

(iii) Assume that c_1 is negative, which implies that $t > 1$ and M is spacelike. Then we can write $c_1 = -c_3^2$ for a positive c_3, which implies that

$$\begin{aligned} I &= \int \left(\frac{1}{1+c_1 s^2} - \frac{1}{1+s^2} \right) ds = \int \left(\frac{1}{1-c_3^2 s^2} - \frac{1}{1+s^2} \right) ds \\ &= \frac{1}{2c_3} \log\left(\frac{1+c_3 s}{1-c_3 s} \right) - \arctan s + c_4 \\ &= \frac{1}{2c_3} \log\left(\frac{\sqrt{1+c_3^2 t^2} + c_3\sqrt{t^2-1}}{\sqrt{1+c_3^2 t^2} - c_3\sqrt{t^2-1}} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1+c_3^2 t^2}} \right) + c_4 \\ &= \frac{1}{c_3} \log\left(\sqrt{1+c_3^2 t^2} + c_3\sqrt{t^2-1} \right) - \arctan\left(\sqrt{\frac{t^2-1}{1+c_3^2 t^2}} \right) - \frac{1}{2c_3} \log(1+c_3^2) + c_4 \end{aligned}$$

where c_4 is a constant. Thus we have

$$Q(t) = \pm\left(\frac{1}{c_3}\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right)\right.$$

$$\left.-\arctan\left(\sqrt{\frac{t^2-1}{1+c_3^2t^2}}\right)-\frac{1}{2c_3}\log\left(1+c_3^2\right)+c_4\right).$$

Let

$$Q_+(t) = \frac{1}{c_3}\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right)-\arctan\left(\sqrt{\frac{t^2-1}{1+c_3^2t^2}}\right)+c_5$$

and

$$Q_-(t) = -\left(\frac{1}{c_3}\log\left(\sqrt{1+c_3^2t^2}+c_3\sqrt{t^2-1}\right)-\arctan\left(\sqrt{\frac{t^2-1}{1+c_3^2t^2}}\right)\right)+c_6$$

for constants c_5, c_6. We can choose c_6 such that

$$\frac{1}{2c_3}\log\left(1+c_3^2\right)+c_5 = -\frac{1}{2c_3}\log\left(1+c_3^2\right)+c_6.$$

In all cases (i), (ii) and (iii), $t=1$ is a boundary point and

$$\lim_{t\to 1} Q'_\pm(t) = 0.$$

So the curves $(t\cos(Q_+(t)), t\sin(Q_+(t)))$ and $(t\cos(Q_-(t)), t\sin(Q_-(t)))$ cannot be connected as a regular curve.

Thus we can give the following corollary.

Corollary 5. *The solution of the zero normal curvature tensor equation* (3.11) *is given by one of the followings*
(i) *for a constant* c_2,

$$Q(t) = \pm\left(\sqrt{t^2-1}-\arctan\left(\sqrt{t^2-1}\right)+c_2\right).$$

In this case, the surface M *is spacelike.*
(ii) *for a positive constant* $c_1 \neq 1$ *and a constant* c_2,

$$Q(t) = \pm\left(\frac{1}{\sqrt{c_1}}\arctan\left(\sqrt{\frac{c_1(t^2-1)}{1-c_1t^2}}\right)-\arctan\left(\sqrt{\frac{t^2-1}{1-c_1t^2}}\right)+c_2\right).$$

When $0 < c_1 < 1$, *the surface* M *is spacelike, and when* $c_1 > 1$, *the surface* M *is timelike.*
(iii) *for constants* $c_3 > 0$ *and* c_4,

$$Q(t) = \pm \left(\frac{1}{c_3} \log \left(\sqrt{1 + c_3^2 t^2} + c_3 \sqrt{t^2 - 1} \right) - \arctan \left(\sqrt{\frac{t^2 - 1}{1 + c_3^2 t^2}} \right) - \frac{1}{2c_3} \log \left(1 + c_3^2 \right) + c_4 \right).$$

In this case, the surface M *is spacelike.*

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