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OSCILLATION THEOREMS FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS WITH SEVERAL SUBLINEAR NEUTRAL TERMS

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Abstract. This paper presents new sufficient conditions for the oscillation of second-order difference equations with several sublinear neutral terms. The results obtained here extend and generalize the existing results in the literature. Several examples are provided to illustrate and highlight the importance and novelty of the main results.

1. Introduction. In this paper, we investigate the oscillatory properties of second-order difference equations with several sublinear neutral terms

$$(E)\qquad \Delta(b(n)\Delta u(n)) + p(n)x^{\gamma}(n-l) = 0,\ n \geq \mathbb{N}(n_0),$$

where $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$ and $u(n) = x(n) + \sum_{i=1}^{m} d_i(n)x^{\alpha_i}(n-k_i)$. Throughout the paper, we assume that

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(H_1) $0 < \alpha_i \le 1$ for $i = 1, 2, \dots, m$ and moreover, each of the α_i, γ is a quotient of odd positive integers;

(H_2) $\{b(n)\}$, $\{d_i(n)\}$, $\{p(n)\}$ are positive real sequences with $\lim_{n\to\infty} d_i(n) = d_i \in (0, 1)$ for $i = 1, 2, \dots, m$ and $\sum_{i=1}^{m} d_i < 1$;

(H_3) k_i and l are positive integers for $i = 1, 2, \dots, m$.

Furthermore, we assume that (E) is in a canonical form, namely,

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty. \tag{1.1}$$

Let $\theta = \max\{k_1, k_2, \dots, k_n, l\}$. By a solution of (E) we mean a real sequence $\{x(n)\}$ defined for all $n \ge n_0 - \theta$ and satisfying (E) for all $n \ge \mathbb{N}(n_0)$. A nontrivial solution of (E) is called *oscillatory* if the terms of the sequence are neither eventually positive nor eventually negative. Otherwise, it is *nonoscillatory.* Equation (E) is called oscillatory if all its solutions are oscillatory.

Neutral type differential/difference equations arise in a number of problems in physics, biology, etc., see [1, 16]. Therefore, the qualitative behavior of the solutions of this type of equations has attracted considerable interest and extensive research has been produced on this subject, see for example,[1, 2, 9, 11, 12, 17, 18, 20] and the references cited therein.

However, there has been only a few results on the oscillation of second-order difference equations with sublinear neutral terms [3, 5, 6, 13, 14, 15, 19, 22, 23]. To the best of our knowledge, explicitly or implicitly, all these results assume that $\lim_{n\to\infty} d_i(n) = 0$. In this paper, we get away with this assumption and establish oscillation criteria for (E) when $\lim_{n\to\infty} d_i(n) = d_i \in (0, 1)$. Thus the results obtained in this paper extend and generalize some of the known results reported in the literature [3, 5, 6, 13, 15, 17]. For related results concerning second-order differential equations with several sub-linear neutral terms, we refer the reader to [4, 7].

2. Main Results. In this section, we derive sufficient conditions for the oscillation of all the solutions of (E). With no loss of generality, we can deal only with the eventually positive solutions of (E).

We begin with the following lemma which can be found in Theorem 41 of [10].

Lemma 2.1. *If a is positive, then*

$$a^{\alpha} \leq \alpha a + (1-\alpha) \quad for \quad 0 < \alpha \leq 1.$$

The equality holds if and only if $a = 1$.

Lemma 2.2. *Let $\{x(n)\}$ be a positive solution of (E). If*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{b(n)} \sum_{t=n}^{\infty} p(t) \right) = \infty \tag{2.1}$$

then the corresponding sequence $\{u(n)\}$ satisfies

(*i*) $u(n) > 0$, $\Delta u(n) > 0$, *and* $\Delta(b(n)(\Delta u(n))^{\beta}) < 0$, $n \geq n_1 \geq n_0$;

(*ii*) $u(n) \to \infty$ *as* $n \to \infty$;

(*iii*) $\dfrac{u(n)}{B(n)}$ *is decreasing where* $B(n) = \sum_{s=n_1}^{n-1} \dfrac{1}{b(s)}$, $n \geq n_1 \geq n_0$.

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (E). Then $u(n) \geq 0$ for all $n \geq n_1 \geq n_0$. It follows from (E) and condition (1.1) that case (i) is true, which implies that

$$u(n) \geq B(n) b(n) \Delta u(n), \ n \geq n_1.$$

Thus, $\left\{ \dfrac{u(n)}{B(n)} \right\}$ is decreasing. We claim that condition (2.1) implies that $u(n) \to \infty$ as $n \to \infty$. Since $\{u(n)\}$ is a positive increasing sequence, there exits a constant $M > 0$ such that

$$\lim_{n \to \infty} u(n) = M > 0. \tag{2.2}$$

Let $\liminf_{n \to \infty} x(n) = c$. Then by Lemma 2.2 of [8], we see that

$$M = c + \sum_{i=1}^{m} d_i c^{\alpha_i}.$$

Since $m \geq 1$ and $d_i > 0$ for $i = 1, 2, \ldots, m$, we claim that $c > 0$. If $c = 0$, then $M = 0$, which contradicts (2.2). Hence there exists an integer $n_1 \geq n_0$ such that

$$x(n-l) \geq \frac{c}{2} > 0, \ n \geq n_1.$$

Using this in (E), we get

$$\Delta(b(n)\Delta u(n)) + p(n)\frac{c^\gamma}{2^\gamma} \le 0. \tag{2.3}$$

Summing up (2.3) from n to ∞, we have

$$\Delta u(n) \ge \frac{c^\gamma}{2^\gamma}\frac{1}{b(n)}\sum_{s=n}^{\infty} p(s).$$

Summing up the above inequality from n_1 to n, we obtain

$$u(n+1) \ge u(n_1) + \frac{b^\gamma}{2^\gamma}\sum_{s=n_1}^{n}\left(\frac{1}{b(s)}\sum_{t=s}^{\infty} p(t)\right),$$

which in view of (2.1) implies that $u(n) \to \infty$ as $n \to \infty$. The proof of the lemma is complete. □

Lemma 2.3. *Let $\{x(n)\}$ be a positive solution of (E) and (2.1) hold. Then*

$$x(n) \ge du(n) \tag{2.4}$$

where $d = \left(1 - \sum_{i=1}^{m} d_i\right) > 0$, for sufficiently large n.

Proof. Let $\{x(n)\}$ be a positive solution of (E). Then there exists an integer $n_1 \ge n_0$ such that $x(n-k_i) > 0$ and $x(n-l) > 0$ for $i = 1, 2, \ldots, m$ for $n \ge n_1$. Therefore, $u(n) > 0$ and from Lemma 2.2 (ii), we see that $u(n) \to \infty$ as $n \to \infty$. From the definition of $u(n)$, we have

$$\begin{aligned} x(n) &= u(n) - \sum_{i=1}^{m} d_i(n)x^{\alpha_i}(n-k_i) \ge u(n) - \sum_{i=1}^{m} d_i u^{\alpha_i}(n) \\ &\ge u(n) - \sum_{i=1}^{m} d_i\left(\alpha_i u(n) + (1-\alpha_i)\right) \\ &\ge u(n) - \sum_{i=1}^{m} d_i\left(\alpha_i + (1-\alpha_i)\right)u(n) = \left(1 - \sum_{i=1}^{m} d_i\right)u(n), \end{aligned}$$

where we have used Lemma 2.1 and $u(n) \ge 1$ for n sufficiently large. Hence

$$x(n) \ge du(n).$$

The proof of the lemma is complete. □

In what follows, we present our first oscillation criterion for the case $\gamma > 1$.

Theorem 2.4. *Assume that* (2.1) *holds and* $\gamma > 1$. *If*

$$\limsup_{n\to\infty}\Bigg\{B^{-\gamma}(n-l)\sum_{s=n_1}^{n-l-1}B(s+1)B^{\gamma}(s-l)p(s)$$

$$+B^{1-\gamma}(n-l)\sum_{n=n-l}^{n-1}p(s)B^{\gamma}(s-l)+B(n-l)\sum_{s=n}^{\infty}p(s)\Bigg\}>0 \tag{2.5}$$

then (E) *is oscillatory.*

P r o o f. Assume that (E) is an eventually positive solution of $\{x(n)\}$. Then there exists an integer $n_1 \geq n_0$ such that $x(n-k_i) > 0$ and $x(n-l) > 0$ for $i = 1, 2, \ldots, m$ for $n \geq n_1$. In what follows $u(n) > 0$ and from Lemma 2.3, we have that

$$x^{\gamma}(n-l) \geq d^{\gamma}u^{\gamma}(n-l),\ n \geq n_1. \tag{2.6}$$

Substituting (2.6) into (E), we have

$$\Delta(b(n)\Delta u(n)) + d^{\gamma}p(n)u^{\gamma}(n-l) \leq 0. \tag{2.7}$$

Summing up of (2.7) yields

$$\begin{aligned}
u(n) &\geq \sum_{s=n_1}^{n-1}\frac{1}{b(s)}\sum_{t=s}^{\infty}p(t)d^{\gamma}u^{\gamma}(t-l)\\
&= \sum_{s=n_1}^{n-1}\frac{1}{b(s)}\sum_{t=s}^{n-1}p(t)d^{\gamma}u^{\gamma}(t-l) + \sum_{s=n_1}^{n-1}\frac{1}{b(s)}\sum_{t=n}^{\infty}p(t)d^{\gamma}u^{\gamma}(t-l)\\
&= \sum_{s=n_1}^{n-1}B(s+1)p(s)d^{\gamma}u^{\gamma}(s-l) + B(n)\sum_{t=n}^{\infty}p(t)d^{\gamma}u^{\gamma}(t-l).
\end{aligned}$$

Therefore

$$\begin{aligned}
u(n-l) &\geq \sum_{s=n_1}^{n-l-1}B(s+1)p(s)d^{\gamma}u^{\gamma}(s-l) + B(n-l)\sum_{t=n-l}^{\infty}p(t)d^{\gamma}u^{\gamma}(t-l)\\
&= \sum_{s=n_1}^{n-l-1}B(s+1)p(s)d^{\gamma}u^{\gamma}(s-l) + B(n-l)\sum_{t=n-l}^{n-1}p(t)d^{\gamma}u^{\gamma}(t-l)\\
&\quad +B(n-l)\sum_{t=n}^{\infty}p(t)d^{\gamma}u^{\gamma}(t-l).
\end{aligned}$$

Since $u(n)$ is increasing and $\dfrac{u(n)}{B(n)}$ is decreasing, we obtain

$$\begin{aligned} u(n-l) \geq \; & \frac{u^{\gamma}(n-l)}{B^{\gamma}(n-l)} \sum_{s=n_1}^{n-l-1} B(s+1)p(s)d^{\gamma}B^{\gamma}(s-l) \\ & + B^{1-\gamma}(n-l)u^{\gamma}(n-l) \sum_{t=n-l}^{n-1} p(t)B^{\gamma}(t-l)d^{\gamma} \\ & + B(n-l)u^{\gamma}(n-l) \sum_{t=n}^{\infty} p(t)d^{\gamma}. \end{aligned}$$

That is,

$$\begin{aligned} \frac{u^{1-\gamma}(n-l)}{d^{\gamma}} \geq \; & B^{-\gamma}(n-l) \sum_{s=n_1}^{n-l-1} B(s+1)B^{\gamma}(s-l)p(s) \\ & + B^{1-\gamma}(n-l) \sum_{t=n-l}^{n-1} p(t)B^{\gamma}(t-l) + B(n-l) \sum_{t=n}^{\infty} p(t). \end{aligned} \tag{2.8}$$

Since $\gamma > 1$ and $u(n) \to \infty$ as $n \to \infty$ taking $\limsup$ as $n \to \infty$ on both sides of the last inequality, we are led to a contradiction with the theorem's assumption. The proof of the theorem is complete. □

The next oscillation result covers the case when $\gamma < 1$.

Theorem 2.5. *Assume that* (2.1) *holds, and* $0 < \gamma < 1$. *If*

$$\sum_{n=n_1}^{\infty} p(n)B^{\gamma}(n-l) = \infty \tag{2.9}$$

and

$$\begin{aligned} \lim_{n\to\infty} \sup \Bigg\{ & \frac{1}{B(n-l)} \sum_{s=n_1}^{n-l-1} B(s+1)p(s)B^{\gamma}(s-l) + \sum_{n=n-l}^{n-1} p(s)B^{\gamma}(s-l) \\ & + B^{\gamma}(n-l) \sum_{s=n}^{\infty} p(s) \Bigg\} > 0 \end{aligned} \tag{2.10}$$

then (*E*) *is oscillatory.*

Proof. Assume that (E) has an eventually positive solution $\{x(n)\}$ for $n \geq n_0$. It is easy to see that $u(n) > 0$ for $n \geq n_1 \geq n_0$. Furthermore $u(n)$ is an increasing sequence and $\frac{u(n)}{B(n)}$ is decreasing. We will claim that (2.9) implies

$$\lim_{n\to\infty} \frac{u(n)}{B(n)} = 0. \tag{2.11}$$

Indeed, assume that

$$\lim_{n\to\infty} \frac{u(n)}{B(n)} = c > 0.$$

Then $\frac{u(n)}{B(n)} \geq \frac{c}{2}$ and so

$$u^\gamma(n-l) \geq \left(\frac{c}{2}\right)^\gamma B^\gamma(n-l).$$

Now using the above inequality in (2.7) and summing up from n_1 to ∞, we obtain

$$b(n_1)\Delta u(n_1) \geq \left(\frac{c}{2}\right)^\gamma d^\gamma \sum_{s=n_1}^{\infty} p(s)B^\gamma(s-l).$$

This contradicts assumption (2.9). Therefore we conclude that (2.11) holds. On the other hand, letting

$$w(n) = \frac{u(n-l)}{B(n-l)}$$

in the inequality (2.8), we get

$$\begin{aligned} w^{1-\gamma}(n) \quad &\geq \quad \frac{1}{B(n-l)} \sum_{s=n_1}^{n-l-1} B(s)p(s)B^\gamma(s-l) \\ &\quad + \sum_{s=n-l}^{n-1} p(s)B^\gamma(s-l) + B^\gamma(n-l)\sum_{s=n}^{\infty} p(s). \end{aligned}$$

Applying $\limsup$ as $n \to \infty$ on both sides of the above inequality we have a contradiction with (2.10). The proof of the theorem is complete. □

For the linear case of (E), we present the following result.

Theorem 2.6. *Assume that* (2.2) *holds and* $\gamma = 1$. *If*

$$\limsup_{n\to\infty}\left\{\frac{1}{B(n-l)}\sum_{s=n_1}^{n-l-1}B(s+1)p(s)B(s-l)+\sum_{n=n-l}^{n-1}p(s)B(s-l)\right.$$
$$\left.+B(n-l)\sum_{s=n}^{\infty}p(s)\right\}>\frac{1}{d} \tag{2.12}$$

then (E) *is oscillatory.*

Proof. Assume that (E) has an eventually positive solution $\{x(n)\}$. Proceeding as in the proof of Theorem 2.4, we are led to (2.8) with $\gamma = 1$. Consequently

$$\frac{1}{d}\geq\frac{1}{B(n-l)}\sum_{s=n_1}^{n-l-1}B(s+1)p(s)B(s-l)+\sum_{n=n-l}^{n-1}p(s)B(s-l)+B(n-l)\sum_{s=n}^{\infty}p(s).$$

Taking lim sup as $n\to\infty$ on both sides of the last inequality, we are led to a contradiction with (2.12). This completes the proof of the theorem. □

3. Examples. In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the second-order difference equation with a couple of sublinear neutral terms

$$\Delta^2\left(x(n)+\frac{1}{2}x^{\frac{1}{3}}(n-1)+\frac{1}{3}x^{\frac{1}{5}}(n-2)\right)+\frac{a}{n(n+1)}x^3(n-2)=0,\ n\geq 1 \tag{3.1}$$

where $a > 0$, and $\gamma = 3$, $l = 2$.

It is easy to verify that

$$B(n)=n-1,\ d=1-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.$$

Therefore, condition (2.1) reduces to

$$\sum_{n=1}^{\infty}\left(\sum_{s=n}^{\infty}\frac{a}{s(s+1)}\right)\approx\sum_{n=1}^{\infty}\frac{a}{n}=\infty.$$

Now condition (2.4) of Theorem 2.4, becomes

$$\limsup_{n\to\infty}\left\{\frac{a}{(n-3)^3}\sum_{s=1}^{n-3}\frac{s(s-3)^3}{s(s+1)}+\frac{a}{(n-3)^2}\sum_{s=n-2}^{n-1}\frac{(s-3)^3}{s(s+1)}\right.$$
$$\left.+a(n-3)\sum_{s=n}^{\infty}\frac{1}{s(s+1)}\right\}=\frac{5a}{3}>0$$

and so we conclude that (3.1) is oscillatory for all $a > 0$.

Example 3.2. Consider the second-order difference equation of the form

$$\Delta\left(x(n)+\frac{1}{4}x^{\frac{1}{3}}(n-1)+\frac{1}{3}x^{\frac{1}{5}}(n-2)\right)+\frac{a}{n^{\frac{4}{3}}}x^{\frac{1}{3}}(n-2)=0,\ n\geq 1 \tag{3.2}$$

where $a > 0$, and $\gamma = \frac{1}{3}$, $l = 2$.

It is easy to verify that $B(n) = n-1$, $d = 1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$. Also conditions (2.1) and (2.9) are satisfied. Furthermore, condition (2.10) of Theorem 2.5, reduces to

$$\limsup_{n\to\infty}\left\{\frac{a}{n-3}\sum_{s=1}^{n-3}s\frac{1}{s^{\frac{4}{3}}}(s-3)^{\frac{1}{3}}+a\sum_{s=n-2}^{n-1}\frac{1}{s^{\frac{4}{3}}}(s-3)^{\frac{1}{3}}+a(n-3)^{\frac{1}{3}}\sum_{s=n}^{\infty}\frac{1}{s^{\frac{4}{3}}}\right\}$$
$$=a+3a=4a>0$$

and so we conclude that (3.2) is oscillatory for all $a > 0$.

Example 3.3. Consider the second-order linear neutral difference equation

$$\Delta\left(x(n)+\sum_{i=1}^{m}d_i x^{\alpha_i}(n-k_i)\right)+\frac{a}{n(n+1)}x(n-2)=0,\ n\geq 1 \tag{3.3}$$

where $0<\alpha_i<1$, $d_i>0$ and $\sum_{i=1}^{m}d_i<1$, k_i is a positive integer and $a>0$.

It is easy to verify that $B(n) = n-1$, $d = 1-\sum_{i=1}^{m}d_i>0$. Also condition (2.1) is satisfied. Furthermore, condition (2.12) of Theorem 2.6, reduces to

$$\lim_{n\to\infty}\sup\left\{\frac{a}{n-3}\sum_{s=1}^{n-3}\frac{s(s-3)}{s(s+1)}+\sum_{s=n-2}^{n-1}\frac{a(s-3)}{s(s+1)}+a(n-3)\sum_{s=n}^{\infty}\frac{1}{s(s+1)}\right\}$$
$$=a+a=2a>\frac{1}{d}.$$

Thus, we conclude that (3.3) is oscillatory, if $ad>\frac{1}{2}$.

Remark 3.4. The existing results reported in [3, 5, 6, 13, 15, 17] cannot be applied to Example 3.3 for $m=1$ since the coefficient of the neutral term does not tend to zero as $n\to\infty$.

4. Conclusion.

In this paper, we have obtained new summation averaging conditions for the oscillation of all the solution of second- order nonlinear difference equations with several sublinear neutral terms. Our criteria improve on the existing literature by generalizing from the condition $d_i(n)\to 0$ as $n\to\infty$ of the existing results to the more general condition that $d_i(n)\to d_i\in(0,1)$. These results, easily, can be extended to apply to the more general nonlinear difference equation

$$\Delta(b(n)(\Delta u(n)))+p(n)f(x(n-l))=0,$$

by employing the condition: $f\in c(-\infty,\infty)$, $uf(u)>0$ for $u\neq 0$ and

$$-f(-uv)\geq f(uv)\geq f(u)f(v)\ \text{ for }\ uv>0.$$

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