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MULTIVARIATE COMPACT LAW OF THE ITERATED LOGARITHM FOR AVERAGED STOCHASTIC APPROXIMATION ALGORITHMS

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Abstract. The aim of this paper is to establish a multivariate compact law of the iterated logarithm for the averaged version of stochastic approximation algorithms. This is achieved by studying the strong convergence rate of a two-time-scale stochastic approximation algorithm, the use of which generalizes the averaging principle of stochastic approximation algorithms. The general result obtained is then applied to the well-known averaged versions of Robbins-Monro's and Kiefer-Wolfowitz's algorithms.

1. Introduction. Approximating the zero of a function is an old problem and a large number of methods have been developed. Among these, the introduction of a random noise was at the origin of the study of the stochastic approximation algorithms with decreasing stepsizes. The first algorithm was introduced by Robbins and Monro [43], and allows the approximation of the zero

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of a function for which observations are available at any level. This pioneering work has been developed and extended in many directions. In particular, the second best-known algorithm is the one introduced by Kiefer and Wolfowitz [15] to approach the maximizer of a regression function which is observable at any level. This algorithm, as well as its generalization to the multivariate framework made by Blum [6], can be seen as an algorithm searching the zero of the gradient of the regression function. However, it is quite different from Robbins-Monro's algorithm, since only the regression function is observable, and not its gradient, which thus must be approximated at each step. Even though both algorithms behave quite differently, their study can be unified by considering the general stochastic algorithm approximating the zero z^* of an unknown function $h : \mathbb{R}^d \to \mathbb{R}^d$, defined by choosing an initial $\mathbb{R}^d$-valued random vector (Z_0) and by setting, for all $n \geq 1$,

$$Z_n = Z_{n-1} + \gamma_n W_n(c_n), \tag{1}$$

where the stepsize (γ_n) is a positive sequence going to zero, and where $W_n(c_n)$ is an observation or an approximation of the function h at the point Z_{n-1}. More precisely,

$$W_n(c_n) = h\left(Z_{n-1}\right) + R_n + c_n^{-1}\varepsilon_n,$$

where (c_n) is either a positive sequence which varies regularly with exponent $-\tau$, $\tau > 0$, or $(c_n) = 1$ (in which case we set $\tau = 0$). Moreover, (R_n) and (ε_n) are two sequences of d-dimensional random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$; (R_n) is a perturbation, and (ε_n) a noise satisfying $\mathbb{E}\left(\varepsilon_n|\mathcal{F}_{n-1}\right) = 0$ a.s. and $\lim_{n\to\infty} \mathbb{E}(\varepsilon_n\varepsilon_n^T|\mathcal{F}_{n-1}) = \Gamma$ a.s. for a positive definite matrix Γ. In the case of Robbins-Monro's algorithm, $R_n = 0$, $c_n = 1$, and ε_n is the noise induced by the observation of $h\left(Z_{n-1}\right)$. In the case of Kiefer-Wolfowitz's algorithm, (c_n) is the sequence introduced to approximate the gradient of the regression function, R_n is the error term due to the approximation of the gradient with the help of the regression function itself, and ε_n is the noise induced by the observations of the regression function made to approximate its gradient.

In addition to including Robbins-Monro's and Kiefer–Wolfowitz's algorithms, the algorithm (1) includes stochastic approximation algorithms with Markovian disturbances: we refer to Benvéniste et al. [1] and Duflo [10] for an overview. Such algorithms have found applications in diverse areas: learning, signal processing and adaptative control (see Kushner and Yin [24] for an exhaustive presentation) and in nonparametric statistics issues (see Duflo [11] and Nevels'on and Has'minskii [36]).

The asymptotic behaviour of the algorithm (1) was extensively studied. Criteria ensuring the almost sure convergence of (Z_n) toward z^* can be found in particular in Nevels'on and Has'minskii [36], Kushner and Clark [22], Hall and Heyde [14], Benvéniste et al. [1], Duflo [10], Ljung et al. [28], Kushner and Yin [24], and in some references therein. The study of the weak convergence rate of the algorithm was also widely studied by the former authors, and it is now well-known that, under suitable assumptions, (Z_n) satisfies the central limit theorem

$$\sqrt{\gamma_n^{-1} c_n^2}\,(Z_n - z^*) \ \stackrel{\mathcal{D}}{\to}\ \mathcal{N}(\mu\ ,\ \Sigma)\,, \tag{2}$$

where $\stackrel{\mathcal{D}}{\to}$ denotes the weak convergence, where the $\mathbb{R}^d$-valued vector μ may be zero, and where the covariance matrix Σ depends on Γ and on the differential matrix H of the function h at the unknown point z^*. The strong convergence rate of the algorithm (1) was studied by many authors; let us cite, among many others, Lai and Robbins [25], Gaposhkin and Krasulina [13], Ruppert [44], Koval [19], Pelletier [38], and Koval and Schwabe [20, 21]. In the case when (2) holds with $\mu = 0$, Mokkadem and Pelletier [30] proved that, with probability one, the sequence $(\sqrt{\gamma_n^{-1} c_n^2/(2\ln\ln n)}\,(Z_n - z^*))$ is relatively compact and its limit set is the ellipsoid $\{u \in \mathbb{R}^d\ ;\ u^T\Sigma^{-1}u \leq 1\}$.

Throughout this paper, we shall say that (Z_n) reaches *the optimal weak convergence rate* if (Z_n) satisfies the central limit theorem (2) with the maximum convergence rate and with the minimal asymptotic covariance matrix; in the particular case when $\mu = 0$, this coincides with the asymptotic efficiency of (Z_n). Similarly, we shall say that (Z_n) reaches *the optimal strong convergence rate* if (Z_n) satisfies the law of the iterated logarithm with the maximum convergence rate and with the smallest limit set. Now, for (Z_n) to reach the optimal weak convergence rate, the stepsize (γ_n) must be matricial and depend on the differential of h at the point z^*; this differential being usually unknown, this choice of stepsize is, most of the time, impossible.

In order to construct algorithms reaching the optimal weak convergence rate, Ruppert [45] and Polyak [41] independently introduced the averaging principle. They considered the case when (1) is Robbins-Monro's algorithm (in which case $(R_n) = 0$ and $(c_n) = 1$). Their procedure consists in constructing an averaged algorithm $(\overline{Z}_n)$ in two steps. First, the algorithm (1) is executed with a stepsize satisfying $\lim_{n\to\infty} n\gamma_n = \infty$ (in which case (Z_n) does not converge at the optimal rate). Then, $\overline{Z}_n$ is defined as the arithmetic average of the Z_k, *i.e.*

$\overline{Z}_n = n^{-1}\sum_{k=1}^{n} Z_k$. The averaged algorithm $(\overline{Z}_n)$ was widely studied, and the averaging principle extended to other algorithms than Robbins-Monro's (see, among many others Yin [47], Delyon and Juditsky [8], Polyak and Juditsky [42], Kushner and Yang [23], Dippon and Renz [9], and Pechtl [37]). Dippon and Renz [9] showed in particular that for the averaged Kiefer-Wolfowitz's algorithm to reach the optimal weak convergence rate, the average of the Z_k must not be arithmetic any more, but weighted by the sequence (c_k^2).

More recently, Mokkadem and Pelletier [34] gave a generalization of the averaging procedure by using a two-time-scale stochastic approximation algorithm. Two-time-scale stochastic approximation algorithms were introduced, and then studied, by Borkar [5], Konda and Borkar [16], Baras and Borkar [2], Bhatnagar *et al.* [3, 4], Konda and Tsitsiklis [17], Mokkadem and Pelletier [30], and are composed by two algorithms, whose stepsizes converge to zero with different rates. To generalize the averaging of the algorithm (1), Mokkadem and Pelletier [34] defined the two-time-scale stochastic approximation algorithm by choosing an initial $\mathbb{R}^{2d}$-valued random vector (Z_0, Y_0) and by setting, for all $n \geq 1$,

$$Z_n = Z_{n-1} + \gamma_n W_n(c_n), \tag{3}$$
$$Y_n = (1-\beta_n)\, Y_{n-1} + \beta_n Z_n, \tag{4}$$

where the stepsizes (γ_n) and (β_n) are chosen such that $\lim_{n\to\infty} n\gamma_n = \infty$ and $\lim_{n\to\infty} n\beta_n = 1-2\tau$. They proved in particular that, under suitable conditions,

$$\sqrt{nc_n^2}\,(Y_n - z^*) \;\stackrel{\mathcal{D}}{\to}\; \mathcal{N}\left(\mu'\ ,\ (1-2\tau)H^{-1}\Gamma[H^{-1}]^T\right), \tag{5}$$

where the $\mathbb{R}^d$-valued vector μ' may be zero, and where H is the differential of the function h at the point z^*. Since the covariance matrix $(1-2\tau)H^{-1}\Gamma[H^{-1}]^T$ is minimum, (Y_n) reaches the optimal weak convergence rate when (c_n) is chosen such as to make the convergence rate $\sqrt{nc_n^2}$ maximum.

The aim of this paper is to establish a multivariate compact law of the iterated logarithm for the sequence (Y_n) defined by the two-time-scale stochastic approximation algorithm (3)–(4). We prove in particular that, under suitable assumptions, with probability one, the sequence

$$(\sqrt{nc_n^2/(2\ln\ln n)}\,(Y_n - z^*))$$

is relatively compact and its limit set is the ellipsoid

$$\mathcal{E} = \left\{ v \in \mathbb{R}^d\ ;\ (1-2\tau)^{-1}\left(v-\mu'\right)^T H^T\Gamma^{-1}H\left(v-\mu'\right) \leq 1 \right\}.$$

The fact that the covariance matrix $(1-2\tau)H^{-1}\Gamma[H^{-1}]^T$ is minimum guarantees that the ellipsoid $\mathcal{E}$ is the smallest possible. (Y_n) then reaches the optimal strong convergence rate when (c_n) is chosen such as to make the convergence rate $\sqrt{nc_n^2/(2\ln\ln n)}$ maximal, and thus for a sequence (c_n), which may be slightly different from that making (Y_n) reach the optimal weak convergence rate.

To conclude this introduction, let us note that, if the stepsize (β_n) in (4) equals (n^{-1}), then Y_n equals the arithmetic average of the Z_k. On the other hand, if (β_n) equals $(c_n^2[\sum_{k=1}^{n} c_k^2]^{-1})$ and if Y_0 is set equal to zero, then Y_n equals the (c_k^2)-average of the Z_k. As a by-product of our result, we thus obtain multivariate compact laws of the iterated logarithm for the averaged Robbins-Monro's and Kiefer-Wolfowitz's algorithms.

The remainder of the paper is organized as follows. Our assumptions and compact law of the iterated logarithm for the two-time-scale stochastic approximation algorithm (3)&(4) are stated in Section 2. Applications to the averaged Robbins-Monro's and Kiefer-Wolfowitz's algorithms are given in Section 3. Section 4 is devoted to the proofs.

2. Assumptions and main result. Throughout this paper, we denote by A^T the transpose of a matrix A. Moreover, we recall that a nonrandom positive sequence $(v_n)_{n\geq 1}$ belongs to $\mathcal{GS}(\gamma)$, $\gamma\in\mathbb{R}$, if

$$\lim_{n\to+\infty} n\left[1-\frac{v_{n-1}}{v_n}\right]=\gamma. \tag{6}$$

Condition (6) was introduced by Galambos and Seneta [12] (see also Bojanic and Seneta [7]); it was used by Mokkadem and Pelletier [32] in the context of stochastic approximation algorithms. Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b\in\mathbb{R}$, $n^\gamma(\log n)^b$, $n^\gamma(\log\log n)^b$, and so on.

Our assumptions on the two-time-scale algorithm (3)&(4) are the following. These assumptions will be discussed through the applications given in Section 3.

(A1) $\lim_{n\to\infty} Z_n = z^*$ a.s.

(A2) (i) There exist $\eta>1$ and a neighbourhood of z^* on which

$$h(z)=H(z-z^*)+O(\|z-z^*\|^\eta).$$

(ii) The largest real part of the eigenvalues of H is negative.

(A3) $(c_n) \in \mathcal{GS}(-\tau)$ with $\tau \in]0, 1/2[$ or $(c_n) = 1$, in which case we set $\tau = 0$.

(A4) (i) $(\gamma_n) \in \mathcal{GS}(-\alpha)$ with $\alpha \in]1/2, 1]$.

(ii) $\lim_{n\to\infty} n\gamma_n [\log(\sum_{k=1}^{n} \gamma_k)]^{-1} = \infty$.

(iii) $\lim_{n\to\infty} n\gamma_n^{\eta} c_n^{2(1-\eta)} [\log(\sum_{k=1}^{n} \gamma_k)]^{\eta} = 0$.

(A5) One of the following conditions is satisfied.

(C1) There exist $\rho \in \mathbb{R}^d$, $\rho \neq 0$, and $(v_n) \in \mathcal{GS}(v^*)$, $v^* > 0$, such that $\lim_{n\to\infty} v_n R_n = \rho$ a.s.

(C2) $\lim_{n\to\infty} \sqrt{nc_n^2 (\ln\ln n)^{-1}} R_n = 0$ a.s., in which case we set $\rho = 0$ and $v_n = \sqrt{nc_n^2 (\ln\ln n)^{-1}}$.

(A6) (i) $\mathbb{E}\left(\varepsilon_{n+1} | \mathcal{F}_n\right) = 0$.

(ii) There exists a nonrandom, positive definite matrix Γ such that $\lim_{n\to\infty} \mathbb{E}(\varepsilon_{n+1}\varepsilon_{n+1}^T | \mathcal{F}_n) = \Gamma$ a.s.

(iii) There exists $m > 2/\alpha$ such that $\sup_{n\geq 0} \mathbb{E}\left(\|\varepsilon_{n+1}\|^m\right) < \infty$.

(A7) $(\beta_n) \in \mathcal{GS}(-1)$ and $\lim_{n\to\infty} n\beta_n = 1 - 2\tau$.

We can now state our main result.

Theorem 1. *Let (Y_n) be defined by the two-time-scale stochastic approximation algorithm* (3)&(4)*, and assume that* (A1)–(A7) *hold.*

1) *If $\lim_{n\to\infty} v_n^{-2} nc_n^2 / \ln\ln n = \infty$, then,*

$$v_n \left(Y_n - z^*\right) \overset{a.s.}{\to} \frac{-(1-2\tau)}{1-2\tau-v^*} H^{-1}\rho. \tag{7}$$

2) *If $\lim_{n\to\infty} v_n^{-2} nc_n^2 / \ln\ln n = 0$, then, with probability one, the sequence*

$$\left(\sqrt{\frac{nc_n^2}{2\ln\ln n}} \left(Y_n - z^*\right)\right)$$

is relatively compact and its limit set is

$$\left\{w \in \mathbb{R}^d \ ; \ (1-2\tau)^{-1} w^T H^T \Gamma^{-1} H w \leq 1\right\}.$$

3) *If there exists $c > 0$ such that* $\lim_{n\to\infty} v_n^{-2} n c_n^2 / \ln\ln n = c$, *then, with probability one, the sequence*

$$\left(\sqrt{\frac{nc_n^2}{2\ln\ln n}}\,(Y_n - z^*)\right)$$

is relatively compact and its limit set is

$$\left\{ w \in \mathbb{R}^d \;;\; (1-2\tau)^{-1}\left(w + \sqrt{2c}H^{-1}\rho\right)^T H^T \Gamma^{-1} H \left(w + \sqrt{2c}H^{-1}\rho\right) \leq 1 \right\},$$

and (Y_n) *reaches the optimal strong convergence rate.*

Remark 1. Since $(v_n^{-2} n c_n^2) \in \mathcal{GS}(-2v^* + [1-2\tau])$, the condition $\lim_{n\to\infty} v_n^{-2} n c_n^2/\ln\ln n = \infty$ in Part 1 of Theorem 1 implies that $-2v^* + [1-2\tau] \geq 0$. It thus follows from (A3) that $-v^* + [1-2\tau] > 0$, which ensures that the limit in (7) is well defined.

The multivariate compact law of the iterated logarithm given in Parts 2 and 3 of Theorem 1 implies laws of the iterated logarithm for the l^p-norms of the sequence $(Y_n - z^*)$ suitably normalized. For any vector $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$, set $\|x\|_p = [\sum_{i=1}^d |x_i|^p]^{1/p}$ for $p \in [1, +\infty[$ and $\|x\|_\infty = \max_{1\leq i\leq d} |x_i|$. Moreover, for any matrix A and any $p \in [1, +\infty]$, let $|||A|||_{2,p}$ denote the matricial norm defined by $|||A|||_{2,p} = \sup_{\|x\|_2 \leq 1} \|Ax\|_p$. Following the proof of Corollary 1 in Mokkadem and Pelletier [30], we obtain the following.

Corollary 1. *Let* (Y_n) *be defined by the two-time-scale stochastic approximation algorithm* (3)&(4), *and assume that* (A1)–(A7) *hold. If there exists* $c \geq 0$ *such that* $\lim_{n\to\infty} v_n^{-2} n c_n^2/\ln\ln n = c$, *then for any* $p \in [1, +\infty]$, *with probability one, the sequence of the* l^p*-norms,*

$$\left(\left\|\sqrt{\frac{nc_n^2}{2\ln\ln n}}\,(Y_n - z^*) + \sqrt{2c}H^{-1}\rho\right\|_p\right)$$

is relatively compact and its limit set is the interval $\left[0, \sqrt{1-2\tau}\delta_p\right]$ *with* $\delta_p = |||([H^{-1}]\Gamma[H^{-1}]^T)^{1/2}|||_{2,p}$.

In particular, δ_2 is the spectral norm of the matrix $([H^{-1}]\Gamma[H^{-1}]^T)^{1/2}$.

3. Applications to the averaged Robbins–Monro's and Kiefer–Wolfowitz's algorithms.

3.1. Averaged Robbins–Monro's algorithm. As said in the introduction, Robbins–Monro's algorithm allows the approximation of the zero z^* of a function $f : \mathbb{R}^d \to \mathbb{R}^d$ for which observations $W(x)$ of $f(x)$ are available at any level x. It is defined by setting $Z_0 \in \mathbb{R}^d$, and, for $n \geq 1$,

$$Z_n = Z_{n-1} + \gamma_n W(Z_{n-1}). \tag{8}$$

Ruppert [45] and Polyak [41] introduced the averaged version of Robbins-Monro's algorithm by setting $\overline{Z}_n = n^{-1} \sum_{k=1}^{n} Z_k$, and proved its asymptotic efficiency. Since $(\overline{Z}_n)$ can be rewritten as

$$\overline{Z}_n = \left(1 - \frac{1}{n}\right) \overline{Z}_{n-1} + \frac{1}{n} Z_n, \tag{9}$$

the averaged Robbins-Monro's algorithm $(\overline{Z}_n)$ can be viewed as defined by the two-time-scale stochastic approximation algorithm (8)&(9). Let us consider the following assumptions.

(H1) $\lim_{n\to\infty} Z_n = z^*$ a.s.

(H2) There exist $\eta > 1$ and a neighbourhood of z^* on which $f(z) = \Phi(z - z^*) + O(\|z - z^*\|^\eta)$, and the largest real part of the eigenvalues of Φ is negative.

(H3) (i) $(\gamma_n) \in \mathcal{GS}(-\alpha)$, $\alpha \in]\max\{1/2, 2/m\}, 1]$.

(ii) $\lim_{n\to\infty} n\gamma_n[\log(\sum_{k=1}^{n} \gamma_k)]^{-1} = \infty$.

(iii) $\lim_{n\to\infty} n\gamma_n^\eta[\log(\sum_{k=1}^{n} \gamma_k)]^\eta = 0$.

(H4) For some positive definite matrix $\Gamma > 0$, $\lim_{x\to z^*} (W(x)[W(x)]^T) = \Gamma$, while, for some $m > 2/\alpha$, $\sup_{x\in\mathbb{R}^d} \mathbb{E}(\|W(x)\|^m) < \infty$.

Equation (8) can be rewritten as $Z_n = Z_{n-1} + \gamma_n [f(Z_{n-1}) + \varepsilon_n]$ with $\varepsilon_n = Y(Z_{n-1}) - f(Z_{n-1})$, and the sequence $(\overline{Z}_n)$ defined by the two-time-scale stochastic approximation algorithm (8)&(9) satisfies the assumptions of Theorem 1. As a matter of fact, (H1) and (H2) ensure that (A1) and (A2) hold with $h = f$ and $H = \Phi$; (A3) holds with $(c_n) = 1$ (and thus $\tau = 0$), Condition (C2) in (A5) is fulfilled. (A4) and (A6) are ensured by (H3) and (H4), respectively, whereas the sequence $(\beta_n) = (n^{-1})$ clearly satisfies (A7). The following corollary thus follows from the application of Part 3 of Theorem 1.

Corollary 2. *Let $(\overline{Z}_n)$ be defined by the two-time-scale stochastic approximation algorithm* (8)&(9). *Under* (H1)–(H4), *with probability one, the sequence $\left(\sqrt{n/(2\ln\ln n)}\left(\overline{Z}_n - z^*\right)\right)$ is relatively compact, its limit set is the ellipsoid $\left\{w \in \mathbb{R}^d \ ; \ w^T\Phi^T\Gamma^{-1}\Phi w \leq 1\right\}$, and $(\overline{Z}_n)$ reaches the optimal strong convergence rate.*

It is worth noting that the averaged Robbins–Monro's algorithm simultaneously reaches the optimal weak and strong convergence rates. Let us also mention that Corollary 2 is an extension law of the iterated logarithm

$$\begin{aligned}\limsup_{n\to\infty}\sqrt{\frac{n}{2\ln\ln n}}w^T\left(\overline{Z}_n - z^*\right) &= -\liminf_{n\to\infty}\sqrt{\frac{n}{2\ln\ln n}}w^T\left(\overline{Z}_n - z^*\right)\\ &= w^T[\Phi^{-1}]\Gamma[\Phi^{-1}]^T w \ a.s.,\end{aligned}$$

established in Pelletier [39] for all vectors w of $\mathbb{R}^d$.

3.2. Averaged Kiefer–Wolfowitz's algorithm. Consider two random variables X and Y with values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively, and unknown common distribution $\mathbb{P}_{X,Y}$. Assume that the regression function $f(.) = \mathbb{E}(Y|X = .) : \mathbb{R}^d \Longrightarrow \mathbb{R}$ exists, is sufficiently smooth, and has a unique maximizer $\theta \in \mathbb{R}^d$, and assume that observations $Y(x)$ of $f(x)$ are available at any level x. Kiefer and Wolfowitz [15] (in the case $d = 1$) and Blum [6] (in the case $d \geq 1$) introduced the following algorithm searching the zero of the gradient ∇f of f to recursively approximate θ:

$$Z_n = Z_{n-1} + \gamma_n W_n, \tag{10}$$

where W_n is a (random) approximation of $\nabla f(Z_{n-1})$. More precisely, let (c_n) be a positive nonrandom sequence that goes to zero, and let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{R}^d$; the approximation W_n is the d-dimensional vector defined as

$$W_n = \frac{1}{2c_n}\{Y\left(Z_{n-1} + c_n e_i\right) - Y\left(Z_{n-1} - c_n e_i\right)\}_{i\in\{1,\ldots,d\}}.$$

Kiefer–Wolfowitz's algorithm was widely studied, as well as its averaged version, which is defined as

$$\overline{Z}_n = \frac{1}{\sum_{k=1}^n c_k^2} \sum_{k=1}^n c_k^2 Z_k, \tag{11}$$

and which reaches the optimal weak convergence rate (see, for instance, Dippon and Renz [9]).

We introduce the two-time-scale stochastic approximation algorithm defined as

$$Z_n = Z_{n-1} + \gamma_n W_n \tag{12}$$

$$\theta_n = (1-\beta_n)\theta_{n-1} + \beta_n Z_n, \tag{13}$$

and give the strong convergence rate of (θ_n) under the following assumptions.

(H1) $\lim_{n\to\infty} Z_n = \theta$ a.s.

(H2) f is three-times continuously differentiable in a neighborhood of θ, and the Hessian $D^2 f(\theta)$ of f at θ is negative definite.

(H3) $(c_n) \in \mathcal{GS}(-\tau)$, $0 < \tau < 1/2$.

(H4) (i) $(\gamma_n) \in \mathcal{GS}(-\alpha)$, $\alpha \in]1/2, 1]$.

(ii) $\lim_{n\to\infty} n\gamma_n [\log(\sum_{k=1}^n \gamma_k)]^{-1} = \infty$.

(iii) $\lim_{n\to\infty} n\gamma_n^2 c_n^{-2} [\log(\sum_{k=1}^n \gamma_k)]^2 = 0$.

(H5) Set $\varepsilon_{n,i}^+ = Y(Z_{n-1} + c_n e_i) - f(Z_{n-1} + c_n e_i)$, $\varepsilon_{n,j}^- = Y(Z_{n-1} - c_n e_j) - f(Z_{n-1} - c_n e_j)$, and let $\mathcal{F}_n$ be the σ-field spanned by $\{\varepsilon_{m,i}^+, \varepsilon_{p,j}^-,\ 1 \le i,j \le d,\ 1 \le m,p \le n-1\}$.

(i) $\varepsilon_{n,i}^+$ and $\varepsilon_{n,j}^-$ $(i,j \in \{1,\dots,d\})$ are independent conditionally on $\mathcal{F}_n$.

(ii) For some $\sigma > 0$, $\mathbb{V}(Y|X=x) = \sigma^2$ for all $x \in \mathbb{R}^d$, while, for some $m > 2/\alpha$, $\sup_{x\in\mathbb{R}^d} \mathbb{E}(|Y|^m | X = x) < \infty$.

(H6) $(\beta_n) \in \mathcal{GS}(-1)$ and $\lim_{n\to\infty} n\beta_n = 1 - 2\tau$.

Remark 2. Theorem 3 in Blum [6] ensures that (H1) holds under (H2)-(H5) and the following additional conditions: (i) $\alpha + \tau > 1$ and $2(\alpha - \tau) > 1$; (ii) $D^2 f$ is bounded; (iii) $\forall \delta > 0, \sup_{\|x-\theta\| \geq \delta} f(x) < f(\theta)$; (iv) $\forall \varepsilon > 0, \exists \rho(\varepsilon) > 0$ such that $\|x - \theta\| \geq \varepsilon \Rightarrow \|\nabla f(x)\| \geq \rho(\varepsilon)$. Similar conditions, but which are less restrictive on α and τ, can be found in Ljung [27] and Hall and Heyde [14]. Another kind of conditions with particular emphasis on control theory applications is given in Ljung [26], Kushner and Clark [22], and Duflo [10]. The approach in these three references is to associate the approximation algorithm (10) with a deterministic differential equation in terms of which conditions are given to ensure (H1).

Equation (12) can be rewritten as $Z_n = Z_{n-1} + \gamma_n \left[\nabla f(Z_{n-1}) + R_n + c_n^{-1}\varepsilon_n\right]$ with $R_n = (2c_n)^{-1}\{f(Z_{n-1} + c_n e_i) - f(Z_{n-1} - c_n e_i)\}_{i\in\{1,\ldots,d\}} - \nabla f(Z_{n-1})$ and $\varepsilon_n = \frac{1}{2}\left(\varepsilon_n^+ - \varepsilon_n^-\right)$, and the sequence (θ_n) be defined by the two-time-scale stochastic approximation algorithm (12)&(13) satisfies the assumptions of Theorem 1. As a matter of fact, (H1)-(H4) ensure that (A1)-(A4) hold with $h = \nabla f$, $H = D^2 f(\theta)$, and $\eta = 2$, and (A1) and (A2) guarantee that the i-th coordinate of R_n satisfies $R_{n,i} = \frac{c_n^2}{6}\frac{\partial^3 f}{\partial x_i^3}(\theta) + o\left(c_n^2\right)$, so that Condition (C1) in (A5) holds with $(v_n) = (c_n^{-2})$ (and thus $v^* = 2\tau$) and $\rho = \frac{1}{6}[D^2 f(\theta)]^{-1}\left\{\frac{\partial^3 f}{\partial x_i^3}(\theta)\right\}_{1\leq i\leq d}$. Moreover, (H5) ensures that (A6) is satisfied with $\Gamma = \frac{\sigma^2}{2}I$, and (A7) follows from (H6). The following corollary is thus a straightforward consequence of the application of Theorem 1.

Corollary 3. *Let (θ_n) be defined by the two-time-scale stochastic approximation algorithm* (12)&(13)*, and assume* (H1)–(H6) *hold.*

1) *If* $\lim_{n\to\infty} nc_n^6/\ln\ln n = \infty$*, then*

$$c_n^{-2}(\theta_n - \theta) \overset{a.s.}{\to} \mathcal{M}(\tau,\theta) = -\frac{1}{6}\left(\frac{1-2\tau}{1-4\tau}\right)[D^2 f(\theta)]^{-1}\left\{\frac{\partial^3 f}{\partial x_i^3}(\theta)\right\}_{1\leq i\leq d}.$$

2) *If* $\lim_{n\to\infty} nc_n^6/\ln\ln n = 0$*, then, with probability one, the sequence*

$$\left(\sqrt{\frac{nc_n^2}{2\ln\ln n}}\,(\theta_n - \theta)\right)$$

is relatively compact and its limit set is

$$\left\{w \in \mathbb{R}^d \ ; \ 2(1-2\tau)^{-1}\sigma^{-2}w^T[D^2f(\theta)]^2 w \leq 1\right\}.$$

3) *If there exists $c > 0$ such that $\lim_{n\to\infty} nc_n^6/\ln\ln n = c$, then, with probability one, the sequence*

$$\left(\sqrt{\frac{nc_n^2}{2\ln\ln n}}\,(\theta_n - \theta)\right)$$

is relatively compact, its limit set is

$$\left\{w \in \mathbb{R}^d \ ; \ 3\sigma^{-2}\left(w+\mu(c)\right)^T \left[D^2 f(\theta)\right]^2 \left(w+\mu(c)\right) \leq 1\right\}$$

with

$$\mu(c) = \frac{\sqrt{2c}}{6}[D^2 f(\theta)]^{-1}\left\{\frac{\partial^3 f}{\partial x_i^3}(\theta)\right\}_{1\leq i\leq d},$$

and (θ_n) reaches the optimal strong convergence rate.

Let us first note that, if (β_n) is chosen equal to $\left(c_n^2[\sum_{k=1}^n c_k^2]^{-1}\right)$ and if θ_0 is set equal to zero, then θ_n equals the weighted averaged $\overline{Z}_n$ defined in (11). Moreover, Assumption (H3) on (c_n) ensures that (H6) holds (*see* Lemma 1 in Mokkadem *et al.* [33]). Thus, Corollary 3 gives in particular the strong convergence rate of the averaged Kiefer–Wolfowitz's algorithm. It also ensures that if (β_n) is chosen equal to $((1-2\tau)n^{-1})$, then the sequence (θ_n) defined by the two-time-scale stochastic approximation algorithm (12)&(13) shares the same strong asymptotic behaviour as the averaged Kiefer–Wolfowitz's algorithm, while having a faster update than $(\overline{Z}_n)$.

Unlike the averaged Robbins-Monro's algorithm, the averaged Kiefer–Wolfowitz's algorithm does not simultaneously reach the optimal weak and strong convergence rates. As a matter of fact, for (θ_n) to reach the optimal weak convergence rate, the sequence (c_n) must be chosen such that $\lim_{n\to\infty} nc_n^6 = c > 0$, in which case Part 2 of Corollary 3 holds. On the other hand, when (c_n) is chosen such as to make (θ_n) reach the optimal strong convergence rate, the asymptotic weak behaviour of (θ_n) is degenerated, since, in this case, the sequence $(c_n^{-2}(\theta_n - \theta))$ converges to $\mathcal{M}\left(\frac{1}{6}, \theta\right)$ in probability.

4. Proof of Theorem 1. Without loss of generality, we assume that $\beta_n < 1$ for all $n \geq 1$, and set $Q_n = \prod_{j=1}^{n}(1-\beta_j)$. Following the proof of Theorem 1 in Mokkadem and Pelletier [34], we note that

$$Y_n - z^* \quad = \quad -\Phi_{n+1} - B_{n+1} - \sum_{i=1}^{3} \mathcal{R}_{n+1}^{(i)} \tag{14}$$

with:

$$\begin{aligned}
\Phi_{n+1} &= Q_n \sum_{k=1}^{n} Q_k^{-1}\beta_k c_{k+1}^{-1} H^{-1}\varepsilon_{k+1}, \\
B_{n+1} &= Q_n \sum_{k=1}^{n} Q_k^{-1}\beta_k H^{-1} R_{k+1}, \\
\mathcal{R}_{n+1}^{(1)} &= Q_n \sum_{k=1}^{n} Q_k^{-1}\beta_k \gamma_{k+1}^{-1} H^{-1}\left[Z_k - Z_{k+1}\right], \\
\mathcal{R}_{n+1}^{(2)} &= Q_n \sum_{k=1}^{n} Q_k^{-1}\beta_k O\left(\|Z_{k+1} - z^*\|^{\eta}\right), \\
\mathcal{R}_{n+1}^{(3)} &= -Q_n(Z_0 - z^*).
\end{aligned}$$

The three following lemmas give the asymptotic strong behaviour of the sequences (Φ_n), (B_n), and $(\mathcal{R}_n^{(i)})$ for $i \in \{1,2,3\}$, respectively.

Lemma 1. *Under Assumptions* (A1)–(A7), *for any* $u \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{\sqrt{nc_n^2}}{\sqrt{2\ln\ln n}} u^T \Phi_{n+1} = \sqrt{(1-2\tau)u^T H^{-1}\Gamma[H^{-1}]^T u} \ \ a.s.$$

Lemma 2. *Let* (A1)–(A7) *hold.*

1) *If* $\lim_{n\to\infty} v_n^{-2} nc_n^2/\ln\ln n \in]0,\infty]$, *then* $\lim_{n\to\infty} v_n B_{n+1} = \dfrac{1-2\tau}{1-2\tau-v^*} H^{-1}\rho$ *a.s.*

2) *If* $\lim_{n\to\infty} v_n^{-2} nc_n^2/\ln\ln n = 0$, *then* $\lim_{n\to\infty} \sqrt{nc_n^2} B_{n+1} = 0$ *a.s.*

Lemma 3. *Let* (A1)–(A7) *hold. For* $i \in \{1,2,3\}$,

$$\left\|\mathcal{R}_{n+1}^{(i)}\right\| = o\left(\max\left\{\sqrt{n^{-1}c_n^{-2}\ln\ln n}; v_n^{-1}\right\}\right) \ \ a.s.$$

The proof of Lemmas 2 and 3 follows step by step that of Lemmas 2 and 3 in Mokkadem and Pelletier [34], and is thus omited. In Section 4.1, we show how Theorem 1 can be deduced from the combination of Lemmas 1, 2, and 3. Section 4.2 is devoted to the proof of Lemma 1.

4.1. Proof of Theorem 1.

• Let us first consider the case when $\lim_{n\to\infty} v_n^{-2} n c_n^2 / \ln\ln n = \infty$. Since Lemma 1 implies that $\Phi_{n+1} = O\left(\sqrt{n^{-1}c_n^{-2}\ln\ln n}\right)$ a.s., the first part of Theorem 1 is a straigthforward consequence of the combination of (14) and of Lemmas 1, 2, and 3.

• Let us now assume that there exists $c \geq 0$ such that $\lim_{\to\infty} v_n^{-2} n c_n^2 / \ln\ln n = c$. We set $\Sigma = (1-2\tau)H^{-1}\Gamma[H^{-1}]^T$, and note that Lemma 1 implies that, for all $w \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{\sqrt{nc_n^2}}{\sqrt{2\ln\ln n}} w^T \Sigma^{-1/2} \Phi_{n+1} = \sqrt{w^T w} \ a.s.$$

In view of (14), and since $1-2\tau = 2v^*$ in the case when $c \neq 0$, the application of Lemmas 2 and 3 yields, for all $w \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{\sqrt{nc_n^2}}{\sqrt{2\ln\ln n}} w^T \Sigma^{-1/2}\left(Y_n - z^*\right) = \sqrt{w^T w} - \sqrt{2c} w^T \Sigma^{-1/2} H^{-1} \rho \ a.s.$$

Now, set

$$W_n = \Sigma^{-1/2}\left[\frac{\sqrt{nc_n^2}}{\sqrt{2\ln\ln n}}\left(Y_n - z^*\right) + \sqrt{2c} H^{-1}\rho\right],$$

so that, for all $w \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} w^T W_n = \sqrt{w^T w} \ a.s. \tag{15}$$

Set $\mathcal{S}_d(0,1) = \{w \in \mathbb{R}^d, \ \|w\|_2 = 1\}$ and $\overline{\mathcal{B}}_d(0,1) = \{w \in \mathbb{R}^d, \ \|w\|_2 \leq 1\}$. Following the proof of Theorem 1 in Mokkadem and Pelletier [30], we get the following claim.

Claim 1. *The univariate law of the iterated logarithm* (15) *for all inner products implies that:*

- *With probability one, the sequence* (W_n) *is relatively compact and its limit set* $\mathcal{U}$ *is included in* $\overline{\mathcal{B}}_d(0,1)$*.*
- $\mathcal{S}_d(0,1)$ *is included in* $\mathcal{U}$*.*

Let (z_n, y_n) be defined by setting $(z_1, y_1) \in \mathbb{R}^2$ and, for $n \geq 1$,

$$\begin{aligned} z_n &= z_{n-1} - \gamma_n z_{n-1} + \gamma_n c_n^{-1} e_n \\ y_n &= (1-\beta_n)\, y_{n-1} + \beta_n z_n \end{aligned}$$

where (γ_n), (c_n), and (β_n) are the sequences defining the two-time-scale algorithm (3)–(4), and where (e_n) is a sequence of independent $\mathcal{N}(0,1)$-distributed random vectors, independent of (Z_n) and of (ε_n). Now, set

$$\tilde{Z}_n = \begin{pmatrix} Z_n \\ z_n \end{pmatrix},\ \tilde{Y}_n = \begin{pmatrix} Y_n \\ y_n \end{pmatrix},\ \tilde{z}^* = \begin{pmatrix} z^* \\ 0 \end{pmatrix},\ \tilde{H} = \begin{pmatrix} H & 0 \\ 0 & -1 \end{pmatrix},\ \tilde{\varepsilon}_n = \begin{pmatrix} \varepsilon_n \\ e_n \end{pmatrix},$$
$$\tilde{\Gamma} = \begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix},\ \tilde{R}_n = \begin{pmatrix} R_n \\ 0 \end{pmatrix},\ \tilde{\rho} = \begin{pmatrix} R \\ 0 \end{pmatrix},$$

and let $\tilde{h} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d+1}$ be the function defined by

$$\tilde{h}(x_1, \ldots, x_{d+1}) = (h(x_1, \ldots, x_d), -x_{d+1}).$$

We note that the sequence $(\tilde{Z}_n, \tilde{Y}_n)$ is then such that $(\tilde{Z}_1, \tilde{Y}_1) = (Z_1, z_1, Y_1, y_1)$ and, for $n \geq 1$,

$$\tilde{Z}_n = \tilde{Z}_{n-1} + \gamma_n \tilde{h}(\tilde{Z}_{n-1}) + \gamma_n c_n^{-1} \tilde{\varepsilon}_n \tag{16}$$
$$\tilde{Y}_n = (1-\beta_n)\, \tilde{Y}_{n-1} + \beta_n \tilde{Z}_n. \tag{17}$$

Since (c_n) and (γ_n) satisfy Assumptions (A3) and (A4), respectively, we have $\lim_{n\to\infty} z_n = 0$ a.s., thus $\lim_{n\to\infty} \tilde{Z}_n = \tilde{z}^*$ a.s., so that $(\tilde{Z}_n)$ fulfills Assumption (A1). Moreover, we clearly have $\lim_{n\to\infty} v_n \tilde{R}_n = \tilde{\rho}$, so that Assumption (A5) is satisfied, and $(\tilde{\varepsilon}_n)$ fulfills Assumption (A6). Since Assumptions (A1)–(A7) are satisfied by the two-time-scale algorithm (16)–(17), Lemma 1 applies. Set $\tilde{\Sigma} = (1 - 2\tau)\tilde{H}^{-1}\tilde{\Gamma}[\tilde{H}^{-1}]^T$ and

$$\tilde{W}_n = \tilde{\Sigma}^{-1/2} \left[\frac{\sqrt{n c_n^2}}{\sqrt{2 \ln \ln n}} \left(\tilde{Y}_n - \tilde{z}^* \right) + \sqrt{2c}\, \tilde{H}^{-1} \tilde{\rho} \right].$$

For all $w \in \mathbb{R}^{d+1}$, we thus have

$$\limsup_{n\to\infty} w^T \tilde{W}_n = \sqrt{w^T w} \ a.s. \tag{18}$$

In view of Claim 1, it can be deduced from (18) that, with probability one, the sequence $(\tilde{W}_n)$ is relatively compact and its limit set $\tilde{\mathcal{U}}$ is such that $\mathcal{S}_{d+1}(0,1) \subset \tilde{\mathcal{U}}$.

Now, let π: $\mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection map defined by $\pi((x_1, \ldots, x_{d+1})^T) = (x_1, \ldots, x_d)^T$. We clearly have $\pi(\mathcal{S}_{d+1}(0,1)) = \overline{\mathcal{B}}_d(0,1)$ and $\pi(\tilde{W}_n) = W_n$. Thus $\overline{\mathcal{B}}_d(0,1)$ is included in the limit set of the sequence (W_n). We then deduce from the first part of Claim 1 that the limit set $\mathcal{U}$ of the sequence (W_n) is the unit ball $\overline{\mathcal{B}}_d(0,1)$. It follows that the limit set of the sequence $(\Sigma^{1/2}W_n)$ is the ellipsoid $\{w \in \mathbb{R}^d \ ; \ w\Sigma^{-1}w \leq 1\}$, and thus that the limit set of the sequence $(\sqrt{nc_n^2/(2\ln\ln n)}\,(Y_n - z^*))$ is the ellipsoid

$$\left\{v \in \mathbb{R}^d \ ; \ (1-2\tau)^{-1}\left(v + \sqrt{2}cH^{-1}\rho\right)^T H^T\Gamma^{-1}H\left(v + \sqrt{2}cH^{-1}\rho\right) \leq 1\right\},$$

which concludes the proof of Parts 2 and 3 of Theorem 1.

4.2. Proof of Lemma 1. In view of Assumption (A6), there exists a sequence (η_n) of independent and $\mathcal{N}(0,\Gamma)$-distributed random variables such that

$$\sum_{k=1}^{n} \varepsilon_{k+1} - \sum_{k=1}^{n} \eta_{k+1} = \left(\sqrt{n\ln\ln n}\right) \ a.s. \tag{19}$$

(*see* Morrow and Philipp [35]). Set

$$\Psi_{n+1} = \sum_{k=1}^{n}\left[\left(\prod_{j=k+1}^{n}(1-\beta_j)\right)\beta_k c_{k+1}^{-1}H^{-1}\eta_{k+1}\right],$$

$u \in \mathbb{R}^d$, and note that $u^T\Psi_{n+1} = Q_n\sum_{k=1}^{n} Q_k^{-1}\beta_k c_{k+1}^{-1}u^T H^{-1}\eta_{k+1}$. Set

$$V_n = \mathbb{V}\left(\sum_{k=1}^{n} Q_k^{-1}\beta_k c_{k+1}^{-1}u^T H^{-1}\eta_{k+1}\right),$$

where $\mathbb{V}(.)$ denotes the variance. We have $V_n = \sum_{k=1}^{n} Q_k^{-2}\beta_k^2 c_{k+1}^{-2}u^T H^{-1}\Gamma[H^{-1}]^T u$, and the application of Lemma 2 in Mokkadem *et al.* [33] ensures that

$$V_n = \left(u^T H^{-1}\Gamma[H^{-1}]^T u\right) Q_n^{-2}\beta_n c_n^{-2}(1+o(1)). \tag{20}$$

It follows that

$$\lim_{n\to\infty}\frac{V_{n+1}}{V_n} = \lim_{n\to\infty}\frac{\beta_{n+1}c_{n+1}^{-2}}{\beta_n c_n^{-2}}(1-\beta_n)^{-2} = 1.$$

Moreover, since $(\beta_n c_n^{-2}) \in \mathcal{GS}(-(1-2\tau))$, we have, in view of (6),

$$[\beta_{n-1}c_{n-1}^{-2}][\beta_n c_n^{-2}]^{-1} = 1 + (1-2\tau)n^{-1} + o(n^{-1}),$$

and, in view of (A7), $\beta_n = (1-2\tau)n^{-1} + o(n^{-1})$. We deduce that

$$\begin{aligned}
\frac{Q_{n-1}^{-2}\beta_{n-1}c_{n-1}^{-2}}{Q_n^{-2}\beta_n c_n^{-2}} &= \frac{\beta_{n-1}c_{n-1}^{-2}}{\beta_n c_n^{-2}}(1-\beta_n)^2 \\
&= \left[1 + \frac{1-2\tau}{n} + o\left(\frac{1}{n}\right)\right]\left[1 - \frac{2(1-2\tau)}{n} + o\left(\frac{1}{n}\right)\right] \\
&= 1 - \frac{(1-2\tau)}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

It follows that the sequence $(Q_n^{-2}\beta_n c_n^{-2})$ belongs to $\mathcal{GS}((1-2\tau))$ with, in view of (A3), $1-2\tau > 0$. It follows in particular that $\lim_{n\to\infty} Q_n^{-2}\beta_n c_n^{-2} = \infty$ and thus, in view of (20), $\lim_{n\to\infty} V_n = \infty$. Since $V_n^{-1/2}u^T\Psi_{n+1}$ is $\mathcal{N}(0,1)$-distributed, Condition (7.24) in Petrov [40] is fulfilled, and Theorem 7.2 applies, so that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n Q_k^{-1}\beta_k c_{k+1}^{-1}u^T H^{-1}\eta_{k+1}}{\sqrt{2V_n \ln\ln V_n}} = 1 \ a.s.$$

Now, the fact that $(Q_n^{-2}\beta_n c_n^{-2}) \in \mathcal{GS}((1-2\tau))$ ensures that $\ln\ln Q_n^{-2}\beta_n c_n^{-2} = \ln\ln n(1+o(1))$; in view of (20), we thus obtain

$$\limsup_{n\to\infty} \frac{\sqrt{\beta_n^{-1}c_n^2}\,Q_n \sum_{k=1}^n Q_k^{-1}\beta_k c_{k+1}^{-1}u^T H^{-1}\eta_{k+1}}{\sqrt{2\left(u^T H^{-1}\Gamma[H^{-1}]^T u\right)\ln\ln n}} = 1 \ a.s.,$$

and thus, in view of (A7),

$$\limsup_{n\to\infty} \frac{\sqrt{nc_n^2}\,u^T\Psi_{n+1}}{\sqrt{2(1-2\tau)\left(u^T H^{-1}\Gamma[H^{-1}]^T u\right)\ln\ln n}} = 1 \ a.s. \tag{21}$$

Now, set $D_0 = 0$ and, for $n \geq 1$, $D_n = \sum_{k=1}^n (\varepsilon_{k+1} - \eta_{k+1})$. We have

$$\Phi_{n+1} - \Psi_{n+1} = Q_n \sum_{k=1}^n Q_k^{-1}\beta_k c_{k+1}^{-1}(\varepsilon_{k+1} - \eta_{k+1})$$

$$\begin{aligned} &= Q_n \sum_{k=1}^{n} Q_k^{-1} \beta_k c_{k+1}^{-1} (D_k - D_{k-1}) \\ &= Q_n \sum_{k=1}^{n} \left[Q_k^{-1} \beta_k c_{k+1}^{-1} - Q_{k+1}^{-1} \beta_{k+1} c_{k+2}^{-1} \right] D_k + \beta_n c_{n+1}^{-1} D_n \\ &= Q_n \sum_{k=1}^{n} Q_k^{-1} \beta_k c_{k+1}^{-1} \left[1 - \frac{\beta_{k+1} c_{k+2}^{-1}}{(1-\beta_k)\beta_k c_{k+1}^{-1}} \right] D_k + \beta_n c_{n+1}^{-1} D_n. \end{aligned}$$

We deduce from (A3) and (A7), and then from (19), that

$$\begin{aligned} \Phi_{n+1} - \Psi_{n+1} &= Q_n \sum_{k=1}^{n} Q_k^{-1} \beta_k c_{k+1}^{-1} \left[1 - \left(1 + O\left(\frac{1}{k} \right) \right) \right] D_k + O\left(\frac{1}{n} \right) c_{n+1}^{-1} D_n \\ &= Q_n \sum_{k=1}^{n} Q_k^{-1} \beta_k o\left(\sqrt{\frac{\ln \ln k}{k c_k^2}} \right) + o\left(\sqrt{\frac{\ln \ln n}{n c_n^2}} \right) \ a.s. \end{aligned}$$

The application of Lemma 2 in Mokkadem *et al.* [33] then yields

$$\Phi_{n+1} - \Psi_{n+1} = o\left(\sqrt{\frac{\ln \ln n}{n c_n^2}} \right) \ a.s.,$$

and Lemma 1 follows from (21).

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