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Serdica Math. J. **46** (2020), 357–386

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
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EXAMPLES OF GROUP AMALGAMATIONS WITH NONTRIVIAL QUASI-KERNELS

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Communicated by V. Drensky

Abstract. We introduce some examples of group amalgamations motivated by the problems of C^*-simplicity and unique trace property. Moreover, we prove that our examples are not inner amenable and identify a relatively large, simple, normal subgroup in each one.

1. Introduction and Preliminaries.

1.1. Introduction. The questions of C^*-simplicity and unique trace property for a discrete group have been studied extensively. By definition, a discrete group G is C^*-simple if the C^*-algebra associated to the left regular representation, $C_r^*(G)$, is simple; likewise, it has the unique trace property if $C_r^*(G)$ has a unique tracial state. An extensive introduction to that topic was given by de la Harpe ([10]). Recently, Kalantar and Kennedy ([13]) gave a necessary and sufficient condition for C^*-simplicity in terms of action on the Furstenberg boundary. After that, Breuillard, Kalantar, Kennedy, and Ozawa ([4]) studied further the question of C^*-simplicity and also showed that a group has the unique trace

2020 *Mathematics Subject Classification:* 22D25, 20E06 (Primary) 46L05, 43A07, 20E08 (Secondary).

Key words: C^*-simplicity, free product of groups with amalgamation, inner amenability.

property if and only if its amenable radical is trivial. They also showed that C^*-simplicity implies the unique trace property. The reverse implication was disproven by examples given by Le Boudec ([15]). In the case of group amalgamations and HNN-extensions, the kernel controls the uniqueness of trace, and the quasi-kernels control the C^*-simplicity.

The notion of inner amenability for discrete groups was introduced by Effros ([8]) as an analogue to Property Γ for II_1 factors that was introduced by Murray and von Neumann ([19]). By definition, a discrete group G is inner amenable if there exists a conjugation invariant, positive, finitely additive, probability measure on $G \setminus \{1\}$. Effros showed that Property Γ implies inner amenability, but the reverse implication doesn't hold, as demonstrated by Vaes ([22]).

Our examples stem from the questions of C^*-simplicity and the unique trace properties for groups. In particular, all of our examples have the unique trace property, and we also determine the C^*-simple ones and the non-C^*-simple ones. The examples of section 3 generalize the example given in [12, Section 4] (which corresponds to the group $G[Sym(3), Sym(3)]$ of section 3). There is a resemblance to the groups introduced by Le Boudec in [14] since they all act on trees. In fact, some of our examples are isomorphic to his, but not all are (see Remark 3.9). The main benefit is that our groups are given concretely by generators and relations, which makes them more tractable to investigate some further properties they possess.

We study some additional analytic properties of our examples. We show that they are all non-inner-amenable by showing that they are finitely fledged – a property that we introduce in section 2. Our groups are examples of non-highly-transitive groups of a special type, as discussed in [9, Section 8.3], where the non-topological-freeness plays an important role (see Remark 3.19). Moreover, using some well known facts, we show that the reduced C^*-algebras of some of the examples have stable rank bigger than one (see Remark 3.23).

We also explore some of the group-theoretical properties of our examples. We remark that they are not finitely presented. Also, under some mild natural assumptions, we show that each group has a relatively large, simple, normal subgroup.

The organization of the paper is as follows: In section 2 we discuss groups acting on trees and inner amenability. We also prove an auxiliary statement about inner amenability using equivariant maps ([2]), which is interesting on its own. In Section 3 we present our examples and study their group-theoretic and analytic structure.

1.2. Preliminaries. For a group Γ acting on a set X, we denote the set-wise stabilizer of a subset $Y \subset X$ by

$$\Gamma_{\{Y\}} \equiv \{g \in \Gamma \mid gY = Y\}$$

and the point-wise stabilizer of a subset $Y \subset X$ by

$$\Gamma_{(Y)} \equiv \{g \in \Gamma \mid gy = y, \ \forall y \in Y\}.$$

For a point $x \in X$, we denote its stabilizer by

$$\Gamma_x = \{g \in \Gamma \mid gx = x\}.$$

Note that, $\Gamma_{\{Y\}}$, $\Gamma_{(Y)}$, and Γ_x are all subgroups of Γ. Also note that

$$g\Gamma_{\{Y\}}g^{-1} = \Gamma_{\{gY\}}, \ g\Gamma_x g^{-1} = \Gamma_{gx} \text{ , and } g\Gamma_{(Y)}g^{-1} = \Gamma_{(gY)}.$$

For a group G and its subgroup H, by $\langle\langle H\rangle\rangle_G$ or by $\langle\langle H\rangle\rangle$, we denote the normal closure of H in G.

For some general references on group amalgamations and HNN-extensions see, e.g., [1], [6], [20], [11], etc.

Let $G_i = \langle X_i \mid R_i\rangle$ for $i = 0, 1$ be two groups with generating sets X_i and relations R_i having a common subgroup H embedded via $j_i : H \hookrightarrow G_i$. Their free product with amalgamation is the group

$$G_0 *_H G_1 \equiv \langle X_0 \sqcup X_1 \mid R_0 \sqcup R_1 \sqcup \{j_0(h) = j_1(h) \mid h \in H\}\rangle.$$

We will assume that the embeddings j_i are self-evident. Every element $g \in G_0 *_H G_1 \setminus H$ can be written in reduced form as

$$g = (g_0)g_1 \cdots g_n, \text{ where } n \in \mathbb{N}_0 \text{ and } g_k \in G_{k \pmod 2} \setminus H.$$

If S_i is a set of left coset representatives for G_i/H, where $i = 0, 1$, satisfy $S_0 \cap S_1 = \{1\}$, then every element $g \in G_0 *_H G_1$ can be uniquely written in normal form as

$$g = (s_0)s_1 \cdots s_n \cdot h, \text{ where } n \in \mathbb{N}_0, \quad s_k \in S_{k \pmod 2} \setminus \{1\}, \text{ and } h \in H.$$

The group amalgamation $G_0 *_H G_1$ is called nontrivial if $G_0 \neq H \neq G_1$ and is called nondegenerate if, moreover, Index$[G_0 : H] \geq 3$ or Index$[G_1 : H] \geq 3$.
The Bass-Serre tree $T[G_0 *_H G_1]$ of $G_0 *_H G_1$ is the graph, that can be shown to be a tree, consisting of a vertex set

$$\text{Vertex}(T[G_0 *_H G_1]) \\ = \{G_0\} \cup \{(s_0)s_1 \cdots s_n G_{n+1 \pmod 2} \mid n \in \mathbb{N}_0,\ s_k \in S_{k \pmod 2} \setminus \{1\}\}$$

and an edge set

$$\text{Edge}(T[G_0 *_H G_1]) \ = \ \{(s_0)s_1 \cdots s_n H \mid n \in \mathbb{N}_0,\ s_k \in S_{k \pmod 2} \setminus \{1\}\}.$$

The vertex $(s_0)s_1 \cdots s_n G_{n+1 \pmod 2}$ is adjacent to the vertex $(s_0)s_1 \cdots s_n s_{n+1} G_{n \pmod 2}$ with connecting edge

$$(s_0)s_1 \cdots s_n G_{n+1 \pmod 2} \cap (s_0)s_1 \cdots s_n s_{n+1} G_{n \pmod 2} \ = \ (s_0)s_1 \cdots s_n s_{n+1} H.$$

Also $G_0 *_H G_1$ acts on $T[G_0 *_H G_1]$ by left multiplication.
Finally, $G_0 *_H G_1$ has the following universal property (see, e.g., [6], page 29 or [1], page 128):

Remark 1.1. Let C be a group, and let $\alpha_i : G_i \longrightarrow C$ be group homomorphisms ($i = 0, 1$) for which $\alpha_0|_H = \alpha_1|_H$. Then there is a unique group homomorphism $\beta : G_0 *_H G_1 \longrightarrow C$ satisfying $\beta|_{G_i} = \alpha_i$ for every $i = 0, 1$.

2. Inner amenability. The barycentric subdivision of a graph T is, by definition, the graph $T^{(1)}$ with vertex and edge sets

$$\text{Vertex}(T^{(1)}) = \text{Vertex}(T) \sqcup \{\ \{e, \bar{e}\} \mid e \in \text{Edge}(T)\},$$

$$\text{Edge}(T^{(1)}) = \text{Edge}(T) \times \{0, 1\},$$

respectively, where $\overline{(e, \varepsilon)} = (\bar{e}, 1 - \varepsilon)$.
It is easy to see that each tree T of finite diameter has a unique center, which is a vertex or an edge (this follows essentially from Exercise 3 on page 21 of [20]). Therefore $T^{(1)}$ has a unique centre, which is a vertex.
Inductively, $T^{(k+1)}$ is defined as $(T^{(k)})^{(1)}$.

For a tree T, a ray is defined as a sequence of vertices of T, $(v_n)_{n=1}^{\infty}$, such that v_n and v_{n+1} are adjacent and $v_n \neq v_{n+2}$ (i.e., there is no backtracking). The boundary of T, ∂T, is defined as the set of all rays mod out by the following equivalence relation:

$$(v_n)_{n=1}^{\infty} \ \sim \ (w_n)_{n=1}^{\infty} \ \text{ if } \ \exists k, m \in \mathbb{N} : \ v_{n+k} = w_{n+m}, \forall n \in \mathbb{N}.$$

The boundaries ∂T and $\partial T^{(k)}$ can be identified in a natual way: A ray from $T^{(k)}$ has a unique subray that belongs to T, and a ray in T can be uniquely extended to a ray in $T^{(k)}$, using the fact that there is a unique path between two vertices in a connected tree.

Let G be a group acting on a tree T. Then it is clear that G acts on ∂T and on the closure $\bar{T} \equiv T \cup \partial T$ in a canonical way. Every element of G is either elliptic, i.e. fixes some some vertex or some edge of T, or is hyperbolic otherwise (see, e.g., [20, I.6]). It is clear that there is a canonical way in which G acts on $T^{(1)}$ and on its boundary $\partial T^{(1)}$. Note that, if an element of G is elliptic, then it necessary fixes a vertex of $T^{(1)}$. Every hyperbolic element g fixes exactly two point the boundary ∂T (and also on the boundary $\partial T^{(1)}$), r_g and s_g, the first one is attractive and the other one is repulsive; i. e., for every open neighborhood U of r_g and every open neighborhood V of s_g there is a large enough number $n \in \mathbb{N}$ satisfying $g^n(\bar{T} \setminus V) \subset U$ and $g^{-n}(\bar{T} \setminus U) \subset V$. By definition, two hyperbolic elements are transverse if they don't have any common fixed points on ∂T. The action of G on T is said to be of general type if there are two transverse hyperbolic elements.

The action of a group G on a set X is called transitive if for every $x, y \in X$, there is an element $g \in G$ satisfying $gx = y$. The action of a group G on a topological space X is called minimal if every G-orbit is dense in X. It is obvious that the action of a group amalgamation or an HNN-extension on their corresponding Bass-Serre trees is transitive and, therefore, minimal.

For an elliptic element $g \in G \setminus \{1\}$, we define a set

$$\Psi_T(g) \;=\; \{t \in T_g^{(1)} \mid t \text{ has a neighbour that is not fixed by } g\}.$$

Clearly, if $g \neq 1$ is elliptic, then $\Psi_T(g) \neq \emptyset$. Let $\Upsilon_T(g)$ be the smallest connected subtree of $T^{(1)}$ containing $\Psi_T(g)$. We propose the following definitions.

Definition 2.1. *Let G act on a tree T. We call an elliptic element $g \in G \setminus \{1\}$ finitely fledged with respect to $G \curvearrowright T$ (or just finitely fledged), if $\Upsilon_T(g)$ has a finite diameter. We call the action of G on T finitely fledged if every elliptic element $g \in G \setminus \{1\}$ is finitely fledged.*

Note that $\Upsilon_T(g)$ has a unique center in $T^{(2)}$ in this situation. Note also that, if $c \in T^{(2)}$ is the center of $\Upsilon_T(g)$ and if $h \in G$, then hc is the center of $\Upsilon_T(hgh^{-1})$.

Definition 2.2. *Let G act on a tree T. We call an elliptic element $g \in G \setminus \{1\}$ infinitely fledged with respect to $G \curvearrowright T$ if $\Upsilon_T(g)$ has an infinite diameter. We call the action of G on T infinitely fledged if there is an infinitely fledged elliptic element $g \in G \setminus \{1\}$.*

We will make use of [2, Proposition 7], which for a group G acting on a tree T states that if the action is of general type and if there is an equivariant (with respect to conjugation) map $\delta : G \setminus \{1\} \to T \cup \partial T$, then G is not inner amenable. Note that $T \cup \partial T$, equipped with the shadow topology, is a compact and totally disconnected topological space (see [17, Section 4.1], [5, Appenix A]).

The following proposition can be regarded as a refinement of [21, Proposition 0.3].

Proposition 2.3. *Let G be a group acting on a tree T, so that the action is of general type. If the action of G on T is finitely fledged, then G is not inner amenable.*

P r o o f. Define $X = T^{(2)} \cup \partial T^{(2)}$. As in the proof of [21, Proposition 0.3], we define a G-equivariant map $\delta : G \setminus \{1\} \longrightarrow X$ in the following way:
- If $g \in G$ is hyperbolic, then $\delta(g)$ is the attractive point of g on $\partial T^{(2)}$;
- If $g \in G \setminus \{1\}$ is elliptic, then $\delta(g)$ is the center of $\Upsilon_T(g)$ in $T^{(2)}$.

The result follows from [2, Proposition 7]. □

It was proven in [21] that the Baumslag-Solitar group

$$BS(2,3) = \{\tau, b \mid \tau^{-1} b^2 \tau = b^3\}$$

is inner amenable, and thus correcting a mistake in [2] (see also [3]). We note that the action of $BS(2,3)$ on its Bass-Serre tree T is infinitely fledged.

To see this, it is easy to verify that the element b^6 fixes the linear subtree

$$T' = \{\dots, \tau^{-1} b \tau b \tau^{-1}\langle b\rangle, \tau^{-1} b \tau \langle b\rangle, \tau^{-1}\langle b\rangle, \langle b\rangle, \tau\langle b\rangle, \tau b \tau^{-1}\langle b\rangle, \tau b \tau^{-1} b \tau\langle b\rangle, \tau b \tau^{-1} b \tau b \tau^{-1}\langle b\rangle, \dots\}.$$

For example,

$$\begin{aligned} b^6 \tau b \tau^{-1} b \tau b \tau^{-1}\langle b\rangle &= \tau b^9 \cdot b \tau^{-1} b \tau b \tau^{-1}\langle b\rangle = \tau b \tau^{-1} b^6 \cdot b \tau b \tau^{-1}\langle b\rangle \\ &= \tau b \tau^{-1} b \tau b^9 \cdot b \tau^{-1}\langle b\rangle = \tau b \tau^{-1} b \tau b \tau^{-1} b^6 \langle b\rangle = \tau b \tau^{-1} b \tau b \tau^{-1}\langle b\rangle. \end{aligned}$$

Each vertex of T' that starts and ends with the same power of τ, however, has a neighbour that is not fixed by b^6. The neighbour $\tau b \cdots \tau^{-1} b \tau \cdot \tau\langle b\rangle$ of the vertex $\tau b \cdots \tau^{-1} b \tau\langle b\rangle$ is not fixed by b^6, and the neighbour $\tau^{-1} b \cdots \tau b \tau^{-1} \cdot \tau^{-1}\langle b\rangle$ of the vertex $\tau^{-1} b \cdots \tau b \tau^{-1}\langle b\rangle$ is not fixed by b^6. To see this, observe, for example, that

$$\begin{aligned} b^6 \tau b \tau^{-1} b \tau b \tau^{-1} b \tau \cdot \tau\langle b\rangle &= \tau b^9 \cdot b \tau^{-1} b \tau b \tau^{-1} b \tau \cdot \tau\langle b\rangle = \tau b \tau^{-1} b^6 \cdot b \tau b \tau^{-1} b \tau \cdot \tau\langle b\rangle \\ &= \tau b \tau^{-1} b \tau b^9 \cdot b \tau^{-1} b \tau \cdot \tau\langle b\rangle = \tau b \tau^{-1} b \tau b \tau^{-1} b^6 \cdot b \tau \cdot \tau\langle b\rangle \\ &= \tau b \tau^{-1} b \tau b \tau^{-1} b \tau b^9 \tau\langle b\rangle = \tau b \tau^{-1} b \tau b \tau^{-1} b \tau b \cdot b^8 \tau\langle b\rangle \\ &= \tau b \tau^{-1} b \tau b \tau^{-1} b \tau b \tau b^{12}\langle b\rangle = \tau b \tau^{-1} b \tau b \tau^{-1} b \tau b \tau\langle b\rangle \\ &\neq \tau b \tau^{-1} b \tau b \tau^{-1} b \tau \cdot \tau\langle b\rangle. \end{aligned}$$

Therefore the action of $BS(2,3)$ on T is infinitely fledged.

The proofs of Proposition 2.3 and [2, Proposition 7] imply

Proposition 2.4. *Let G be a group acting on a tree T, so that the action is of general type. If m is a conjugation invariant mean on $G \setminus \{1\}$, then m is supported on the infinitely fledged elliptic elements with respect to $G \curvearrowright T$.*

Proof. Let $F \subset G \setminus \{1\}$ be the set of all hyperbolic and finitely fledged elliptic elements. Let $I \subset G \setminus \{1\}$ be the set of all infinitely fledged elliptic elements. Then $F \sqcup I = G \setminus \{1\}$. It is important to note that both sets F and I are conjugation invariant. Let X and $\delta : F \to X$ are as in Proposition 2.3, i.e., $X = T^{(2)} \cup \partial T^{(2)}$ and

- If $\gamma \in F$ is hyperbolic, then $\delta(\gamma)$ is the attractive point of γ on $\partial T^{(2)}$;
- If $\gamma \in F$ is elliptic, then $\delta(\gamma)$ is the center of $\Upsilon_T(\gamma)$ in $T^{(2)}$.

Let g and h be two transverse hyperbolic elements of G with attractive points r_g and r_h, respectively, and repulsive points s_g and s_h, respectively. Since, clearly, X is Hausdorff in the shadow topology, we can chose two by two non-intersecting neighborhoods R_g, R_h, S_g, and S_h of r_g, r_h, s_g, and s_h, respectively. These neighborhoods will constitute our 'paradoxical' decomposition. By the hyperbolicity of g and h, it follows that we can find an integer n such that $g^n(S_g) \supset X \setminus R_g$ and $h^n(S_h) \supset X \setminus R_h$.

Denote $R'_g = \delta^{-1}(R_g)$, $R'_h = \delta^{-1}(R_h)$, $S'_g = \delta^{-1}(S_g)$, and $S'_h = \delta^{-1}(S_h)$.

Let m be a conjugation invariant mean with $m(F) > 0$. We can assume that m is supported on F, i.e., $m(I) = 0$ and $m(F) = 1$. Then

$$
\begin{aligned}
1 = m(F) &\geq m(R'_g \sqcup R'_h \sqcup S'_g \sqcup S'_h) \\
&= m(R'_g)+m(R'_h)+m(S'_g)+m(S'_h) = m(R'_g)+m(R'_h)+m(g^n S'_g g^{-n})+m(h^n S'_h h^{-n}) \\
&\geq m(R'_g \cup g^n S'_g g^{-n}) + m(R'_h \cup h^n S'_h h^{-n}) \;=\; m(F) + m(F) = 2.
\end{aligned}
$$

This contradiction implies that the existence of a mean with $m(F) > 0$ is impossible. □

Remark 2.5. The proposition implies that a conjugation invariant mean is supported on the se of elliptic element that are infinitely fledged with respect to every action on a tree.

To conclude the section, we give an easy application of the finite-fledgeness:

Let G_0 and G_1 be two discrete groups and let $G_0 * G_1$ be their free product. Then the elliptic elements of $G_0 * G_1 \curvearrowright T[G_0 * G_1]$ belong to the conjugates of G_0 and G_1 (see, e.g., [20, I.4.3, Proposition 18]). If $g_0 \in G_0 \setminus \{1\}$, then g_0 acts freely on the set of neighbours $\{h_0G_1 \mid h_0 \in G_0\}$ of G_0 in $T[G_0 * G_1]$. Therefore g_0 fixes only one vertex, G_0. The same argument applies to every element of every conjugate of G_0 and G_1. This shows that every nontrivial elliptic element of the action $G_0 * G_1 \curvearrowright T[G_0 * G_1]$ fixes exactly one vertex of $T[G_0 * G_1]$. Therefore $G_0 * G_1$ is finitely fledged. In the case when G_0 and G_1 are nontrivial and not both of order two, we can apply Proposition 2.3 to derive the well known result that $G_0 * G_1$ is not inner amenable.

3. Group amalgamations.

3.1. Notations, definitions, quasi-kernels. We introduce notations, some of which appear in [12]:

Let $G = G_0 *_H G_1$ be a nontrivial amalgam. We define sets $T_{j,n} \subset G$ as follows: Let $T_{0,0} = T_{1,0} = H$. For $j = 0, 1$ and $n \geq 1$, let

$$T_{j,n} = \{g_j \cdots g_{j+n-1} \mid g_i \in G_{i \pmod 2} \setminus H\}.$$

Now, for $j = 0, 1$ and $n \geq 0$, let

$$C_{j,n} = \bigcap_{g \in T_{j,n}} gHg^{-1}. \tag{1}$$

Next, we consider the quasi-kernels defined in [12]:

$$K_0 = \bigcap_{n \geq 0} C_{0,n} \quad \text{and} \quad K_1 = \bigcap_{n \geq 0} C_{1,n}. \tag{2}$$

Note that

$$\ker G = K_0 \cap K_1, \tag{3}$$

where the kernel of G is defined as

$$\ker G = \bigcap_{g \in G} gHg^{-1}.$$

Also, for $j = 0, 1$, $n \geq 1$, and $g_i \in G_{i \pmod 2} \setminus H$, let

$$K(g_{j+n} \cdots g_j) = g_{j+n} \cdots g_j K_j g_j^{-1} \cdots g_{j+n}^{-1}.$$

It follows from [12, Proposition 3.1] that G has the unique trace property if and only if $\ker G$ has the unique trace property. It also follows from [5, Theorem 3.9] and from Proposition 3.5 (i) (below) that G is C^*-simple if and only if K_0 or K_1 is trivial or non-amenable provided G is a nondegenerate amalgam and $\ker G$ is trivial.

We need the following results.

Lemma 3.1. *K_0 and K_1 are normal subgroups of H. If $\ker G$ is trivial, then K_0 and K_1 have a trivial intersection and mutually commute.*

Proof. First statement follows from the observation that for each $h \in H$ and each $j = 0, 1$,

$$h \cdot T_{j,n} = \{h \cdot g_j \cdots g_{j+n-1} \mid g_i \in G_{i \pmod 2} \setminus H\} = T_{j,n}.$$

For the second statement, (3) implies $K_0 \cap K_1 = \ker G = \{1\}$. Take $k_j \in K_j$, $j = 0, 1$. Since $K_j \triangleleft H$ for each $j = 0, 1$, it follows $k_0 k_1^{-1} k_0^{-1} \in K_1$ and $k_1 k_0 k_1^{-1} \in K_0$. Therefore,

$$K_0 \ni (k_1 k_0 k_1^{-1}) k_0^{-1} = k_1 (k_0 k_1^{-1} k_0^{-1}) \in K_1,$$

so $k_1 k_0 k_1^{-1} k_0^{-1} \in K_0 \cap K_1 = \{1\}$. □

Lemma 3.2. *Let $j \in \{0, 1\}$ and $g_i \in G_{i \pmod 2} \setminus H$ for each i. Then $K(g_{j+n} \cdots g_j)$ is a subgroup of $K_{j+n+1 \pmod 2}$.*

Proof. For $m \geq 1$ and $h \in H$, observe that

$$\begin{aligned} g_j \cdot T_{j,m} &= \{g_j \cdot \gamma_j \cdots \gamma_{j+m-1} \mid \gamma_i \in G_{i \pmod 2} \setminus H\} \\ &\supset \{g_j \cdot (g_j)^{-1} h \gamma_{j+1} \cdots \gamma_{j+m-1} \mid \gamma_i \in G_{i \pmod 2} \setminus H\} \\ &= T_{j+1 \pmod 2, m-1}. \end{aligned}$$

It follows by induction on $n \geq 0$ that for $m \geq n+1$, one has

$$\begin{aligned} g_{j+n} \cdots g_j \cdot T_{j,m} &= \{g_{j+n} \cdots g_j \cdot \gamma_j \cdots \gamma_{j+m-1} \mid \gamma_i \in G_{i \pmod 2} \setminus H\} \\ &\supset \{g_{j+n} \cdots g_j \cdot (g_j^{-1} \cdots g_{j+n}^{-1}) h \gamma_{j+n+1} \cdots \gamma_{j+m-1} \mid \\ &\qquad\qquad \gamma_i \in G_{i \pmod 2} \setminus H\} \\ &= T_{j+n+1 \pmod 2, m-n-1}. \end{aligned}$$

The assertion follows from equations (1) and (2). □

Lemma 3.3. *Let $j \in \{0, 1\}$ and $g_i, g_i' \in G_{i \pmod 2} \setminus H$ for each i. Then the following hold:*
(i) If $(g_j')^{-1} \cdots (g_{j+n}')^{-1} g_{j+n} \cdots g_j \in H$, then $K(g_{j+n} \cdots g_j) = K(g_{j+n}' \cdots g_j')$.
(ii) If $\ker G$ is trivial and if $(g_j')^{-1} \cdots (g_{j+n}')^{-1} g_{j+n} \cdots g_j \notin H$, then $K(g_{j+n} \cdots g_j)$ and $K(g_{j+n}' \cdots g_j')$ have a trivial intersection and mutually commute.

Proof. Denote $\gamma = (g_j')^{-1} \cdots (g_{j+n}')^{-1} g_{j+n} \cdots g_j$.
(i) If $\gamma \in H$, then

$$\begin{aligned}(g_j')^{-1}\cdots(g_{j+n}')^{-1}K(g_{j+n}\cdots g_j)g_{j+n}'\cdots g_j' \\ = (g_j')^{-1}\cdots(g_{j+n}')^{-1}g_{j+n}\cdots g_jK_jg_j^{-1}\cdots g_{j+n}^{-1}g_{j+n}'\cdots g_j' \\ = \gamma K_j\gamma^{-1} = K_j,\end{aligned}$$

where the last equality follows from Lemma 3.1.
(ii) Note that, if $\gamma \notin H$, then γ starts and ends with elements of $G_j\backslash H$ (or itself is an element of $G_j\backslash H$). Therefore Lemma 3.2 implies

$$\begin{aligned}(g_j')^{-1}\cdots(g_{j+n}')^{-1}K(g_{j+n}\cdots g_j)g_{j+n}'\cdots g_j' \\ = (g_j')^{-1}\cdots(g_{j+n}')^{-1}g_{j+n}\cdots g_jK_jg_j^{-1}\cdots g_{j+n}^{-1}g_{j+n}'\cdots g_j' \\ = \gamma K_j\gamma^{-1} = K(\gamma) < K_{j+1 \pmod 2}.\end{aligned}$$

This, combined with

$$(g_j')^{-1}\cdots(g_{j+n}')^{-1}K(g_{j+n}'\cdots g_j')g_{j+n}'\cdots g_j' = K_j$$

and Lemma 3.1, yield the last statement. □

Remark 3.4. Consider the Bass-Serre tree $T = T[G]$ of the group $G = G_0 *_H G_1$, and consider the edge H with ends G_0 and G_1. For each $j = 0, 1$, denote by T_j the full subtree of T consisting of all vertices $v \in T$ for which $\mathrm{dist}(v, G_j) < \mathrm{dist}(v, G_{j+1 \pmod 2})$. Note that, if S_i are coset representatives for G_i/H, then

$$(4)\quad T_j = \{G_j\} \cup \{s_j\cdots s_{j+n}G_{j+n+1 \pmod 2} \mid n \geq 0,\ s_k \in S_{k \pmod 2} \setminus \{1\}\}.$$

Proposition 3.5. *With the notation of the previous remark, the following hold:*
(i) $K_j = G_{(T_j)}$.
(ii) $K(g_{j+n}\cdots g_j) = G_{(g_{j+n}\cdots g_jT)}$.

P r o o f. (i) We have

$$\begin{aligned} h \in K_j &\iff \\ g_{j+n-1}^{-1}\cdots g_j^{-1}hg_j\cdots g_{j+n-1} \in H,\ \forall n \geq 0,\ \forall g_k \in G_{k \pmod 2}\setminus H &\iff \\ hg_j\cdots g_{j+n-1} \in g_j\cdots g_{j+n-1}H,\ \forall n \geq 0,\ \forall g_k \in G_{k \pmod 2}\setminus H &\iff \\ hg_j\cdots g_{j+n-1}H = g_j\cdots g_{j+n-1}H,\ \forall n \geq 0,\ \forall g_k \in G_{k \pmod 2}\setminus H &\iff \\ h \text{ fixes every edge of } T_j &\iff \\ h \in G_{(T_j)}. \end{aligned}$$

(ii) As in (i), we have

$$\begin{aligned} h \in K(g_{j+n} \cdots g_j) &\iff \\ h \in g_{j+n} \cdots g_j K_j g_j^{-1} \cdots g_{j+n}^{-1} &\iff \\ g_j^{-1} \cdots g_{j+n}^{-1} h g_{j+n} \cdots g_j \in K_j &\iff \\ g_j^{-1} \cdots g_{j+n}^{-1} h g_{j+n} \cdots g_j \in G_{(T_j)} &\iff \\ h \in g_{j+n} \cdots g_j G_{(T_j)} g_j^{-1} \cdots g_{j+n}^{-1} &\iff \\ h \in G_{(g_{j+n} \cdots g_j T_j)}. & \end{aligned}$$

□

Now, chose representatives S_j of G_j/H containing $\{1\}$ for $j = 0, 1$, and denote $S'_j = S_j \setminus \{1\}$.

Assume that $\ker G = \{1\}$.

Then, from Lemmas 3.1 and 3.2, it follows that $K(s_j) \cap K_j = \{1\}$ and that $K(s_j)$ and K_j mutually commute for $s_j \in S'_j$ and for $j = 0, 1$. Also, from Lemma 3.3, it follows that if $s_j, t_j \in S'_j$ are different, then $K(s_j)$ and $K(t_j)$ have a trivial intersection and mutually commute. Likewise, it follows from Lemma 3.3 that for $n \geq 1$ and for $s_i, t_i \in S'_{i \pmod 2}$, $K(s_{j+n}s_{j+n-1} \cdots s_j) = K(t_{j+n}t_{j+n-1} \cdots t_j)$ if and only if $(s_{j+n}, \ldots, s_j) = (t_{j+n}, \ldots, t_j)$. If $(s_{j+n}, \ldots, s_j) \neq (t_{j+n}, \ldots, t_j)$, then $K(s_{j+n}s_{j+n-1} \cdots s_j)$ and $K(t_{j+n}t_{j+n-1} \cdots t_j)$ have a trivial intersection and mutually commute. Moreover, from Lemmas 3.1 and 3.2, it follows that $K(s_{j+n+1}s_{j+n} \cdots s_j)$ and $K(t_{j+n}t_{j+n-1} \cdots t_j)$ have a trivial intersection and mutually commute for any choice of s_i and t_i.

Thus, if we consider for $j = 0, 1$ and for $n \in \mathbb{N}_0$ the following subgroups of H:

$$\mathcal{K}'(j, n) = \bigoplus_{\substack{s_i \in S'_{i \pmod 2}, \\ i = j, \ldots, j+n}} K(s_j \cdots s_{j+n}) \text{ and} \tag{5}$$

$$\mathcal{K}(j, n) = \bigoplus_{\substack{\gamma_j \in S_j, \\ s_i \in S'_{i \pmod 2}, \\ i = j+1, \ldots, j+n}} K(\gamma_j s_{j+1} \cdots s_{j+n}), \tag{6}$$

then it is easy to see that:

Proposition 3.6. *$\mathcal{K}(j, n)$ is a normal subgroup of G_j, and $\mathcal{K}'(j, n)$ is a normal subgroup of H.*

Remark 3.7. From Lemma 3.2, it follows that there are subgroups of K_1 isomorphic to K_0 and vice versa. Consequently, $K_0 = \{1\}$ if and only if $K_1 = \{1\}$. In this situation, the groups $\mathcal{K}(j,n)$ and $\mathcal{K}'(j,n)$ are all trivial.

3.2. A family of examples. For $j = 0,1$, consider nonempty sets I'_j with $I_j = I'_j \sqcup \{\iota_j\}$ and transitive permutation groups Γ_j on I_j with stabilizer groups $\Gamma'_j \equiv (\Gamma_j)_{\iota_j}$ that are not both trivial. We define a family of groups that depend on Γ_0 and Γ_1 as follows:

$$G[\Gamma_0,\Gamma_1] \equiv G[I_0,I_1;\iota_0,\iota_1;\Gamma_0,\Gamma_1] \equiv G_0 *_H G_1,$$

where $H = \langle Q_0, Q_1\rangle$ and where

$$Q_j = \langle \ \{ \ h_j(i_j,i_{j+1},\ldots,i_{j+n};\sigma_{j+n+1}) \mid n\in\mathbb{Z}_0,\ i_k\in I'_{k \pmod 2}, \\ \sigma_{j+n+1}\in\Gamma'_{j+n+1 \pmod 2} \ \} \ \cup \ \{ \ g_j(\sigma_j) \mid \sigma_j\in\Gamma'_j \ \} \ \rangle$$

for $j = 0,1$. Finally, $G_j = \langle H \cup \{ \ g_j(\sigma_j) \mid \sigma_j \in \Gamma_j \setminus \Gamma'_j \ \}\rangle$ with the following relations (there are redundancies):
(R1) The groups Q_0 and Q_1 mutually commute.
(R2) For $j = 0,1$, $0 \le m \le n$, $i_k, s_k \in I'_{k \ (\mathrm{mod}\ 2)}$ with $(i_j,\ldots,i_{j+m}) \ne (s_j,\ldots,s_{j+m})$, and for $\sigma_k \in \Gamma'_{k \ (\mathrm{mod}\ 2)}$, the elements

$$h_j(s_j,\ldots,s_{j+m};\sigma_{j+m+1}) \quad \text{and} \quad h_j(i_j,\ldots,i_{j+m},\ldots,i_{j+n};\sigma_{j+n+1})$$

commute.
(R3) For $j = 0,1$, $0 \le m < n$, $i_k \in I'_{k \ (\mathrm{mod}\ 2)}$, and for $\sigma_k \in \Gamma'_{k \ (\mathrm{mod}\ 2)}$, the following holds

$$h_j(i_j,\ldots,i_{j+m};\sigma_{j+m+1})h_j(i_j,\ldots,i_{j+m},i_{j+m+1}\ldots,i_{j+n};\sigma_{j+n+1}) \\ \times h_j(i_j,\ldots,i_{j+m};\sigma_{j+m+1})^{-1} \\ = h_j(i_j,\ldots,i_{j+m},\sigma_{j+m+1}(i_{j+m+1}),\ldots,i_{j+n};\sigma_{j+n+1}).$$

(R4) For $j = 0,1$, $m \in \mathbb{Z}_0$, $i_k \in I'_{k \ (\mathrm{mod}\ 2)}$, and $\sigma_{j+m+1}, \tilde\sigma_{j+m+1} \in \Gamma'_{j+m+1 \ (\mathrm{mod}\ 2)}$, the following hold

$$h_j(i_j,\ldots,i_{j+m};\mathrm{id}) = 1,$$

$$h_j(i_j,\ldots,i_{j+m};\tilde\sigma_{j+m+1})h_j(i_j,\ldots,i_{j+m};\sigma_{j+m+1}) \\ = h_j(i_j,\ldots,i_{j+m};\tilde\sigma_{j+m+1}\sigma_{j+m+1}),$$

and

$$h_j(i_j, \dots, i_{j+m}; \sigma_{j+m+1})^{-1} = h_j(i_j, \dots, i_{j+m}; \sigma_{j+m+1}^{-1}).$$

(R5) For $j = 0, 1$ and $\sigma_j, \tilde{\sigma}_j \in \Gamma_j$, the following hold

$$g_j(\mathrm{id}) = 1, \qquad g_j(\sigma_j) g_j(\tilde{\sigma}_j) = g_j(\sigma_j \tilde{\sigma}_j), \qquad \text{and} \qquad g_j(\sigma_j)^{-1} = g_j(\sigma_j^{-1}).$$

(R6) For $j = 0, 1$, $m \in \mathbb{Z}_0$, $i_k \in I'_{k \pmod 2}$, $\sigma_j \in \Gamma_j$, and $\sigma'_{j+m+1} \in \Gamma'_{j+m+1 \pmod 2}$, the following holds

$$g_j(\sigma_j) h_j(i_j, \dots, i_{j+m}; \sigma'_{j+m+1}) g_j(\sigma_j)^{-1} \; = \begin{cases} g_{j+1 \pmod 2}(\sigma'_{j+m+1}) \text{ if } \sigma_j(i_j) = \iota_j \text{ and } \; m = 0, \\ h_{j+1 \pmod 2}(i_{j+1}, \dots, i_{j+m}; \sigma'_{j+m+1}) \text{ if } \sigma_j(i_j) = \iota_j \text{ and } \; m \geq 1, \\ h_j(\sigma_j(i_j), i_{j+1}, \dots, i_{j+m}; \sigma'_{j+m+1}) \text{ if } \sigma_j(i_j) \neq \iota_j, \text{ and } \; m \geq 0. \end{cases}$$

3.3. Some basic properties of the examples and their quasi-kernels. For a group $G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$, let's note that $\mathrm{Index}[G_j : H] = \#(I_j)$, $j = 0, 1$. To see this, recall that Γ_j acts transitively on I_j, and for $i \in I'_j$, let $\tau_j^i \in \Gamma_j$ be such that $\tau_j^i(\iota_j) = i$. Let's denote $\gamma_j^i \equiv g_j(\tau_j^i)$, and take an element $\sigma \in \Gamma_j \setminus \Gamma'_j$ with $\sigma(\iota_j) = i$. Then $(\tau_j^i)^{-1} \circ \sigma(\iota_j) = \iota_j$, so $g_j((\tau_j^i)^{-1} \circ \sigma(\iota_j)) \in H$, and therefore $g_j(\sigma) \in g_j(\tau_j^i) H = \gamma_j^i H$. Thus

$$G_j \; = \; H \sqcup \bigsqcup_{i \in I'_j} \gamma_j^i H. \tag{7}$$

Consider the canonical action of $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$ on its Bass-Serre tree $T = T[G]$. Adjacent vertices to the vertex G_j different from $G_{j+1 \pmod 2}$ can be indexed by the set I'_j, so we denote the vertex G_j by $v(\iota_j)$, and for $i \in I'_j$, we denote the vertex $\gamma_j^i G_{j+1 \pmod 2}$ by $v(\iota_j, i)$. Also, for $i_t \in I'_{t \pmod 2}$, we denote the vertex $\gamma_j^{i_j} \gamma_{j+1 \pmod 2}^{i_{j+1}} \cdots \gamma_{j+k \pmod 2}^{i_{j+k}} G_{j+k+1 \pmod 2}$ by $v(\iota_j, i_j, \dots, i_{j+k})$. Note that, using notation from Remark 3.4, T_j is the full subtree of T containing the vertex $v(\iota_j)$ and the vertices $v(\iota_j, i_j, \dots, i_{j+k})$, where $i_t \in I'_{t \pmod 2}$ and $j = 0, 1$.

Remark 3.8. Notice that the case $\#(I_0) = \#(I_1) = 2$ is impossible because of the requirement of non-triviality of the stabilizers, so $\#(I_0) \geq 3$ or $\#(I_1) \geq 3$. Therefore the corresponding Bass-Serre tree is not a linear tree and the amalgam is nondegenerate (see [11, Proposition 19]).

Remark 3.9. There is a resemblance of our examples with the groups introduced by Le Boudec in [14]. In fact, the example from [12, Section 4] is isomorphic to one of the groups from [14] (see [12, Remark 4.7]). It can be shown that all of our examples satisfy the conditions of [15, Theorem A]. Whenever $\#(I_0) \neq \#(I_1)$, however, none of the groups $G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$ appears in [14] or in [15, Theorem C].

Remark 3.10. It is immediate from [1, Theorem VI.9], that our examples are not finitely presented since H is never finitely generated.

We need some easy facts about $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$.

Lemma 3.11. *(i) Let $m \geq 0$, $i_t \in I'_{t\,(\mathrm{mod}\ 2)}$, and $\sigma'_{j+m+1} \in \Gamma'_{j+m+1\ (\mathrm{mod}\ 2)}$. Then*

$$h_j(i_j, \dots, i_{j+m}; \sigma'_{j+m+1}) = \\ = \gamma_j^{i_j} \cdots \gamma_{j+m\ (\mathrm{mod}\ 2)}^{i_{j+m}} g_{j+m+1\ (\mathrm{mod}\ 2)}(\sigma'_{j+m+1})(\gamma_{j+m\ (\mathrm{mod}\ 2)}^{i_{j+m}})^{-1} \cdots (\gamma_j^{i_j})^{-1}.$$

(ii) Every element $h \in Q_j$ can be written as

$$h = g_j(\sigma'_j) \prod_{k=1}^{m} h_j(i_j^k, \dots, i_{j+n_k}^k; \sigma''_k),$$

where $m \geq 1$, $\sigma'_j \in \Gamma'_j$, $\sigma''_k \in \Gamma'_{j+n_k+1\ (\mathrm{mod}\ 2)}$, $0 \leq n_1 \leq \cdots \leq n_m$, and $i_t^k \in I'_{t\ (\mathrm{mod}\ 2)}$.

(iii) Every element $g \in T_{j,n}$ can be written as

$$g = \gamma_j^{k_j} \cdots \gamma_{j+n-1\ (\mathrm{mod}\ 2)}^{k_{j+n-1}} h,$$

where $h \in H$ and $k_t \in I'_{t\ (\mathrm{mod}\ 2)}$.

Proof. (i) For $\sigma_j \in \Gamma_j$, with $\sigma_j(\iota_j) = i_j$, (R6), read backwards, gives

$$h_j(i_j, \dots, i_{j+m}; \sigma'_{j+m+1}) = g_j(\sigma_j) h_{j+1\ (\mathrm{mod}\ 2)}(i_{j+1}, \dots, i_{j+m}; \sigma'_{j+m+1}) g_j(\sigma_j)^{-1}.$$

Now (i) follows by induction.
(ii) follows from (R4) and (R5) applied several times.
(iii) follows from equation (7) and the structure of the amalgams. □

Lemma 3.12. *Let $m \geq 0$, $i_t \in I'_{t\ (\mathrm{mod}\ 2)}$, $\sigma'_j \in \Gamma'_j$, and $\sigma'_{j+m+1} \in \Gamma'_{j+m+1\ (\mathrm{mod}\ 2)}$. Then:*

(*i*) *For $n > m$, the following holds*

$$h_j(i_j,\dots,i_{j+m};\sigma'_{j+m+1})v(\iota_j,i_j,\dots,i_{j+m},i_{j+m+1},i_{j+m+2},\dots,i_{j+n}) = \\ = v(\iota_j,i_j,\dots,i_{j+m},\sigma'_{j+m+1}(i_{j+m+1}),i_{j+m+2},\dots,i_{j+n});$$

Also $g_j(\sigma'_j)v(\iota_j,k_j) = v(\iota_j,\sigma'_j(k_j))$ *for* $k_j \in I'_j$.

(*ii*) $g_j(\sigma'_j) \in G_{v(\iota_j)}$ *and* $h_j(i_j,\dots,i_{j+m};\sigma'_{j+m+1}) \in G_{v(\iota_j,i_j,\dots,i_{j+m})}$.

(*iii*) $g_j(\sigma'_j) \in K_{j+1 \pmod 2}$.

(*iv*) $h_j(i_j,\dots,i_{j+m};\sigma'_{j+m+1}) \in K_{j+1 \pmod 2}$.

(*v*) *For* $n \geq m$ *and for* $s_t \in I'_{t \pmod 2}$ *with* $(i_j,\dots,i_{j+m}) \neq (s_j,\dots,s_{j+m})$, *we have*

$$h_j(s_j,\dots,s_{j+m};\sigma'_{j+m+1}) \in G_{v(\iota_j,i_j,\dots,i_{j+m},\dots,i_{j+n})}.$$

P r o o f. (i) First note that, by the structure of the amalgams and from Lemma 3.11 (i) and (iii), it follows that there are $t_l \in I'_l$ and a $\chi \in H$ satisfying

$$(\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2})^{-1} = \chi\gamma^{t_{j+n}}_{j+n \pmod 2}\cdots\gamma^{t_{j+m+2}}_{j+m+2 \pmod 2}.$$

Then note that

$$\sigma \equiv (\tau^{\sigma'_{j+m+1}(i_{j+m+1})}_{j+m+1})^{-1}\circ\sigma'_{j+m+1}\circ\tau^{i_{j+m+1}}_{j+m+1} \in \Gamma'_{j+m+1 \pmod 2}$$

since σ fixes $\iota_{j+m+1 \pmod 2}$. Then it follows that

$$(\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2})^{-1}g_{j_m+1}(\sigma)\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2} \\ = \chi h_{j+n}(t_{j+n},\dots,t_{j+m+2};\sigma)\chi^{-1}.$$

Therefore,

$$\begin{aligned} &h_j(i_j,\dots,i_{j+m};\sigma'_{j+m+1})v(\iota_j,i_j,\dots,i_{j+m},i_{j+m+1},i_{j+m+2},\dots,i_{j+n}) \\ &= \gamma^{i_j}_j\cdots\gamma^{i_{j+m}}_{j+m \pmod 2}g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})(\gamma^{i_{j+m}}_{j+m \pmod 2})^{-1}\cdots(\gamma^{i_j}_j)^{-1} \\ &\cdot\gamma^{i_j}_j\cdots\gamma^{i_{j+m}}_{j+m \pmod 2}\gamma^{i_{j+m+1}}_{j+m+1 \pmod 2}\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2}G_{j+n+1 \pmod 2} \\ &= \gamma^{i_j}_j\gamma^{i_{j+1}}_{j+1 \pmod 2}\cdots\gamma^{i_{j+m}}_{j+m \pmod 2}g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})\gamma^{i_{j+m+1}}_{j+m+1 \pmod 2} \\ &\quad\cdot\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2}G_{j+n+1 \pmod 2} = \\ &= \gamma^{i_j}_j\gamma^{i_{j+1}}_{j+1 \pmod 2}\cdots\gamma^{i_{j+m}}_{j+m \pmod 2}\gamma^{\sigma'_{j+m+1}(i_{j+m+1})}_{j+m+1}g_{j+m+1 \pmod 2}(\sigma) \\ &\quad\cdot\gamma^{i_{j+m+2}}_{j+m+2 \pmod 2}\cdots\gamma^{i_{j+n}}_{j+n \pmod 2}G_{j+n+1 \pmod 2} \end{aligned}$$

$$= \gamma_j^{i_j}\gamma_{j+1\ (\mathrm{mod}\ 2)}^{i_{j+1}}\cdots\gamma_{j+m\ (\mathrm{mod}\ 2)}^{i_{j+m}}\gamma_{j+m+1}^{\sigma'_{j+m+1}(i_{j+m+1})}\gamma_{j+m+2\ (\mathrm{mod}\ 2)}^{i_{j+m+2}}\cdots\gamma_{j+n\ (\mathrm{mod}\ 2)}^{i_{j+n}}$$
$$(\gamma_{j+m+2\ (\mathrm{mod}\ 2)}^{i_{j+m+2}}\cdots\gamma_{j+n\ (\mathrm{mod}\ 2)}^{i_{j+n}})^{-1}g_{j+m+1\ (\mathrm{mod}\ 2)}(\sigma)\gamma_{j+m+2\ (\mathrm{mod}\ 2)}^{i_{j+m+2}}\cdots\gamma_{j+n\ (\mathrm{mod}\ 2)}^{i_{j+n}}$$
$$\times G_{j+n+1\quad(\mathrm{mod}\ 2)}$$
$$= \gamma_j^{i_j}\gamma_{j+1\ (\mathrm{mod}\ 2)}^{i_{j+1}}\cdots\gamma_{j+m\ (\mathrm{mod}\ 2)}^{i_{j+m}}\gamma_{j+m+1}^{\sigma'_{j+m+1}(i_{j+m+1})}\gamma_{j+m+2\ (\mathrm{mod}\ 2)}^{i_{j+m+2}}\cdots\gamma_{j+n\ (\mathrm{mod}\ 2)}^{i_{j+n}}$$
$$\chi h_{j+n}(t_{j+n},\ldots,t_{j+m+2};\sigma)\chi^{-1}G_{j+n+1\quad(\mathrm{mod}\ 2)}$$
$$= v(\iota_j,i_j,\ldots,i_{j+m},\sigma'_{j+m+1}(i_{j+m+1}),i_{j+m+2},\ldots,i_{j+n}).$$

Second statement is clear.
(ii) First claim is obvious. Second claim follows from

$$h_j(i_j,\ldots,i_{j+m};\sigma'_{j+m+1})v(\iota_j,i_j,\ldots,i_{j+m})$$
$$= \gamma_j^{i_j}\cdots\gamma_{j+m\quad(\mathrm{mod}\ 2)}^{i_{j+m}}g_{j+m+1\quad(\mathrm{mod}\ 2)}(\sigma'_{j+m+1})(\gamma_{j+m\quad(\mathrm{mod}\ 2)}^{i_{j+m}})^{-1}\cdots(\gamma_j^{i_j})^{-1}$$
$$\times\gamma_j^{i_j}\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{i_{j+1}}\cdots\gamma_{j+m\quad(\mathrm{mod}\ 2)}^{i_{j+m}}G_{j+m+1\quad(\mathrm{mod}\ 2)}$$
$$= \gamma_j^{i_j}\cdots\gamma_{j+m\quad(\mathrm{mod}\ 2)}^{i_{j+m}}g_{j+m+1\quad(\mathrm{mod}\ 2)}(\sigma'_{j+m+1})G_{j+m+1\quad(\mathrm{mod}\ 2)}$$
$$= v(\iota_j,i_j,\ldots,i_{j+m}).$$

(iii) It follows by Proposition 3.5 (i) that this is equivalent to $g_j(\sigma'_j)\in G_{(T_{j+1(\mathrm{mod}\ 2)})}$. From $g_j(\sigma'_j)\in H$, it immediately follows that $g_j(\sigma'_j)\in G_{v(\iota_{j+1(\mathrm{mod}\ 2)})}=G_{j+1(\mathrm{mod}\ 2)}$. It remains to show that, for any $n\geq 1$ and any $i_t\in I'_t$, it follows $g_j(\sigma'_j)\in G_{v(\iota_{j+1\ (\mathrm{mod}\ 2)},i_{j+1},\ldots,i_{j+n})}$. From the argument at the beginning of the proof of point (i), it follows that

$$(\gamma_{j+n\quad(\mathrm{mod}\ 2)}^{i_{j+n}})^{-1}\cdots(\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{i_{j+1}})^{-1}=\chi'\gamma_{j+n\quad(\mathrm{mod}\ 2)}^{k_{j+n}}\cdots\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{k_{j+1}}$$

for some $k_l\in I'_l$ and some $\chi'\in H$. Consequently,

$$(\gamma_{j+n\quad(\mathrm{mod}\ 2)}^{i_{j+n}})^{-1}\cdots(\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{i_{j+1}})^{-1}g_j(\sigma'_j)\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{i_{j+1}}\cdots\gamma_{j+n\quad(\mathrm{mod}\ 2)}^{i_{j+n}}$$
$$= \chi h_{j+n\quad(\mathrm{mod}\ 2)}(\iota_{j+n\quad(\mathrm{mod}\ 2)},k_{j+n\quad(\mathrm{mod}\ 2)},\ldots,k_{j+1\quad(\mathrm{mod}\ 2)};\sigma'_j)\chi^{-1}.$$

Finally,

$$g_j(\sigma'_j)v(\iota_{j+1\quad(\mathrm{mod}\ 2)},i_{j+1},\ldots,i_{j+n})$$
$$= g_j(\sigma'_j)\gamma_{j+1\quad(\mathrm{mod}\ 2)}^{i_{j+1}}\cdots\gamma_{j+n\quad(\mathrm{mod}\ 2)}^{i_{j+n}}G_{j+n+1\quad(\mathrm{mod}\ 2)}$$

$$= \gamma_{j+1 \pmod 2}^{i_{j+1}} \cdots \gamma_{j+n \pmod 2}^{i_{j+n}} \cdot (\gamma_{j+n \pmod 2}^{i_{j+n}})^{-1} \cdots (\gamma_{j+1 \pmod 2}^{i_{j+1}})^{-1}$$
$$\cdot g_j(\sigma'_j) \gamma_{j+1 \pmod 2}^{i_{j+1}} \cdots \gamma_{j+n \pmod 2}^{i_{j+n}} G_{j+n+1 \pmod 2}$$
$$= \gamma_{j+1 \pmod 2}^{i_{j+1}} \cdots \gamma_{j+n \pmod 2}^{i_{j+n}}$$
$$\cdot \chi h_{j+n \pmod 2}(\iota_{j+n \pmod 2}, k_{j+n \pmod 2}, \ldots, k_{j+1 \pmod 2}; \sigma'_j) \chi^{-1} G_{j+n+1 \pmod 2}$$
$$= \gamma_{j+1 \pmod 2}^{i_{j+1}} \cdots \gamma_{j+n \pmod 2}^{i_{j+n}} G_{j+n+1 \pmod 2}$$
$$= v(\iota_{j+1 \pmod 2}, i_{j+1}, \ldots, i_{j+n}).$$

(iv) From Lemma 3.11 (i), we write

$$h_j(i_j, \ldots, i_{j+m}; \sigma'_{j+m+1}) =$$
$$= \gamma_j^{i_j} \cdots \gamma_{j+m \pmod 2}^{i_{j+m}} g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})(\gamma_{j+m \pmod 2}^{i_{j+m}})^{-1} \cdots (\gamma_j^{i_j})^{-1}.$$

By (iii), we have $g_{j+m+1 \pmod 2}(\sigma'_{j+m+1}) \in K_{j+m \pmod 2}$, and therefore, by definition and by Lemma 3.2, it follows

$$h_j(i_j, \ldots, i_{j+m}; \sigma'_{j+m+1}) \in K(\gamma_j^{i_j} \cdots \gamma_{j+m \pmod 2}^{i_{j+m}}) < K_{j+1 \pmod 2}.$$

(v) Note that in $\gamma \equiv (\gamma_{j+m \pmod 2}^{s_{j+m}})^{-1} \cdots (\gamma_j^{s_j})^{-1} \gamma_j^{i_j} \cdots \gamma_{j+n \pmod 2}^{i_{j+n}}$ there is an even number $0 \le 2r < 2m+2$ of cancellations. Therefore

$$\gamma \in T_{j+m \pmod 2, m+n+2-2r-1} = T_{j+m \pmod 2, m+n+1-2r}$$

starts with an element of $G_{j+m \pmod 2}$ and ends with an element of $G_{j+n \pmod 2}$. By Lemma 3.11 (iii), γ can be written as

$$\gamma = \gamma_{j+m \pmod 2}^{k_{j+m}} \cdots \gamma_{j+2m+n-2r \pmod 2}^{k_{j+2m+n-2r}} h = \gamma_{j+m \pmod 2}^{k_{j+m}} \cdots \gamma_{j+n \pmod 2}^{k_{j+2m+n-2r}} h$$

for some $k_t \in I'_t$. Then,

$$h_j(s_j, \ldots, s_{j+m}; \sigma'_{j+m+1}) \in G_{v(\iota_j, i_j, \ldots, i_{j+m}, \ldots, i_{j+n})} \iff$$
$$\gamma_j^{s_j} \cdots \gamma_{j+m \pmod 2}^{s_{j+m}} g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})(\gamma_{j+m \pmod 2}^{s_{j+m}})^{-1} \cdots$$
$$(\gamma_j^{s_j})^{-1} \in G_{v(\iota_j, i_j, \ldots, i_{j+m}, \ldots, i_{j+n})} \iff$$
$$g_{j+m+1 \pmod 2}(\sigma'_{j+m+1}) \in (\gamma_{j+m \pmod 2}^{s_{j+m}})^{-1} \cdots$$
$$(\gamma_j^{s_j})^{-1} G_{v(\iota_j, i_j, \ldots, i_{j+m}, \ldots, i_{j+n})} \gamma_j^{s_j} \cdots \gamma_{j+m \pmod 2}^{s_{j+m}} \iff$$
$$g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})$$

$$\in G_{(\gamma_{j+m \pmod 2}^{s_{j+m}})^{-1}\cdots(\gamma_j^{s_j})^{-1}v(\iota_j,i_j,\ldots,i_{j+m},\ldots,i_{j+n})} \iff$$
$$g_{j+m+1 \pmod 2}(\sigma'_{j+m+1})$$
$$\in G_{(\gamma_{j+m \pmod 2}^{s_{j+m}})^{-1}\cdots(\gamma_j^{s_j})^{-1}\gamma_j^{i_j}\cdots\gamma_{j+n \pmod 2}^{i_{j+n}}G_{j+n+1 \pmod 2}} \iff$$
$$g_{j+m+1 \pmod 2}(\sigma'_{j+m+1}) \in G_{\gamma_{j+m \pmod 2}^{k_{j+m}}\cdots\gamma_{j+n \pmod 2}^{k_{j+2m+n-2r}}hG_{j+n+1 \pmod 2}} \iff$$
$$g_{j+m+1 \pmod 2}(\sigma'_{j+m+1}) \in G_{v(\iota_{j+m \pmod 2},k_{j+m}\ldots,k_{j+2m+n-2r})}.$$

Last line holds according to (iii). □

Proposition 3.13. *For a group $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$, the following hold:*
(i) $K_j = Q_{j+1 \pmod 2}$.
(ii) $\ker G = \{1\}$.

Proof. (i) It follows by Lemma 3.12 (iii) and (iv) that $Q_j < K_{j+1 \pmod 2}$ and by Lemma 3.1 that $K_{j+1 \pmod 2} \lhd H$. Therefore

$$Q_j < K_{j+1 \pmod 2} \lhd H.$$

Since Q_0 and Q_1 commute and together generate H, it is enough to show that if $h \in Q_j$, then $h \notin K_j$.

For $\sigma'_j \in \Gamma'_j \setminus \{1\}$ (if this is nonempty), let $\rho, \kappa \in I'_j$ be such that $\rho \neq \kappa$ and $\sigma'_j(\rho) = \kappa$. Then, $g_j(\sigma'_j)v(\iota_j, \rho) = v(\iota_j, \kappa)$ by Lemma 3.12 (i). Therefore, $g_j(\sigma'_j) \notin G_{(T_j)} = K_j$.

Take an element $h \in Q_j$, and assume $h \neq g_j(\sigma'_j)$. Then, by Lemma 3.11 (ii) and (R2), it can be written as

$$h = g_j(\sigma'_j) \cdot h_1 \cdot h_2 \cdot h_j(s_j, \ldots, s_{j+n}; \omega'),$$

where

$$h_1 = \prod_{k=1}^{m} h_j(i_j^k, \ldots, i_{j+n_k}^k; \theta'_k) \text{ and } h_2 = \prod_{l=m+1}^{r} h_j(i_j^l, \ldots, i_{j+n}^l; \xi'_l).$$

In the above expression, $r \geq m \geq 0$, $\sigma'_j \in \Gamma'_j$, $\theta'_k \in \Gamma'_{j+n_k+1 \pmod 2}$, $\omega', \xi'_l \in \Gamma'_{j+n+1 \pmod 2}$, $0 \leq n_1 \leq \cdots \leq n_m < n$, $i_t^p \in I'_{t \pmod 2}$, and $(i_j^l, \ldots, i_{j+n}^l) \neq (s_j, \ldots, s_{j+n})$, $\forall l \in \{m+1, \ldots, r\}$.

Since $\omega' \in \Gamma'_{j+n+1 \pmod 2}$ is nontrivial, there exist $\rho, \kappa \in I'_{j+n+1 \pmod 2}$ with $\rho \neq \kappa$ that satisfy $\omega'(\rho) = \kappa$. It follows from Lemma 3.12 (i) that

$$h_j(s_j, \ldots, s_{j+n}; \omega')v(\iota_j, s_j, \ldots, s_{j+n}, \rho) = v(\iota_j, s_j, \ldots, s_{j+n}, \kappa).$$

It follows from Lemma 3.12 (v) that

$$h_2 v(\iota_j, s_j, \ldots, s_{j+n}, \kappa) = v(\iota_j, s_j, \ldots, s_{j+n}, \kappa)$$

and again from Lemma 3.12 (i) that

$$g_j(\sigma'_j) h_1 v(\iota_j, s_j, \ldots, s_{j+n}, \kappa) = v(\iota_j, p_j \ldots, p_{j+n}, \kappa)$$

for some $p_t \in I'_t$.

We conclude that

$$hv(\iota_j, s_j, \ldots, s_{j+n}, \rho) = v(\iota_j, p_j \ldots, p_{j+n}, \kappa) \;\neq\; v(\iota_j, s_j, \ldots, s_{j+n}, \rho)$$

and therefore $h \notin G_{(T_j)} = K_j$.

(ii) We have by the definition of the group $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$ that $Q_1 \cap Q_0 = \{1\}$. Therefore $\ker G = K_0 \cap K_1 = Q_1 \cap Q_0 = \{1\}$. $\square$

We turn to the structure of the groups $K_j = Q_{j+1 \pmod 2}$ for $j = 0, 1$. From the construction of $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$, it is clear that the following holds:

$$\mathcal{K}'(j,n) = \langle\ \{\ h_j(i_j, i_{j+1}, \ldots, i_{j+n}; \sigma_{j+n+1}) \mid n \in \mathbb{Z}_0,\ i_k \in I'_{k \pmod 2},\ \sigma_{j+n+1} \in \Gamma'_{j+n+1 \pmod 2}\ \}\ \rangle.$$

Therefore there is a representation

$$Q_{j+1 \pmod 2} \cong \mathcal{K}'(j,1) \rtimes \Gamma'_j \tag{8}$$

with the obvious action. It can be written also 'recursively' as the wreath product

$$Q_{j+1 \pmod 2} \cong Q_j \wr_{I'_j} \Gamma'_j, \tag{9}$$

where Γ'_j acts on $\#(I'_j)$ coppies of $K_{j+1 \pmod 2} \cong Q_j$ in the obvious way.

We want to express Q_j as a direct limit of wreath products. Let's denote

$$Q_j(0) \;=\; \{\ g_j(\sigma'_j) \mid \sigma'_j \in \Gamma'_j\ \} \;\cong\; \Gamma'_j.$$

For $n \geq 1$, let's denote

$$Q_j(n) = \langle\ \{\ h_j(i_j, i_{j+1}, \ldots, i_{j+n}; \sigma_{j+n+1}) \mid i_k \in I'_{k \pmod 2},\ \sigma_{j+n+1} \in \Gamma'_{j+n+1 \pmod 2}\ \}\ \rangle.$$

Note that $Q_j(n)$ is isomorphic to a direct sum of copies of $\Gamma'_{j+n+1 \pmod 2}$. Finally, let's denote

$$Q_j[n] \;=\; \langle\, Q_j(0) \cup Q_j(1) \cup \cdots \cup Q_j(n) \,\rangle.$$

Then it is easy to see by the construction that $Q_j(1)$ is isomorphic to the direct sum of $\#(I'_j)$ coppies of $\Gamma'_{j+1 \pmod 2}$ and that $Q_j[1] \cong \Gamma'_{j+1 \pmod 2} \wr_{I'_j} \Gamma'_j$. Relation (R3) immediately implies that $Q_j(n) \lhd Q_j[n]$ and that there is an extension

$$\{1\} \longrightarrow Q_j(n) \longrightarrow Q_j[n] \longrightarrow Q_j[n-1] \longrightarrow \{1\}. \tag{10}$$

We can actually write

$$Q_j[n] \;\cong \Gamma'_{j+n \pmod 2} \wr_{I'_{j+n-1 \pmod 2}} \Gamma'_{j+n-1 \pmod 2} \wr_{I'_{j+n-2 \pmod 2}} \cdots \wr_{I'_{j+1 \pmod 2}} \Gamma'_{j+1 \pmod 2} \wr_{I'_j} \Gamma'_j.$$

Since we have the natural embeddings $Q_j[m] \hookrightarrow Q_j[n]$ for $0 \le m \le n$, it is clear that Q_j is the direct limit group of the groups $Q_j[n]$, i.e.,

$$Q_j \;=\; \varinjlim_n Q_j[n]. \tag{11}$$

Now we observe that

Lemma 3.14. *Q_0 is amenable if and only if Q_1 is amenable, if and only if both groups Γ'_0 and Γ'_1 are amenable.*

Proof. If we suppose that, say, Γ'_0 is not amenable, then by equation (9) for $j = 0$, it will follow that Q_0 is not amenable, and by equation (9) for $j = 1$, it will follow that Q_1 is not amenable.

Conversely, suppose that both Γ'_0 and Γ'_1 are amenable. Then $Q_j(n)$ is amenable being isomorphic to a direct sum of copies of the group $\Gamma'_{j+n+1 \pmod 2}$. Also $Q_j[0] = Q_j(0) \cong \Gamma'_j$ is amenable, and by equation (10) for $n = 1$, it follows that $Q_j[1]$ is also amenable.

Equation (10) and an easy induction establish the amenability of $Q_j[n]$ for each $n \in \mathbb{Z}_0$.

Finally, we see from the direct limit representation of Q_j (equation (11)) that Q_j is amenable. □

3.4. Group-theoretic structure. First, we need an easy lemma re-expressing our conditions.

Lemma 3.15. *Let G be a group, and let H be its subgroup. Assume that $G = \langle aHa^{-1} \mid a \in G\rangle$. Then:*
(i) $G = \langle H \cup [G,G]\rangle$.
(ii) $[G,G] = \langle ghg^{-1}h^{-1} \mid g \in G,\ h \in \bigcup_{a\in G} aHa^{-1}\rangle$.

Proof. (i) Every element $g = a_1h_1a_1^{-1}a_2h_2a_2^{-1}a_3h_3a_3^{-1}\cdots a_nh_na_n^{-1}$ can be written as

$$g = (a_1h_1a_1^{-1}h_1^{-1})h_1(a_2h_2a_2^{-1}h_2^{-1})h_2\cdots(a_nh_na_n^{-1}h_n^{-1})h_n.$$

(ii) Take $g \in G$ and $g' = h_1\dots h_n$, where $h_i \in \bigcup_{a\in G} aHa^{-1}$. Then $g' = h_1 \cdot h'$, where $h' = h_2\cdots h_n$ is a product of $n-1$ members of $\bigcup_{a\in G} aHa^{-1}$. Then

$$gg'g^{-1}(g')^{-1} = gh_1h'g^{-1}(h')^{-1}h_1^{-1} = (gh_1g^{-1}h_1^{-1})h_1(gh'g^{-1}(h')^{-1})h_1^{-1} = (gh_1g^{-1}h_1^{-1})\cdot(h_1gh_1^{-1})(h_1h'h_1^{-1})(h_1gh_1^{-1})^{-1}(h_1h'h_1^{-1})^{-1}.$$

In the last expression, $h_1h'h_1^{-1}$ is a product of $n-1$ members of $\bigcup_{a\in G} aHa^{-1}$. A simple induction completes the proof. □

Theorem 3.16. *Let $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$. Assume that:*
(i) Γ_0 and Γ_1 are 2-transitive, that is, all stabilizers $(\Gamma_k)_{i_k}$ are transitive on the sets $I_k \setminus \{i_k\}$, respectively, for all $i_k \in I_k$, $k = 0, 1$.
(ii) Either $\Gamma_k = \langle(\Gamma_k)_{i_k} \mid i_k \in I_k\rangle$, or $\Gamma_k = Sym(2)$ for $k = 0, 1$.
Then there is a group extension

$$1 \longrightarrow N \longrightarrow G \stackrel{\theta}{\longrightarrow} (\Gamma_0/[\Gamma_0,\Gamma_0]) \times (\Gamma_1/[\Gamma_1,\Gamma_1]) \longrightarrow 1,$$

where N is a simple, normal subgroup of G and where θ is defined on the generators by

$$\theta(g_0(\sigma_0)) = ([\sigma_0]_0, 1),\ \theta(g_1(\sigma_1)) = (1, [\sigma_1]_1),\ \text{and}$$

$$\theta(h_j(i_j, i_{j+1}, \dots, i_{j+n}; \sigma_{j+n+1})) = \begin{cases} ([\sigma_{j+n+1}]_0, 1) & \text{if } j+n+1 \text{ is even,} \\ (1, [\sigma_{j+n+1}]_1) & \text{if } j+n+1 \text{ is odd.} \end{cases}$$

Here $[\sigma]_k$ denotes the image of the permutation $\sigma \in \Gamma_k$ in $\Gamma_k/[\Gamma_k,\Gamma_k]$ for $k = 0, 1$.

Proof. First, we can define homomorphisms $\theta_0 : G_0 \to \Gamma_0/[\Gamma_0,\Gamma_0]$ and $\theta_1 : G_1 \to \Gamma_1/[\Gamma_1,\Gamma_1]$ given on the generators by

$$\theta_0(g_0(\sigma_0)) = ([\sigma_0]_0, 1),\ \theta_1(g_1(\sigma_1)) = (1, [\sigma_1]_1),\ \text{and}$$

$$\theta_k(h_j(i_j, i_{j+1}, \ldots, i_{j+n}; \sigma_{j+n+1})) = \begin{cases} ([\sigma_{j+n+1}]_0, 1) & \text{if } j+n+1 \text{ is even,} \\ (1, [\sigma_{j+n+1}]_1) & \text{if } j+n+1 \text{ is odd,} \end{cases}$$

where $k = 0, 1$. This is possible since the respective commutators are in the kernels. Next, we observe that $\theta_0|_H = \theta_1|_H$ and use to the universal property of the group amalgamations (Remark 1.1) to define θ.

Observe that for $g \in G$, according to the definition of θ, we have

$$\theta(g) = \left(\prod\{[\sigma]_0 \mid \sigma \in \Gamma_0 \text{ is presented in } g\}, \prod\{[\tau]_1 \mid \tau \in \Gamma_1 \text{ is presented in } g\}\right).$$

Therefore, by Lemma 3.11, it easily follows that N is generated by the set

$$\begin{gathered} \{g_j(\sigma_j) \mid \sigma_j \in \Gamma_j,\ [\sigma_j]_j = 1,\ j = 0,1\} \cup \\ \{g_j(\sigma_j)h_k(i_k, \ldots, i_{2l+j-1}; \tau_{j+2l}) \mid l \in \mathbb{N}_0,\ \sigma_j, \tau_{j+2l} \in \Gamma'_j,\ [\sigma_j\tau_{j+2l}]_j = 1, \\ i_m \in I'_{m \pmod 2},\ j,k = 0,1\} \cup \\ \{h_j(i_j, \ldots, i_{2t+j-1}; \sigma_{j+2t})h_k(s_k, \ldots, s_{2l+j-1}; \tau_{j+2l}) \mid \\ [\sigma_{j+2t}\tau_{j+2l}]_j = 1,\ t,l \in \mathbb{N}_0,\ j,k = 0,1,\ \sigma_{j+2t}, \tau_{j+2l} \in \Gamma'_j, i_m, s_m \in I'_{m \pmod 2}\}. \end{gathered}$$

Consequently, it is easy to see that the following set generates N:

$$\begin{gathered} \{g_k(\sigma_k) \mid \sigma_k \in \Gamma_k,\ [\sigma_k]_k = 1,\ k = 0,1\} \cup \\ \{g_k(\sigma_k)h_{k+1 \pmod 2}(i_{k+1 \pmod 2}; \sigma_k^{-1}) \mid \sigma_k \in \Gamma'_k,\ k = 0,1\} \cup \\ \{h_0(i_0, \ldots, i_n; \sigma_{n+1})h_1(s_1, \ldots, s_n; \sigma_{n+1}^{-1}) \mid n \in \mathbb{N}, \\ i_k, s_k \in I'_{k \pmod 2}, \sigma_{n+1} \in \Gamma'_{n+1 \pmod 2}\} \cup \\ \{h_0(i_0, \ldots, i_{n-1}; \sigma_n)h_1(s_1, \ldots, s_{n+1}; \sigma_n^{-1}) \mid n \in \mathbb{N}, \\ i_k, s_k \in I'_{k \pmod 2},\ \sigma_n \in \Gamma'_{n \pmod 2}\}. \end{gathered} \tag{12}$$

Now, let $a \in N \setminus \{1\}$ be arbitrary. We need to show that the normal closure, $\langle\langle\ a\ \rangle\rangle_N$, of $a \in N$ coincides with N. Relation (R3) shows that shorter h's modify longer ones, so this observation, together with relation (R6), show that for an element $h_0(i_0, \ldots, i_m; \sigma_{m+1})$ with a large enough length $m \in \mathbb{N}$ and appropriate i_k's, we have

$$ah_0(i_0, \ldots, i_m; \sigma_{m+1}) = h_j(s_j, \ldots, s_{2n+m}; \sigma_{m+1})a$$

for some $j \in \{0, 1\}$, some $n \in \mathbb{Z}$, and some s_k's. Therefore, after observing that $h_0(i_0, \ldots, i_m; \sigma_{m+1})$ and $h_j(s_j, \ldots, s_{2n+m}; \sigma_{m+1})$ commute for appropriate

i_k's (and are different), it is easy to see that for this choice of i_k's, we obtain the following element of $\langle\langle\ a\ \rangle\rangle_N$ (let's call it a'):

$$\begin{aligned}&h_0(i_0,\dots,i_m;\sigma_{m+1})h_j(s_j,\dots,s_{2n+m};\sigma_{m+1}^{-1})ah_j(s_j,\dots,s_{2n+m};\sigma_{m+1})\\ &\quad\times h_0(i_0,\dots,i_m;\sigma_{m+1}^{-1})a^{-1}=h_0(i_0,\dots,i_m;\sigma_{m+1})h_j(s_j,\dots,s_{2n+m};\sigma_{m+1}^{-1})\\ &\qquad\qquad\qquad\qquad\times h_l(t_l,\dots,t_{2p+m};\sigma_{m+1})h_j(s_j,\dots,s_{2n+m};\sigma_{m+1}^{-1}),\end{aligned}$$

where $l\in\{0,1\}$, $p\in\mathbb{Z}$, and t_k's. Next, take $n>m$, an element $h_j(x_j,\dots,x_{j+n};\sigma)$ that doesn't commute with a', and a $h_k(y_k,\dots,y_{j+n+2w};\sigma^{-1})$ that does commute with a'. Then we obtain the following element of $\langle\langle\ a\ \rangle\rangle_N$:

$$\begin{aligned}&h_j(x_j,\dots,x_{j+n};\sigma)h_k(y_k,\dots,y_{j+n+2w};\sigma^{-1})a'h_k(y_k,\dots,y_{j+n+2w};\sigma)\\ &\times h_j(x_j,\dots,x_{j+n};\sigma^{-1})(a')^{-1}=h_j(x_j,\dots,x_{j+n};\sigma)h_j(x'_j,\dots,x'_{j+n};\sigma^{-1})\ \equiv\ a''.\end{aligned}$$

Relation (R6) implies that we can find $\gamma\in G$, and after eventually multiplying γ by an element of the form $g_0(\sigma'_0)g_1(\sigma'_1)$, we can have $\gamma\in N$, and, finally, obtain the following element of $\langle\langle\ a\ \rangle\rangle_N$:

$$\gamma a''\gamma^{-1}=g_{j+n+1\pmod 2}(\sigma)h_r(z_r,\dots,z_{2q+j+n};\sigma^{-1})$$

for some $r\in\{0,1\}$, $\varepsilon\in\{-1,1\}$ and z_k's and for arbitrary $\sigma\in\Gamma'_{j+n+1\pmod 2}$. Since $n\in\mathbb{N}$ is also a large, arbitrary number, $\langle\langle\ a\ \rangle\rangle_N$ contain elements of the form

$$(13)\qquad g_0(\sigma'_0)h_r(z_r,\dots,z_{2q-1};(\sigma'_0)^{-1}),\ g_1(\sigma'_1)h_d(z'_d,\dots,z'_{2u};(\sigma'_1)^{-1}),$$

where $\sigma'_0\in\Gamma'_0$ and $\sigma'_1\in\Gamma'_1$ are arbitrary.

Take $\sigma_0\in\Gamma_0\setminus\Gamma'_0$ and consider $f=g_0(\sigma_0)h_r(z'_r,z_{r+1},\dots,z_{2q-1};(\sigma'_0)^{-1})\in G$, where $z''_r=z_r$ if $r=1$ and $z''_r=\sigma_0^{-1}\circ\sigma'_0(z_r)$ if $r=0$. Then

$$\begin{aligned}b&\equiv f^{-1}g_0(\sigma'_0)h_r(z_r,\dots,z_{2q-1};(\sigma'_0)^{-1})f\\ &=g_0(\sigma_0^{-1}\sigma'_0\sigma_0)h_r(z''_r,z_{r+1},\dots,z_{2q-1};(\sigma'_0)^{-1})\in\langle\langle\ a\ \rangle\rangle_N.\end{aligned}$$

Let $n\geq 1$. We can assume $w_0=\sigma_0^{-1}(\sigma'_0)^{-1}\sigma_0(\iota_0)\in I'_0$ because $\sigma'_0\in\Gamma'_0$ is arbitrary. Choose j_k's and v_k's such that $h_0(w_0,j_1,\dots,j_n;\tau_{n+1})$ and $h_0(\sigma_0^{-1}(\iota_0),v_1,\dots,v_{2m+n};\tau_{n+1}^{-1})$ both commute with $h_r(z'_r,z_{r+1},\dots,z_{2q-1};(\sigma'_0)^{-1})$ (by altering σ_0, j's, v's, or even the γ above if needed). Observe that $h_0(\sigma_0^{-1}(\iota_0),v_1,\dots,v_{2m+n};\tau_{n+1}^{-1})$ commutes with b. Define

$$c=h_0(w_0,j_1,\dots,j_n;\tau_{n+1})h_0(\sigma_0^{-1}(\iota_0),v_1,\dots,v_{2m+n};\tau_{n+1}^{-1})\ \in\ N.$$

Then $d = cbc^{-1}b^{-1} = h_0(w_0, j_1, \ldots, j_n; \tau_{n+1})h_1(j_1, \ldots, j_n; \tau_{n+1}^{-1}) \in \langle\langle\ a\ \rangle\rangle_N$.

In the above expression $\tau_{n+1} \in \Gamma'_{n+1 \pmod 2}$ is arbitrary. Relation (R3) can be used repeatedly on d to infer that the third set of (12) belongs to $\langle\langle\ a\ \rangle\rangle_N$.

The element $g_1(\sigma'_1)h_d(z'_d, \ldots, z'_{2u}; (\sigma'_1)^{-1}) \in \langle\langle\ a\ \rangle\rangle_N$ can be used in analogous way to infer that the fourth set of (12) belongs to $\langle\langle\ a\ \rangle\rangle_N$.

The second set of (12) is formed by products of the third and the fourth set of (12) with the elements (13), so it also belongs to (12).

Finally, we need to show that the first set of (12) is a subset of $\langle\langle\ a\ \rangle\rangle_N$. Let $k \in \{0, 1\}$. If $\Gamma_k = \mathrm{Sym}(2)$, then $[\Gamma_k, \Gamma_k]$ is trivial, so there is nothing to prove. Let's assume that $\Gamma_k \neq \mathrm{Sym}(2)$. In this situation, I'_k has at least two elements.

Now, take arbitrary $\gamma_k, \tau_k \in \Gamma_k$ with representations (from Lemma 3.15 (i))

$$\begin{aligned} \gamma_k^{-1}\tau_k\gamma_k &= a_k^1 b_k^1 \cdots a_k^n b_k^n, \text{ where } a_k^j \in [\Gamma_k, \Gamma_k],\ b_k^j \in \Gamma'_k \text{ and} \\ \gamma_k &= c_k^1 d_k^1 \cdots c_k^m d_k^m, \text{ where } c_k^j \in [\Gamma_k, \Gamma_k],\ d_k^j \in \Gamma'_k, \end{aligned}$$

and let $\sigma_k \in \Gamma'_k$ be arbitrary. Consider

$$p_k = h_k(i_k, i_{k+1 \pmod 2}, i_k, i_{k+1 \pmod 2}; b_k^n)g_k(a_k^n b_k^n) \in N \text{ and}$$

$$q_k = g_k(\sigma_k)h_k(i'_k, i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; (\sigma_k)^{-1}) \in \langle\langle\ a\ \rangle\rangle_N,$$

where $i_{k+1 \pmod 2} \in I'_{k+1 \pmod 2}$, $i_k, i'_k, i''_k \in I'_k$, $i'''_k \equiv a_k^n b_k^n \sigma_k^{-1}(a_k^n b_k^n)^{-1}(i_k) \neq \iota_k$, and where $a_k^n b_k^n(i'_k) \neq \iota_k$. Then,

$$\begin{aligned} \langle\langle\ a\ \rangle\rangle_N \ni\ & p_k q_k p_k^{-1} \\ &= g_k(a_k^n b_k^n \sigma_k (a_k^n b_k^n)^{-1})h_k(a_k^n b_k^n(i'_k), i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; (\sigma_k)^{-1}) \\ \cdot h_k(i'''_k, & i_{k+1 \pmod 2}, i_k, i_{k+1 \pmod 2}; b_k^n)h_k(i_k, i_{k+1 \pmod 2}, i_k, i_{k+1 \pmod 2}; (b_k^n)^{-1}). \end{aligned}$$

In the last expression, the product of the last two factors belongs to $\langle\langle\ a\ \rangle\rangle_N$, and therefore

$$g_k(a_k^n b_k^n \sigma_k (a_k^n b_k^n)^{-1})h_k(a_k^n b_k^n(i'_k), i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; (\sigma_k)^{-1}) \in \langle\langle\ a\ \rangle\rangle_N.$$

Next, we can play the same game with p_k replaced by

$$h_k(i_k, i_{k+1 \pmod 2}, i_k, i_{k+1 \pmod 2}; b_k^{n-1})g_k(a_k^{n-1} b_k^{n-1})$$

and q_k replaced by

$$g_k(a_k^n b_k^n \sigma_k (a_k^n b_k^n)^{-1}) h_k(i'_k, i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; (\sigma_k)^{-1})$$

for appropriate i's. After n steps, we will get

$$g_k(\gamma_k^{-1}\tau_k\gamma_k\sigma_k\gamma_k^{-1}\tau_k^{-1}\gamma_k) h_k(i'_k, i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; (\sigma_k)^{-1}) \ \in \ \langle\langle\ a\ \rangle\rangle_N.$$

Finally, after multiplying the last element on the right by

$$h_k(i'_k, i_{k+1 \pmod 2}, i''_k, i_{k+1 \pmod 2}; \sigma_k) g_k(\sigma_k^{-1}) \ \in \ \langle\langle\ a\ \rangle\rangle_N,$$

we will arrive at

$$g_k(\gamma_k^{-1}\tau_k\gamma_k\sigma_k\gamma_k^{-1}\tau_k^{-1}\gamma_k\sigma_k^{-1}) \ \in \ \langle\langle\ a\ \rangle\rangle_N.$$

This procedure can be repeated using $c_k^m d_k^m, \ \ldots \ , c_k^1 d_k^1$, so we conclude that

$$g_k(\tau_k \cdot (\gamma_k\sigma_k\gamma_k^{-1}) \cdot \tau_k^{-1} \cdot (\gamma_k\sigma_k^{-1}\gamma_k^{-1})) \ \in \ \langle\langle\ a\ \rangle\rangle_N.$$

Remembering that in the last expression, τ_k and γ_k were arbitrary elements of Γ_k and σ_k was an arbitrary element of Γ'_k, and using Lemma 3.15 (ii), we conclude that for, $k = 0, 1$,

$$\{g_k(\sigma_k) \mid [\sigma_k]_k = 1, \ \sigma_k \in \Gamma_k\} \ \subset \ \langle\langle\ a\ \rangle\rangle_N.$$

We have established that $\langle\langle\ a\ \rangle\rangle_N$ contains all sets from (12), and therefore $N = \langle\langle\ a\ \rangle\rangle_N$. □

Remark 3.17. Note that the example introduced in [12, Section 4] corresponds to the special case $\Gamma_0 \cong \Gamma_1 \cong \mathrm{Sym}(3)$, and [12, Proposition 4.5] is the respective version of Theorem 3.16.

Remark 3.18. This remark is related to the simplicity criteria of [14, Section 4] and uses notations thereof. It is easy to see that our groups satisfy the edge independence property, so [14, Corollary 4.6] can be applied. It follows that N contains the group $\langle [G_e, G_e] \mid e \text{ is an edge of } T\rangle$. Since the last group is normal in G and since N is simple, it follows that

$$N \ = \ \langle [G_e, G_e] \mid e \text{ is an edge of } T\rangle.$$

This can also be obtained directly.

Remark 3.19. Our groups provide examples that are not highly transitive. By definition, a group G is highly transitive if there is a faithful action on a set Ω with the following property: For each $n \in \mathbb{N}$ and every two n-tuples $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ there is an element $g \in G$, s. t. $g(i_k) = j_k$, $\forall k = 1, \ldots, n$. It is shown in [9, Corollary 7.7] the a group amalgamation $G = G_0 *_H G_1$ is highly transitive if H is core-free in one of the groups G_0 or G_1. If, instead, one requires that H is core-free in G, G may be not highly transitive. This is the case when the action of G on its Bass-Serre tree is is minimal, of general type, and not topologically free, as shown in [16, Theorem 1.4]. Therefore our examples are not highly transitive. See also [9, Section 8.3].

3.5. Analytic structure.

Lemma 3.20. *The action of each group $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$ on its Bass-Serre tree is minimal and of general type.*

Proof. Since the action is transitive, it is minimal. Also, the Bass-Serre tree is not a linear tree by Remark 3.8. The result now follows from [11, Proposition 19 (ii)]. □

Theorem 3.21. *The amalgamated free product $G = G[\Gamma_0, \Gamma_1]$ has the unique trace property. It is C^*-simple if and only if either one of the groups Γ_0' or Γ_1' is non-amenable.*

Proof. Since G is a nondegenerate amalgam by Remark 3.8, Proposition 3.13 (ii) and [12, Proposition 3.1] establish the first part. Since the action of G on its Bass-Serre tree is minimal and of general type by Lemma 3.20, [5, Theorem 3.9], Proposition 3.13 (i), Proposition 3.5, and Lemma 3.14 establish the second part. □

Now, we prove

Theorem 3.22. *The amalgamated free product $G = G[\Gamma_0, \Gamma_1]$ in not inner amenable.*

Proof. Lemma 3.20 allows us to apply Proposition 2.3. Therefore we need to show that the action of $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$ on its Bass-Serre is finitely fledged.

For this, take any elliptic element $g \in G \setminus \{1\}$. Since g fixes some vertex, it is a conjugate of an element of G_j, where either $j = 0$, or $j = 1$. The finite fledgedness property is conjugation invariant, so we can assume $g \in G_j \setminus \{1\}$.

It follows from Lemma 3.11 (ii) and (iii) that we can write $g = g_j(\sigma_j)h_0h_1$, where $\sigma_j \in \Gamma_j$,

$$h_0 = \prod_{k=1}^{m} h_0(i_0^k, \ldots, i_{0+n_k}^k; \theta_k'), \quad h_1 = \prod_{l=m+1}^{r} h_1(i_1^l, \ldots, i_{1+n_l}^l; \xi_l'),$$

$r \geq m \geq 0$, $\theta_k' \in \Gamma'_{j+n_k+1 \pmod 2}$, $\xi_l' \in \Gamma'_{j+n_l+1 \pmod 2}$, and $i_z^p \in I'_{z \pmod 2}$. We also require $0 \leq n_1 \leq \cdots \leq n_m$ and $0 \leq n_{m+1} \leq \cdots \leq n_r$.

Let's assume that g fixes a vertex $v = v(\iota_t, i_t, \ldots, i_{t+n})$, where $n \geq \max\{n_m, n_r\} + 1$, and take $w = v(\iota_t, i_t, \ldots, i_{t+n}, j_{t+n+1}, \ldots, j_{t+n+d})$ for any $d \geq 1$ and any $j_k \in I'_{k \pmod 2}$. We note that, $h_{t+1 \pmod 2}$ fixes w and h_t modifies only indices with numbers no greater than $\{n_m, n_r\} + 1 \leq n$. Therefore

$$h_t v = v(\iota_t, i_t', \ldots, i_{t+n}')\ , \text{ so } h_t w = v(\iota_t, i_t', \ldots, i_{t+n}', j_{t+n+1}, \ldots, j_{t+n+d})$$

for some $i_k' \in I'_{k \pmod 2}$. It follows from our assumption that

$$v \ = \ g_j(\sigma_j) h_t v \ = \ g_j(\sigma_j) v(\iota_t, i_t', \ldots, i_{t+n}').$$

Form the way in which $g_j(\sigma_j)$ acts on the vertices, it follows that $v = v(\iota_t, i_t', \ldots, i_{t+n}')$ and that $g_j(\sigma_j)v = v$. Consequently,

$$h_t w \ = \ w \quad \text{and} \quad g_j(\sigma_j) w \ = \ w,$$

so $gw = w$.

It is easy to see that this concludes the proof. □

Remark 3.23. In the situation when G_0 and G_1 are exact (e.g., amenable), $G[\Gamma_0, \Gamma_1] = G_0 *_H G_1$ is an exact group by [7, Corollary 3.3]. Additionally, when the quasi-kernels K_0 and K_1 are amenable (i.e., Γ_0 and Γ_1 are amenable), then $G[\Gamma_0, \Gamma_1]$ is not C^*-simple but has the unique trace property. In this situation, since the canonical tracial state is faithful, [18, Theorem 5] implies that the reduced C^*-algebras of such groups have stable ranks bigger than one. By definition, a C^*-algebra A has stable rank one if the set of invertible elements of A is dense in A.

For the next corollay, we recall that we called a group amenablish if it has no nontrivial C^*-simple quotients ([12, Definition 7.1]). We showed in [12] that the class on amenablish groups is a radical class, so every group has a unique maximal normal amenablish subgroup, the amenablish radical. Also, the class of amenablish groups is closed under extensions. The amenablish radical 'detects' C^*-simplicity the same way as the amenable radical 'detects' the unique trace property (see [12, Corollary 7.3] and [4, Theorem 1.3]).

Corollary 3.24. *Let* $G = G[I_0, I_1; \iota_0, \iota_1; \Gamma_0, \Gamma_1]$, *and suppose that the assumptions of Theorem 3.16 hold. Then:*
(i) If either Γ_0 *or* Γ_1 *is non-amenable, then the amenablish radical of* G *is trivial.*
(ii) If Γ_0 *and* Γ_1 *are both amenable, then* G *is amenablish.*

Proof. Let N be the simple, normal subgroup of Theorem 3.16. If we show that the centralizer $C_G(N)$ is trivial, then [4, Theorem 1.4] will imply that G is C^*-simple if and only if N is C^*-simple.

To do so, assume that there is a nontrivial $g \in C_G(N)$. Then g can be written as in Lemma 3.11 (iii), and using relations (R3), and (R6), we can find a non-trivial element of N

$$h_0(i_0, \ldots, i_{n-1}, j_n, \ldots, j_{n+m-1}; \sigma_{n+m}) \cdot h_1(i_1, \ldots, i_{n+1}, j'_n, \ldots, j'_{n+m-1}; \sigma^{-1}_{n+m})$$

that does not commute with g, a contradiction.

(i) If G and N are C^*-simple, then, since N is simple, its amenablish radical will be trivial. It follows that the amenablish radical of G will be trivial too because all nontrivial quotients of G are abelian. The result follows from Theorem 3.21.

(ii) If G and N are not C^*-simple, then, since N is simple, it will be amenablish. In that case G will also be amenablish because the class of amenablish groups is closed under extensions and since, by Theorem 3.16, G is an extension of the group N and the abelian group $(\Gamma_0/[\Gamma_0, \Gamma_0]) \times (\Gamma_1/[\Gamma_1, \Gamma_1])$. The result follows again fromTheorem 3.21. □

Acknowledgments. I want to thank Tron Omland for some useful comments and suggestions.

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Received April 4, 2021