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CYCLIC PENTAGONS AND HEXAGONS WITH INTEGER SIDES, DIAGONALS AND AREAS

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Abstract. A polygon all of whose sides, diagonals and the area are given by integers is called a Heron polygon. A cyclic Heron polygon is called a Brahmagupta polygon. In this paper we obtain parametrized families of Brahmagupta pentagons and hexagons, that is, cyclic pentagons and hexagons with integer sides, diagonals and areas.

1. Introduction. A polygon is called a cyclic polygon if all its vertices lie on the circumference of a circle. A polygon all of whose sides, diagonals and the area are given by integers is called a Heron polygon. A cyclic Heron polygon is called a Brahmagupta polygon. Further, a polygon with rational sides, rational diagonals and rational area is called a rational polygon. This paper is concerned with Brahmagupta pentagons and hexagons, that is, cyclic pentagons and hexagons with integer sides, diagonals and areas.

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It would be recalled that the area K of a triangle is given by Heron's formula,

$$K = \sqrt{s(s-a)(s-b)(s-c)} \tag{1.1}$$

where a, b, c are the sides of the triangle and s is the semi-perimeter given by $s = (a+b+c)/2$. Further, the formulae for the area K and the two diagonals d_1 and d_2 of a cyclic quadrilateral, given by Brahmagupta [3, p. 187] in the seventh century A. D., are as follows:

$$\begin{aligned} K &= \sqrt{(s-a_1)(s-a_2)(s-a_3)(s-a_4)}, \\ d_1 &= \sqrt{(a_1a_2+a_3a_4)(a_1a_3+a_2a_4)/(a_1a_4+a_2a_3)}, \\ d_2 &= \sqrt{(a_1a_3+a_2a_4)(a_1a_4+a_2a_3)/(a_1a_2+a_3a_4)}, \end{aligned} \tag{1.2}$$

where a_1, a_2, a_3, a_4 are the consecutive sides of the cyclic quadrilateral, and s is the semi-perimeter, that is, $s = (a_1 + a_2 + a_3 + a_4)/2$.

Euler gave a complete parametrization of all rational triangles (see Dickson [2, p. 193]). More recently, Sastry [7] has given a complete parametrization of all cyclic quadrilaterals whose sides, diagonals, circumradius and area are all given by integers.

The formulae for the area of cyclic pentagons and cyclic hexagons are very complicated and were discovered only towards the end of the twentieth century by Robbins [5, 6]. In fact, if K is the area of a cyclic pentagon/ hexagon, then K^2 satisfies an equation of degree 7 whose coefficients are symmetric functions of the sides of the cyclic pentagon/ hexagon. Similarly, the formulae for the diagonals of cyclic pentagons and hexagons are given by equations of degree 7 [1, p. 37 and p. 41]. In view of these complicated formulae, the areas and the diagonals of cyclic pentagons and hexagons with rational sides are, in general, not rational. There has thus been considerable interest in constructing cyclic pentagons and hexagons whose sides, diagonals and the area are all given by rational numbers.

Euler [2, p. 221] and Sastry [8] have independently described methods of constructing cyclic polygons with five or more sides such that all the sides, the diagonals and the area of the polygon are rational. Recently, Buchholz and MacDougall [1] have carried out computational searches for rational cyclic pentagons and hexagons. While these computational searches yielded several rational cyclic pentagons, Buchholz and MacDougall did not find even a single hexagon with rational sides and area and with all nine diagonals rational. Till date a parametrization of rational cyclic pentagons and hexagons does not appear to have been published.

We note that geometric figures such as triangles, quadrilaterals and polygons with rational sides, diagonals (where applicable), and areas yield, on appropriate scaling, similar geometric figures with integer sides, diagonals (where applicable), and areas. It therefore suffices to obtain cyclic pentagons and hexagons with rational sides, diagonals and areas.

In this paper we obtain a cyclic pentagon with rational sides, diagonals and area all of which are expressed in terms of rational functions of several arbitrary rational parameters. We also show how additional rational cyclic pentagons may be obtained. The complete parametrization of all rational cyclic pentagons is given by a finite number of such parametrizations. We also give a parametrization of cyclic hexagons whose sides, the nine diagonals and the area are all rational.

2. Parametrizations of rational triangles and quadrilaterals. We will construct a cyclic pentagon by juxtaposing a rational triangle and a rational cyclic quadrilateral having the same circumradius. Similarly we will construct a cyclic hexagon by juxtaposing two rational cyclic quadrilaterals having the same circumradius. Accordingly, we give below, in the following lemmas, the already known complete parametrization of rational triangles and cyclic quadrilaterals and related formulae about their areas and circumradii. We will use these lemmas to obtain parametrized families of rational cyclic pentagons and hexagons.

Lemma 1. *The lengths of the three sides a, b, c, the area K and the circumradius R of an arbitrary rational triangle may be expressed as follows:*

$$
\begin{aligned}
a &= k(m^2+n^2)/(mn),\\
b &= k(p^2+q^2)/(pq),\\
c &= k(pn+qm)(pm-qn)/(pqmn),
\end{aligned} \tag{2.1}
$$

$$K = k^2(pn+qm)(pm-qn)/(pqmn), \tag{2.2}$$

$$R = k(m^2+n^2)(p^2+q^2)/(4mnpq). \tag{2.3}$$

where m, n, p, q and k are arbitrary rational parameters.

P r o o f. The formulae (2.1) for the sides a, b, c of an arbitrary rational triangle were proved by Euler (see Dickson [2, p. 193]). Now, on using Heron's formula (1.1), we immediately get the formula (2.2) for the area of a triangle whose sides a, b, c are given by (2.1).

Further, the circumradius R of a triangle, whose sides are a, b, c, is given by

$$R = abc/\{4\sqrt{s(s-a)(s-b)(s-c)}\},$$

where s is the semi-perimeter, that is, $s = (a+b+c)/2$. Thus, the circumradius R is given explicitly in terms of the sides a, b, c, by the formula,

$$R = abc/\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}. \tag{2.4}$$

The formula (2.3) for the circumradius of a triangle, whose sides a, b, c, are given by (2.1), now readily follows from (2.4). □

Lemma 2. *The lengths of the sides a_1, a_2, a_3, a_4, the diagonals d_1, d_2, the area K and the circumradius R of an arbitrary rational cyclic quadrilateral are given by*

$$\begin{aligned} a_1 &= \{t(u+v)+(1-uv)\}\{u+v-t(1-uv)\},\\ a_2 &= (1+u^2)(v-t)(1+tv),\\ a_3 &= t(1+u^2)(1+v^2),\\ a_4 &= (1+v^2)(u-t)(1+tu),\\ d_1 &= u(1+t^2)(1+v^2),\\ d_2 &= v(1+t^2)(1+u^2),\\ K &= uv\{2t(1-uv)-(u+v)(1-t^2)\}\\ &\qquad \times \{2(u+v)t+(1-uv)(1-t^2)\},\\ R &= (1+t^2)(1+u^2)(1+v^2)/4, \end{aligned} \tag{2.5}$$

where t, u and v are arbitrary rational parameters.

Proof. The formulae (2.5) have been proved by Sastry [7, p. 170]. □

Lemma 3. *The circumradius R of a cyclic quadrilateral with sides a_1, a_2, a_3 and a_4 is given by the formula,*

$$\begin{aligned} R = {}& \{(a_1a_2+a_3a_4)(a_1a_3+a_2a_4)(a_1a_4+a_2a_3)\}^{1/2}\{(-a_1+a_2+a_3+a_4)\\ &\times (a_1-a_2+a_3+a_4)(a_1+a_2-a_3+a_4)(a_1+a_2+a_3-a_4)\}^{-1/2}. \end{aligned} \tag{2.6}$$

Proof. Paramesvara [4] proved the following formula for the circumradius R of a cyclic quadrilateral with sides a_1, a_2, a_3 and a_4:

$$R = \frac{1}{4}\sqrt{\frac{(a_1a_2+a_3a_4)(a_1a_3+a_2a_4)/(a_1a_4+a_2a_3)}{(s-a_1)(s-a_2)(s-a_3)(s-a_4)}}, \tag{2.7}$$

where s is the semi-perimeter of the quadrilateral, that is, $s = (a_1+a_2+a_3+a_4)/2$. On substituting the value of s in the formula (2.7), we get the formula (2.6) explicitly in terms of the sides a_1, a_2, a_3, a_4 of the quadrilateral. □

3. Cyclic pentagons with integer sides, diagonals and area.

Theorem 1. *The lengths of the sides $s_i, i = 1, 2, \dots, 5$, of a parametrized family of rational cyclic pentagons may be written as follows:*

$$\begin{aligned} s_1 &= (-quv + pu + pv + q)(puv + qu + qv - p)/q^2, \\ s_2 &= (u^2 + 1)(qv - p)(pv + q)/q^2, \\ s_3 &= mn(u^2 + 1)(v^2 + 1)(p^2 + q^2)/\{q^2(m^2 + n^2)\}, \\ s_4 &= (u^2 + 1)(v^2 + 1)(mq + np)(mp - nq)/\{q^2(m^2 + n^2)\}, \\ s_5 &= (v^2 + 1)(qu - p)(pu + q)/q^2, \end{aligned} \tag{3.1}$$

where m, n, p, q, u and v are arbitrary rational parameters. On appropriate scaling, we obtain a family of Brahmagupta pentagons.

Proof. We will construct a rational cyclic pentagon and show that the lengths of its sides may be written as $s_i, i = 1, 2, \dots, 5$, where the values of s_i are given by (3.1).

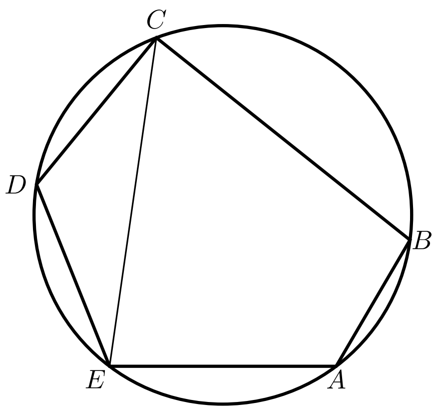

Fig. 1. Cyclic pentagon

We note that any diagonal of a pentagon divides it into two parts: a triangle and a quadrilateral with a common side which is one of the diagonals of the pentagon. Accordingly we may consider a cyclic pentagon $ABCDE$ (see Figure 1) with rational sides and diagonals as being made up by juxtaposing the cyclic quadrilateral $ABCE$ and the triangle CDE.

Since the lengths of the sides of the quadrilateral $ABCE$ and the triangle CDE are rational, it follows from the formulae (1.1) and (1.2) that the areas of the quadrilateral $ABCE$ and the triangle CDE are expressible as $\sqrt{r_1}$ and $\sqrt{r_2}$, where r_1 and r_2 are nonzero rational numbers. Since we want the area of the pentagon to be rational, $\sqrt{r_1} + \sqrt{r_2}$ must be some rational number r. It follows that $r - \sqrt{r_1} = \sqrt{r_2}$, or, $r^2 + r_1 - 2r\sqrt{r_1} = r_2$, and hence $\sqrt{r_1}$ must be rational. Similarly, $\sqrt{r_2}$ must be rational, and it is thus seen that the areas of both the quadrilateral $ABCE$ and the triangle CDE must be rational. It follows that the quadrilateral $ABCE$ is a rational cyclic quadrilateral and the triangle CDE is a rational triangle.

The lengths of the sides of the quadrilateral $ABCE$ and the triangle CDE can be chosen in several ways. For the proof of the theorem we shall consider one of the cases.

Since $ABCE$ is a rational cyclic quadrilateral, we may take the sides AB, BC, CE, and EA as having the lengths a_1, a_2, a_3, a_4, respectively, as given by (2.5) while the lengths of the three sides of the triangle CDE may be taken as given by (2.1).

Since the quadrilateral $ABCE$ and the triangle CDE are inscribed in the same circle, the circumradii of the quadrilateral $ABCE$ and the triangle CDE are equal, and hence, on using the formulae (2.3) and (2.5), we get,

$$k(m^2 + n^2)(p^2 + q^2)/(4mnpq) = (t^2 + 1)(v^2 + 1)(u^2 + 1)/4. \tag{3.2}$$

Now CE is also a side of the triangle CDE and therefore, its length may be taken, without loss of generality, either as a or c as given by (2.1).

We first consider the case when CE has length a as given by (2.1). Equating the two lengths of CE, we get

$$k(m^2 + n^2)/(mn) = t(u^2 + 1)(v^2 + 1). \tag{3.3}$$

Eqs. (3.2) and (3.3) are readily solved, and we get the following two solutions:

$$t = p/q, \quad k = pmn(u^2 + 1)(v^2 + 1)/\{q(m^2 + n^2)\}, \tag{3.4}$$

and

$$t = q/p, \quad k = qmn(u^2 + 1)(v^2 + 1)/\{p(m^2 + n^2)\}. \tag{3.5}$$

Using the solution (3.4) and the formulae (2.1) and (2.5), we readily find that the lengths of the sides AB, BC, CD, DE and EA of the pentagon are given by s_1, $s_2, \ldots$, s_5, respectively, as stated in the theorem.

The circumradius R of the pentagon, given by either side of (3.2), may be written explicitly as follows:

$$R = (u^2 + 1)(v^2 + 1)(p^2 + q^2)/(4q^2). \tag{3.6}$$

The area of the pentagon is rational and is easily worked out, being the sum of the rational areas of the triangle CDE and the quadrilateral $ABCE$.

As regards the diagonals, the length of the diagonal CE, which is the common side of the triangle CDE and the quadrilateral $ABCE$ diagonal CE, is readily found and is given by

$$CE = p(u^2 + 1)(v^2 + 1)/q.$$

The two diagonals AC and BE, being diagonals of the cyclic quadrilateral $ABCE$, may be found using the formulae (2.5) and are given by

$$AC = u(p^2 + q^2)(v^2 + 1)/q^2, \quad BE = v(p^2 + q^2)(u^2 + 1)/q^2.$$

To find the length d of the diagonal AD, we note that the circumradius of the triangle ADE is given by (3.6), and since the sides of the triangle ADE have lengths d, s_4 and s_5, in view of the formula (2.4), we must have

$$\begin{aligned} ds_4s_5/\sqrt{(d + s_4 + s_5)(d + s_4 - s_5)(s_4 + s_5 - d)(s_5 + d - s_4)} \\ = (u^2 + 1)(v^2 + 1)(p^2 + q^2)/(4q^2), \end{aligned} \tag{3.7}$$

where s_4, s_5 are given by (3.1).

We also note that the circumradius of the cyclic quadrilateral $ABCD$ is given by (3.6). Since the sides of the quadrilateral $ABCD$ have lengths s_1, s_2, s_3 and d, it follows from the formula (2.6) that

$$\begin{aligned} &\{(s_1s_2 + s_3d)(s_1s_3 + s_2d)(s_1d + s_2s_3)\}^{1/2}\{(-s_1 + s_2 + s_3 + d) \\ &\quad \times (s_1 - s_2 + s_3 + d)(s_1 + s_2 - s_3 + d)(s_1 + s_2 + s_3 - d)\}^{-1/2} \\ &\qquad = (u^2 + 1)(v^2 + 1)(p^2 + q^2)/(4q^2), \end{aligned} \tag{3.8}$$

where s_1, s_2, s_3 are given by (3.1).

Each of the two equations (3.7) and (3.8) is satisfied by four rational values of d but there is one common root which gives the length d of the diagonal AD. We thus get

$$AD = (p^2 + q^2)(v^2 + 1)(mu - n)(nu + m)/\{(m^2 + n^2)q^2\}.$$

Finally, the diagonal BD may be obtained just as the diagonal AD by solving two equations obtained by equating the circumradius of the pentagon first to the circumradius of the triangle BCD and then to the circumradius of the quadrilateral $ABDE$ and finding the common root. We thus get

$$BD = (u^2+1)(mqv+npv-mp+nq)(mpv-nqv+mq+np)/\{(m^2+n^2)q^2\}.$$

Thus, the cyclic pentagon $ABCDE$ with rational sides s_i, $i = 1, \ldots, 5$, given by (3.1) in terms of arbitrary rational parameters m, n, p, q, u and v has rational circumradius (3.6), rational area and rational diagonals as seen above. On appropriate scaling, we get a cyclic pentagon whose sides, circumradius, area and diagonals are all expressible by integers.

We note that we have obtained the above cyclic pentagon just by equating the circumradii and the side CE of the triangle CDE and the quadrilateral $ABDE$. We also need to ensure that the triangle and the quadrilateral can actually be constructed on opposite sides of CE.

If for certain values of the parameters, the common side CE is twice the circumradius, then CE is a diameter of the circumcircle and if s_i, $i = 1, \ldots, 5$, are all positive, the triangle CDE and the quadrilateral $ABDE$ can always be constructed on opposite sides of the diameter CE. These conditions are satisfied when $p = q$, $u > 1$, $m > n$ and $(u+1)/(u-1) > v > 1$.

When the side CE divides the circle into two unequal segments, there may exist certain values of the parameters for which it is possible that both the triangle and the quadrilateral can be constructed only in the same segment and, in such a case, the desired cyclic pentagon cannot be constructed. To ensure that we actually get a cyclic pentagon, we observe that there is no loss of generality in taking the triangle to be in the minor segment and the quadrilateral in the major segment.

If we choose the parameters m, n, p, q satisfying the conditions

$$q/p > \min(u,v) > p/q > (uv-1)/(u+v), \quad \text{and} \quad qm > pn,$$

then it is easily seen that the sides CD and DE are both shorter than CE and hence the triangle is necessarily in the minor segment of the circumcircle. Further, one of the diagonals of the quadrilateral is longer than CE and so the quadrilateral can be constructed in the major segment, and we will actually have a cyclic pentagon with the desired properties. □

As a numerical example, when $p = q = 1$, $u = 3$, $v = 3/2$, $m = 2$, $n = 1$, we get a cyclic pentagon whose sides AB, BC, CD, DE, EA have lengths 8, 25/2, 26, 39/2, 26, respectively, the diagonals have lengths 39/2, 65/2, 63/2, 30, 65/2, the circumradius is 65/4 and the area of the pentagon is 537.

As a second example, taking $m = 3$, $n = 1$, $p = 2$, $q = 3$, $u = 1$, $v = 2$, we get a cyclic pentagon whose sides AB, BC, CD, DE, EA have lengths 11/3, 56/9, 13/3, 11/3, 25/9, respectively, the diagonal CE has length 20/3, the remaining diagonals have lengths 65/9, 52/9, 323/45, 52/9, the circumradius is 65/18 and the area of the pentagon is 28.

We note that using the solution (3.5) of Eqs. (3.2) and (3.3), we also get a cyclic pentagon but this is essentially equivalent to the pentagon that we have already obtained using the first solution (3.4).

We now consider the case when the side CE of the triangle CDE is taken as c given by (2.1) while, as before, the sides AB, BC, CE, and EA of the quadrilateral have lengths a_1, a_2, a_3, a_4, respectively, given by (2.5). Proceeding as before, we find the sides $s_i, i = 1, \dots, 5$, and the circumradius of the pentagon. It turns out, however, that by a suitable transformation of the parameters, this solution is seen to be equivalent to the solution already obtained above.

In the proof of Theorem 1 we have considered CE as a side of the quadrilateral $ACDE$ as well as a side of the triangle CDE and equated the two lengths. While we had taken the length of the side CE of the quadrilateral $ABCE$ as a_3, given by (2.5), we can also choose the side CE to have length a_1 or a_2 or a_4 as given by (2.5). Since the lengths a_2 and a_4 get interchanged if we interchange the parameters u and v, there is no loss of generality in taking the length of CE either as a_1 or a_2. If we consider CE as one of the sides of the triangle CDE, we have already noted that without loss of generality, we may take its length either as a or c as given by (2.1).

We thus get four cases according to the choice of lengths of the side CE. If we now proceed as in the proof of Theorem 1 and impose the two conditions that the circumradii of the triangle CDE and the quadrilateral $ABCE$ are equal, and the two lengths of CE are also equal, it is noteworthy that in each case, the two conditions are readily solvable, and further, in each case, the five diagonals of the pentagon are also rational, and their lengths can be computed as in Theorem 1

We thus get a finite number of parametrizations for a cyclic pentagon with rational sides, diagonals and area. All such cyclic pentagons arise from one of these parametrizations. We do not give these remaining parametrizations explicitly as any interested reader can readily find them by following the method described in Theorem 1.

4. Cyclic hexagons with integer sides, diagonals and area.

Theorem 2. *The lengths of the sides h_i, $i = 1, 2, \ldots, 6$, of a parametrized family of rational cyclic hexagons may be written as follows:*

$$
\begin{aligned}
h_1(m,\,t,\,u_1,\,u_2) &= \{(u_1+u_2)((u_1+u_2)t-u_1u_2+1)m^2 \\
&\quad -((2u_1^2u_2-4u_1-2u_2)t+2u_1^2+4u_1u_2-2)m \\
&\quad -(u_1^2+2u_1u_2-u_2^2-2)t+u_1^2u_2-u_1u_2^2 \\
&\quad -3u_1-u_2\}\{(u_1+u_2)((u_1u_2-1)t+u_1+u_2)m^2 \\
&\quad +((2u_1^2+4u_1u_2-2)t-2u_1^2u_2+4u_1+2u_2)m \\
&\quad -(u_1^2u_2-u_1u_2^2-3u_1-u_2)t \\
&\quad -u_1^2-2u_1u_2+u_2^2+2\}, \\
h_2(m,\,t,\,u_1,\,u_2) &= -(u_1^2+1)\{(u_1+u_2)(t-u_2)m^2 \\
&\quad +((2-2u_1u_2)t-2u_1-2u_2)m-(u_1+u_2)t \\
&\quad +u_1u_2-u_2^2-2\}\{(u_1+u_2)(u_2t+1)m^2 \\
&\quad +((2u_1+2u_2)t-2u_1u_2+2)m \\
&\quad +(-u_1u_2+u_2^2+2)t-u_1-u_2\}, \\
h_3(m,\,t,\,u_1,\,u_2) &= (u_2^2+1)(m^2+1)(u_1-t)(tu_1+1)\{(u_1+u_2)^2m^2 \\
&\quad +4(u_1+u_2)m+u_1^2-2u_1u_2+u_2^2+4\}, \\
h_4(m,\,t,\,u_1,\,u_2) &= h_1(m,\,t,\,u_2,\,u_1), \\
h_5(m,\,t,\,u_1,\,u_2) &= h_2(m,\,t,\,u_2,\,u_1), \\
h_6(m,\,t,\,u_1,\,u_2) &= h_3(m,\,t,\,u_2,\,u_1),
\end{aligned}
\tag{4.1}
$$

where m, t, u_1 and u_2 are arbitrary rational parameters. On appropriate scaling, we obtain a family of Brahmagupta hexagons.

Proof. We will construct a rational cyclic hexagon by juxtaposing two cyclic quadrilaterals which have the same circumradius and which have a common side which becomes a diagonal of the hexagon (see Figure 2). The lengths of the sides of both the cyclic quadrilaterals are easily seen to be rational, and hence their areas are expressible as $\sqrt{r_1}$ and $\sqrt{r_2}$, where r_1 and r_2 are rational numbers. The area of the hexagon, that is, $\sqrt{r_1} + \sqrt{r_2}$, is rational, and hence, as in the case of the pentagon, we see that both $\sqrt{r_1}$ and $\sqrt{r_2}$ must be rational numbers, and thus the areas of both the quadrilaterals must be rational. It follows that the two quadrilaterals must be rational cyclic quadrilaterals.

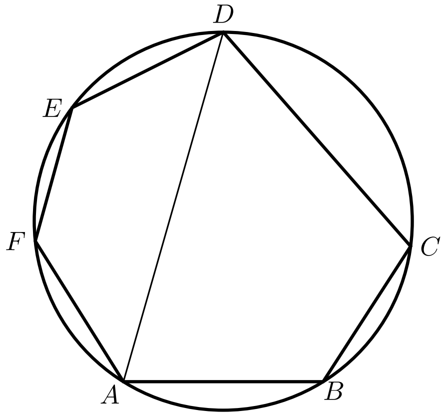

Fig. 2. Cyclic hexagon

It has been shown in [1] that merely juxtaposing two rational cyclic quadrilaterals with the same circumradius and a common side does not generally yield a cyclic hexagon with rational diagonals. We will, however, choose the sides of the two quadrilaterals in such a manner that we are able to obtain a cyclic hexagon whose sides, diagonals, circumradius and area are all rational. We will show that the lengths of the sides of our hexagon may be written as $h_i(m, t, u_1, u_2)$, $i = 1, 2, \ldots, 6$, where the values of $h_i(m, t, u_1, u_2)$, $i = 1, 2, \ldots, 6$, are given by (4.1).

We will take the sides of the cyclic quadrilateral $ABCD$ as being given by Sastry's formulae (2.5) with the parameters t, u, v, having values t, u_1 and v_1, respectively. We thus get the length of the circumradius of the quadrilateral $ABCD$ as $(t^2+1)(u_1^2+1)(v_1^2+1)/4$. We now specifically take the side AD as having length a_3, that is, we take, $AD = t(u_1^2+1)(v_1^2+1)$.

For the second cyclic quadrilateral $ADEF$, we take the sides as being given by the formulae (2.5) with the parameters t, u, v, having values t, u_2 and v_2, respectively. We thus get the length of the circumradius of the quadrilateral $AFED$ as $(t^2+1)(u_2^2+1)(v_2^2+1)/4$. Further, we again take the side AD as having length a_3 given by (2.5), so that we get $AD = t(u_2^2+1)(v_2^2+1)$.

It is now readily seen that the two conditions, that both the cyclic quadilaterals have the same circumradius and that they have a side in common, are simultaneously satisfied if the parameters u_i, v_i, satisfy the following condition:

$$(u_1^2+1)(v_1^2+1) = (u_2^2+1)(v_2^2+1). \tag{4.2}$$

Now Eq. (4.2) may be written as,

$$\begin{aligned} &(u_1v_1+1)^2 + (u_1-v_1)^2 = (u_2v_2+1)^2 + (u_2-v_2)^2, \\ \text{or,}\quad &(u_1v_1+1)^2 - (u_2v_2+1)^2 = (u_2-v_2)^2 - (u_1-v_1)^2, \\ \text{or,}\quad &(u_1v_1+u_2v_2+2)(u_1v_1-u_2v_2) = -(u_1+u_2-v_1-v_2) \\ &\qquad\qquad \times (u_1-u_2-v_1+v_2), \end{aligned}$$

and is therefore equivalent to the following two linear equations in v_1, v_2:

$$u_1v_1 + u_2v_2 + 2 = -m(u_1 + u_2 - v_1 - v_2),$$
$$m(u_1v_1 - u_2v_2) = u_1 - u_2 - v_1 + v_2,$$

where m is an arbitrary rational parameter. We thus get the following solution of Eq. (4.2):

$$\begin{aligned} v_1 &= \{(u_1 + u_2)u_2m^2 + (2u_1 + 2u_2)m - u_1u_2 + u_2^2 + 2\} \\ &\quad \times \{(u_1 + u_2)m^2 + (-2u_1u_2 + 2)m - u_1 - u_2\}^{-1}, \\ v_2 &= \{(u_1 + u_2)u_1m^2 + (2u_1 + 2u_2)m + u_1^2 - u_1u_2 + 2\} \\ &\quad \times \{(u_1 + u_2)m^2 + (-2u_1u_2 + 2)m - u_1 - u_2\}^{-1}. \end{aligned}$$

With these values of v_1, v_2, we can readily work out the sides of the two cyclic quadilaterals $ABCD$ and $ADEF$ as well as their common circumradius in terms of arbitrary rational parameters m, t, u_1 and u_2. After appropriate scaling, the lengths of the sides AB, BC, CD, DE and EF of the hexagon $ABCDEF$ may be written as $h_i(m, t, u_1, u_2)$, $i = 1, \ldots, 6$, respectively, where the values of $h_i(m, t, u_1, u_2)$, $i = 1, \ldots, 6$, are as stated in the theorem.

The circumradius R of the hexagon is given by

$$\begin{aligned} R &= (u_1^2 + 1)(u_2^2 + 1)(t^2 + 1)(m^2 + 1) \\ &\quad \times \{(u_1 + u_2)^2m^2 + (4u_1 + 4u_2)m + u_1^2 - 2u_1u_2 + u_2^2 + 4\}/4. \end{aligned} \tag{4.3}$$

The area of the hexagon, being the sum of the rational areas of the two cyclic quadrilaterals $ABCD$ and $ADEF$ is naturally rational, and is easily determined.

We will now determine all the 9 diagonals of the hexagon $ABCDEF$. We first find the three central diagonals AD, BE, CF, each of which divides the hexagon into two cyclic quadrilaterals. We have, in fact, already found the diagonal AD since it is the common side of the two quadrilaterals $ABCD$ and $ADEF$, and its length is given by

$$\begin{aligned} AD &= t(u_1^2 + 1)(u_2^2 + 1)(m^2 + 1) \\ &\quad \times \{(u_1 + u_2)^2m^2 + 4(u_1 + u_2)m + u_1^2 - 2u_1u_2 + u_2^2 + 4\}. \end{aligned}$$

To find the length d of the central diagonal BE, we note that d is one of the sides of each of the two quadrilaterals $BCDE$ and $ABEF$ both of which have circumradius R. Thus d must satisfy the condition that the circumradius of the quadrilateral $BCDE$ is R. Writing the sides BC, CD, DE simply as h_2, h_3, h_4, and using the formula (2.6), we get the condition,

$$\begin{aligned}(4.4)\quad R = \{(dh_2 + h_3h_4)(dh_3 + h_2h_4)(dh_4 + h_2h_3)\}^{1/2}\{(-d + h_2 + h_3 + h_4) \\ \times (d - h_2 + h_3 + h_4)(d + h_2 - h_3 + h_4)(d + h_2 + h_3 - h_4)\}^{-1/2}.\end{aligned}$$

The values of h_2, h_3, h_4 and R are given by (4.1) and (4.3), and on solving Eq. (4.4) for d, we get four rational roots. Similarly, d must satisfy the condition that the circumradius of the quadrilateral $ABEF$ is R, and this condition also results in an equation that is satisfied by four rational values of d. The common root gives the length d and we thus get,

$$\begin{aligned}BE = \{&(u_1+u_2)^2m^2 + 4(u_1+u_2)m + u_1^2 - 2u_1u_2 + u_2^2 + 4\} \\ \times \{&((u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^3 + 6(u_1+u_2)(u_1u_2-1)t^2 \\ &- 3(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t - 2(u_1+u_2)(u_1u_2-1))m^2 \\ &+ (4(u_1+u_2)(u_1u_2-1)t^3 - 6(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^2 \\ &- 12(u_1+u_2)(u_1u_2-1)t + 2(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1))m \\ &- (u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^3 - 6(u_1+u_2)(u_1u_2-1)t^2 \\ &+ 3(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t + 2(u_1+u_2)(u_1u_2-1)\} \\ \times \{&(2(u_1+u_2)(u_1u_2-1)t^3 - 3(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^2 \\ &- 6(u_1+u_2)(u_1u_2-1)t + (u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1))m^2 \\ &+ (-2(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^3 - 12(u_1+u_2)(u_1u_2-1)t^2 \\ &+ 6(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t + 4(u_1+u_2)(u_1u_2-1))m \\ &- 2(u_1+u_2)(u_1u_2-1)t^3 + 3(u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)t^2 \\ &+ 6(u_1+u_2)(u_1u_2-1)t - (u_1u_2+u_1+u_2-1)(u_1u_2-u_1-u_2-1)\} \\ &\times \{(t^2+1)^2(m^2+1)(u_1^2+1)(u_2^2+1)\}^{-1}.\end{aligned}$$

The length of the central diagonal CF is similarly computed since CF is a common side of the two quadrilaterals $ABCF$ and $CDEF$, and equating the circumradii of these two quadrilaterals to R as before, we get two equations for the unknown length CF, each equation has four rational roots and the common root gives the diagonal CF, which is as follows:

$$CF = (m^2+1)\{(u_1+u_2)t - u_1u_2 + 1\}\{(1-u_1u_2)t - u_1 - u_2\} \\ \times \{(u_1+u_2)^2m^2 + 4(u_1+u_2)m + u_1^2 - 2u_1u_2 + u_2^2 + 4\}.$$

The six minor diagonals of the hexagon, that is, AC, BD, AE, DF, BF and CE are also diagonals of the cyclic quadrilaterals $ABCD$, $ADEF$, $ABCF$ and $CDEF$. Since all the four sides of these quadrilaterals are already known, these diagonals are readily found using the formulae (1.2). We also note that each minor diagonal divides the hexagon into two parts, one of which is a triangle whose circumradius is R and hence the formula (2.4) may also be used to determine the lengths of these diagonals.

The three diagonals AC, BD and BF are given by

$$\begin{aligned}
AC &= u_1(t^2+1)(u_2^2+1)(m^2+1) \\
&\quad \times \{(u_1+u_2)^2m^2 + 4(u_1+u_2)m + u_1^2 - 2u_1u_2 + u_2^2 + 4\}, \\
BD &= \{(u_1+u_2)((u_1u_2-1)t^2 + (2u_1+2u_2)t - u_1u_2 + 1)m^2 \\
&\quad + ((2u_1^2 + 4u_1u_2 - 2)t^2 + (-4u_1^2u_2 + 8u_1 + 4u_2)t - 2u_1^2 - 4u_1u_2 + 2)m \\
&\quad + (-u_1^2u_2 + u_1u_2^2 + 3u_1 + u_2)t^2 + (-2u_1^2 - 4u_1u_2 + 2u_2^2 + 4)t \\
&\quad + u_1^2u_2 - u_1u_2^2 - 3u_1 - u_2\}\{(u_1+u_2)((u_1+u_2)t^2 + (-2u_1u_2+2)t \\
&\quad - u_1 - u_2)m^2 + ((-2u_1^2u_2 + 4u_1 + 2u_2)t^2 + (-4u_1^2 - 8u_1u_2 + 4)t \\
&\quad + 2u_1^2u_2 - 4u_1 - 2u_2)m + (-u_1^2 - 2u_1u_2 + u_2^2 + 2)t^2 + (2u_1^2u_2 \\
&\quad - 2u_1u_2^2 - 6u_1 - 2u_2)t + u_1^2 + 2u_1u_2 - u_2^2 - 2\}(t^2+1)^{-1}, \\
BF &= -\{(u_1+u_2)((u_1u_2^2 - u_1 - 2u_2)t^2 + (4u_1u_2 + 2u_2^2 - 2)t - u_1u_2^2 \\
&\quad + u_1 + 2u_2)m^2 + (4(u_1+u_2)(u_1u_2-1)t^2 - 4(u_1u_2 + u_1 + u_2 - 1) \\
&\quad \times (u_1u_2 - u_1 - u_2 - 1)t - 4(u_1+u_2)(u_1u_2-1))m + \\
&\quad (-u_1^2u_2^2 + u_1u_2^3 + u_1^2 + 5u_1u_2 - 2)t^2 - 2(u_1+u_2)(2u_1u_2 - u_2^2 - 3)t \\
&\quad + u_1^2u_2^2 - u_1u_2^3 - u_1^2 - 5u_1u_2 + 2\}\{(u_1+u_2)((2u_1u_2 + u_2^2 - 1)t^2 \\
&\quad + (-2u_1u_2^2 + 2u_1 + 4u_2)t - 2u_1u_2 - u_2^2 + 1)m^2 + (-2(u_1u_2 + u_1 \\
&\quad + u_2 - 1)(u_1u_2 - u_1 - u_2 - 1)t^2 - 8(u_1+u_2)(u_1u_2-1)t \\
&\quad + 2(u_1u_2 + u_1 + u_2 - 1)(u_1u_2 - u_1 - u_2 - 1))m - (u_1+u_2) \\
&\quad \times (2u_1u_2 - u_2^2 - 3)t^2 + (2u_1^2u_2^2 - 2u_1u_2^3 - 2u_1^2 - 10u_1u_2 + 4)t \\
&\quad + (u_1+u_2)(2u_1u_2 - u_2^2 - 3)\}\{(u_2^2+1)(t^2+1)\}^{-1}.
\end{aligned}$$

The lengths of the three minor diagonals DF, AE, CE are given by formulae obtained by interchanging u_1 and u_2 in the formulae for the lengths of the diagonals AC, BD and BF, respectively.

Thus the circumradius, all the 9 diagonals and the area of the cyclic hexagon, whose sides are given by (4.1), are all rational. On suitable scaling, we get a cyclic hexagon whose sides, circumradius, diagonals and area are all integers.

As in the case of the pentagon, it is possible that for certain values of the parameters, we may not get an actual hexagon since it may not be possible to construct the two quadrilaterals $ABCD$ and $ADEF$ on opposite sides of the common side AD. If AD is twice the circumradius, it becomes a diameter of the circumcircle and the quadrilaterals $ABCD$ and $ADEF$ can always be constructed on opposite sides of AD. If AD divides the circumcircle into two unequal segments, and both the diagonals of one of the two quadrilaterals are shorter than AD while one of the diagonals of the second quadrilateral is longer than AD, then also the quadrilaterals $ABCD$ and $ADEF$ can be constructed on opposite sides of AD and we get a cyclic hexagon with the desired properties. □

As a numerical example, if we take the parameters m, t, u_1 and u_2 as 5, 1, 2 and -3, respectively, we get a cyclic hexagon $ABCDEF$ whose sides AB, BC, CD, DE and EF are 2044, 2880, 3315, 3124, 235, 4420, respectively, the circumradius is 5525/2, the central diagonal AD is 5525 and is thus a diameter of the circumcircle, the area of the heaxgon is 17227230 and the remaining 8 diagonals have lengths 1751561/325, 5304, 4420, 5133, 26664/5, 3315, 4557, and 26167/5.

As a second numerical example, if we take $(m, t, u_1, u_2) = (2, 2, -8, 3)$, we get a hexagon whose six sides are 12075, 2795, 55500, 24747, 13080 and 16835, the circumradius is 60125/2, the area is 672750078 while the 9 diagonals have lengths 48100, 12667468/325, 30525, 14800, 54365, 28084, 36075, 144943/5 and 533817/13.

In both the numerical examples given above, on appropriate scaling, we can easily obtain a cyclic hexagon with integer sides, diagonals and area.

As in the case of the pentagon, we can try to obtain additional parametrizations of a cyclic hexagon by choosing the sides of the two quadrilaterals $ABCD$ and $ADEF$ in different ways and imposing the two conditions that the quadrilaterals have the same circumradius and a common side. In some cases, we can obtain a parametrization for a rational cyclic hexagon but the solution is much more complicated as compared to the solution given above.

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