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THE STRUCTURE OF $U(F(C_n \times D_8))$

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Abstract. Let D_n be the dihedral group of order n. The structures of the unit groups of the finite group algebras FD_8 and $F(C_2 \times D_8)$ over a field F of characteristic 2 are given in: L. Creedon, J. Gildea. The structure of the unit group of the group algebra $F_{2^k}D_8$. *Canad. Math. Bull.* **54**, *2* (2011), 237–243 and J. Gildea. Units of the group algebra $F_{2^k}(C_2 \times D_8)$. *J. Algebra Appl.* **10**, *4* (2011), 643–647, respectively. In this article, we establish the structure of the unit group of the group algebra $F(C_n \times D_8)$, $n \geq 1$ over a finite field F of characteristic p containing $q = p^k$ elements.

1. Introduction. Let $U(FG)$ be the unit group of the group algebra FG of a group G over a finite field F of characteristic p and let $V(FG)$ be the group of the normalized units of FG. It is well known that $U(FG) \cong V(FG) \times U(F)$. We denote by $J(FG)$, the Jacobson radical of FG. If G and H are groups, then $F(G \times H) \cong (FG)H$, the group ring of H over the ring FG, see [6, Chap 3, Page 134].

One of the interesting problems which arises in the study of group algebras is to find precisely the structure of the unit group of the group algebra. Explicit

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calculations in $U(FG)$ are usually difficult, even when G is fairly small. For a finite abelian group G, the structure of the semisimple group algebra FG and hence that of $U(FG)$ is given by Perlis and Walker in [8]. The unit group of a modular group algebra of a finite abelian p-group has been studied by Sandling in [9].

Let $D_{2n} = \langle x, y \mid x^n = y^2 = xyxy = 1\rangle$ be the dihedral group of order $2n$. The structures of $U(FD_8)$ and $U(F(C_2 \times D_8))$, where F is a finite field of characteristic 2 were obtained by Gildea, et al. in [1, 2], respectively. The unit group of the semisimple group algebra FD_{2^n} is given in [5]. In this article, our aim is to study the structure of $U(F(C_n \times D_8))$.

Throughout the paper, C_n is the cyclic group of order n, C_n^k is the direct product of k copies of C_n, F_n is the extension field of F of degree n and $GL(n, F)$ is the general linear group of degree n over F. For coprime integers l and m, $\mathrm{ord}_m(l)$ denotes the multiplicative order of l modulo m.

2. Preliminaries. We shall be using the following results for our work, so we enlist them here.

Theorem 2.1 ([5]). *Let F be a finite field of characteristic $p > 2$ containing $q = p^k$ elements. Then*

$$U(FD_8) \cong C_{q-1}^4 \times GL(2, F).$$

Theorem 2.2 ([4, Theorem 2.1]). *Let F be a finite field of characteristic p containing $q = p^k$ elements. If $(n, p) = 1$, then*

$$U(FC_n) \cong C_{q-1} \times \left(\prod_{l>1,\ l|n} C_{q^{d_l}-1}^{e_l} \right)$$

where $d_l = \mathrm{ord}_l(q)$ and $e_l = \dfrac{\phi(l)}{d_l}$.

Theorem 2.3 ([4, Theorem 2.4]). *Let F be a finite field of characteristic p containing $q = p^k$ elements. If $n = p^m n_1$, where $(n_1, p) = 1$ and $m \geq 1$, then*

$$U(FC_n) \cong U(FC_{p^m}) \times \left(\prod_{l>1,\ l|n_1} U(F_{d_l} C_{p^m})^{e_l} \right)$$

where $d_l = \mathrm{ord}_l(p^k)$ and $e_l = \dfrac{\phi(l)}{d_l}$.

It is not difficult to see that Theorem 2.4 in [9] proved for a field with p elements can be easily adopted for a field with p^k elements. Hence we have the following result:

Theorem 2.4. *Let F be a finite field of characteristic p containing $q = p^k$ elements and let G be a finite abelian p-group. Then*

$$U(FG) \cong C_{q-1} \times \prod_{i=1}^{e} C_{p^i}^{n_i}$$

where

1. *p^e is the exponent of G.*
2. *$n_i = k\left(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}|\right)$ for all $1 \le i \le e$.*

As a consequence of the previous theorem we have:

Theorem 2.5. *Let F be a finite field of characteristic 2 containing 2^k elements and let $G = C_2 \times C_{2^r}$. Then*

$$U(FG) \cong \begin{cases} C_2^k \times C_{2^k-1}. & \text{if } r = 0; \\ C_2^{3k} \times C_{2^k-1}, & \text{if } r = 1; \\ C_2^{5k} \times C_4^k \times C_{2^k-1}, & \text{if } r = 2; \\ C_2^{5.2^{r-2}k} \times \prod_{t=1}^{r-2} C_{2^{r-t}}^{2^{t-1}k} \times C_{2^r}^k \times C_{2^k-1}, & \text{if } r > 2. \end{cases}$$

3. Main results. Let $G = C_n \times D_8$ be presented as $G = \langle x, y, z \mid x^4 = y^2 = z^n = xyxy = 1, xz = zx, zy = yz\rangle$.

Theorem 3.1. *Let F be a finite field of characteristic 2 containing 2^k elements and let $G = C_n \times D_8$.*

1. *If n is odd, then*

$$U(FG) \cong (((C_4^{nk} \times C_2^{nk}) \rtimes C_4^{nk}) \times C_2^{nk}) \rtimes U(F(C_n \times C_2)).$$

2. *If n is even, then*

$$U(FG) \cong (((C_4^{\frac{n}{2}k} \times C_2^{2nk}) \rtimes C_4^{\frac{n}{2}k}) \times C_2^{2nk}) \rtimes U(F(C_n \times C_2)).$$

Proof. Let $K = \langle x \rangle$. Then $G/K \cong H \cong \langle y, z \mid y^2 = z^n = 1, yz = zy \rangle$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{j=0}^{n-1} \sum_{i=0}^{3} x^i z^j (a_{i+4j} + a_{i+4(n+j)} y) \mapsto \sum_{j=0}^{n-1} \sum_{i=0}^{3} z^j (a_{i+4j} + a_{i+4(n+j)} y)$$

we get a group epimorphism $\theta : U(FG) \to U(FH)$.

Further, from the inclusion map $i : FH \to FG$, we have $i : U(FH) \to U(FG)$ such that $\theta i = 1_{U(FH)}$. Therefore $U(FG)$ is a split extension of $U(FH)$ by $ker(\theta)$. Hence $U(FG) \cong V \rtimes U(FH)$, where $V = ker(\theta)$.

Let $u = \sum_{j=0}^{n-1} \sum_{i=0}^{3} x^i z^j (a_{i+4j} + a_{i+4(n+j)} y) \in U(FG)$, then $u \in V$ if and only if $\sum_{i=0}^{3} a_i = 1$ and $\sum_{i=0}^{3} a_{i+4j} = 0$ for $j = 1, 2, \ldots, (2n-1)$. Therefore

$$V = \{1 + \sum_{j=0}^{n-1} \sum_{i=1}^{3} (x^i - 1) z^j (b_{i+3j} + b_{i+3(n+j)} y) \mid b_i \in F\}$$

and $|V| = 2^{6nk}$. Since $ker(\theta)^4 = 0$, $V^4 = 1$. We now study the structure of V in the following steps:

Step 1: Let S be the subset of V consisting of elements of the form

$$1 + \sum_{i=0}^{n-1} p_i z^i (x + x^3) + \sum_{i=0}^{n-1} q_i z^i (y + x^3 y) + \sum_{i=0}^{n-1} r_i z^i (xy + x^2 y),$$

where $p_i, q_i, r_i \in F$. Then S is an abelian group and

$$S \cong \begin{cases} C_4^{nk} \times C_2^{nk}, & \text{if } n \text{ is odd;} \\ C_4^{\frac{n}{2}k} \times C_2^{2nk}, & \text{if } n \text{ is even.} \end{cases}$$

Let

$$s_1 = 1 + \sum_{i=0}^{n-1} p_i z^i (x + x^3) + \sum_{i=0}^{n-1} q_i z^i (y + x^3 y) + \sum_{i=0}^{n-1} r_i z^i (xy + x^2 y) \in S$$

and

$$s_2 = 1 + \sum_{i=0}^{n-1} a_i z^i (x + x^3) + \sum_{i=0}^{n-1} b_i z^i (y + x^3 y) + \sum_{i=0}^{n-1} c_i z^i (xy + x^2 y) \in S,$$

where $p_i, q_i, r_i, a_i, b_i, c_i \in F$. Then

$$\begin{aligned}s_1 s_2 =& 1 + \sum_{i=0}^{n-1}[(p_i + a_i)z^i + \gamma_1](x + x^3) + \sum_{i=0}^{n-1}[(q_i + b_i)z^i + \gamma_2](y + x^3 y)\\ &+ \sum_{i=0}^{n-1}[(r_i + c_i)z^i + \gamma_2](xy + x^2 y) \in S,\end{aligned}$$

where

$$\gamma_1 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(q_i b_j + q_i c_j + r_i b_j + r_i c_j)z^{i+j},$$
$$\gamma_2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(p_i b_j + p_i c_j + q_i a_j + r_i a_j)z^{i+j}.$$

Thus S is an abelian subgroup of V and $|S| = 2^{3nk}$. Let $S \cong C_4^{l_1} \times C_2^{l_2}$. Then $3nk = 2l_1 + l_2$. To obtain l_1 and l_2, let

$$W = \{s_1 \in S \mid s_1^2 = 1 \text{ and there exist } s_2 \in S \text{ such that } s_1 = s_2^2\}.$$

If $s_1 \in S$ is as defined above, then

$$s_1^2 = \begin{cases} 1 + \sum_{i=0}^{n-1}(q_i + r_i)^2 z^{2i}(x + x^3), & \text{if } n \text{ is odd;}\\ 1 + \sum_{i=0}^{\frac{n}{2}-1}(q_i + q_{i+\frac{n}{2}} + r_i + r_{i+\frac{n}{2}})^2 z^{2i}(x + x^3), & \text{if } n \text{ is even.}\end{cases}$$

Now we have two cases:

Case 1: If n is odd, then $s_1^2 = 1$ and $s_1 = s_2^2$, yield $q_i = r_i = 0$ for $i = 0, 1, \ldots, n-1$. Thus

$$W = \{1 + \sum_{i=0}^{n-1} p_i z^i (x + x^3) \mid p_i \in F\},$$

$l_1 = nk$ and $S \cong C_4^{nk} \times C_2^{nk}$.

Case 2: If n is even, then $s_1^2 = 1$ and $s_1 = s_2^2$, yield $q_i = r_i = 0$ for $i = 0, 1, \ldots, n-1$ and $p_i = 0$ for $i = 1, 3, \ldots, n-1$. Thus

$$W = \{1 + \sum_{i=0}^{\frac{n}{2}-1} p_{2i} z^{2i}(x + x^3) \mid p_{2i} \in F\},$$

$l_1 = \frac{n}{2}k$ and $S \cong C_4^{\frac{n}{2}k} \times C_2^{2nk}$.

Step 2: Let U be the subset of V consisting of elements of the form

$$1 + \sum_{j=1}^{n}\sum_{i=1}^{3}(1+x^i)s_{j_i}z^{j-1}(1+y)$$

where $s_{j_i} \in F$. Then U is an abelian group and

$$U \cong \begin{cases} C_4^{nk} \times C_2^{nk}, & \text{if } n \text{ is odd;} \\ C_4^{\frac{n}{2}k} \times C_2^{2nk}, & \text{if } n \text{ is even.} \end{cases}$$

Let

$$u_1 = 1 + \sum_{j=1}^{n}\sum_{i=1}^{3}(1+x^i)s_{j_i}z^{j-1}(1+y) \in U$$

and

$$u_2 = 1 + \sum_{j=1}^{n}\sum_{i=1}^{3}(1+x^i)t_{j_i}z^{j-1}(1+y) \in U,$$

where $s_{j_i}, t_{j_i} \in F$. Then

$$\begin{aligned} u_1u_2 =& 1 + \sum_{j=1}^{n}\sum_{i=1}^{3}(1+x^i)(s_{j_i}+t_{j_i})z^{j-1}(1+y) \\ &+ \widehat{x}\sum_{j=1}^{n}\sum_{k=1}^{n}(s_{j_1}+s_{j_3})(t_{k_1}+t_{k_3})z^{j+k-2}(1+y) \in U. \end{aligned}$$

So U is an abelian subgroup of V and $|U| = 2^{3nk}$. Let $U \cong C_4^{k_1} \times C_2^{k_2}$. Then $3nk = 2k_1 + k_2$. To obtain k_1 and k_2, let

$$T = \{u_1 \in U \mid u_1^2 = 1 \text{ and there exist } u_2 \in U \text{ such that } u_1 = u_2^2\}.$$

If $u_1 \in U$ is as defined above, then

$$u_1^2 = \begin{cases} 1 + \widehat{x}\sum_{i=1}^{n}(s_{i_1}+s_{i_3})^2 z^{2i-2}(1+y), & \text{if } n \text{ is odd;} \\ 1 + \widehat{x}\sum_{i=1}^{\frac{n}{2}}(s_{i_1}+s_{(i+\frac{n}{2})_1}+s_{i_3}+s_{(i+\frac{n}{2})_3})^2 z^{2i-2}(1+y), & \text{if } n \text{ is even.} \end{cases}$$

Now we have two cases:

Case 1: If n is odd, then $u_1^2 = 1$ and $u_1 = u_2^2$, yield $s_{i_1} = s_{i_2} = s_{i_3}$ for $i = 1, 2 \dots, n$. Thus

$$T = \{1 + \widehat{x} \sum_{i=1}^{n} s_i z^{i-1}(1+y) \mid s_i \in F\},$$

$k_1 = nk$ and $U \cong C_4^{nk} \times C_2^{nk}$.

Case 2: If n is even, then $u_1^2 = 1$ and $u_1 = u_2^2$, yield $s_{i_1} = s_{i_2} = s_{i_3}$ for $i = 1, 2 \dots, n$ and $s_{i_j} = 0$ for $i = 2, 4, \dots, n$ and $j = 1, 2, 3$. Thus

$$T = \{1 + \widehat{x} \sum_{i=1}^{\frac{n}{2}} s_{2i-1} z^{2i-2}(1+y) \mid s_{2i-1} \in F\},$$

$k_1 = \frac{n}{2}k$ and $U \cong C_4^{\frac{n}{2}k} \times C_2^{2nk}$.

Step 3: $V \cong \begin{cases} ((C_4^{nk} \times C_2^{nk}) \rtimes C_4^{nk}) \times C_2^{nk}, & \text{if } n \text{ is odd;} \\ ((C_4^{\frac{n}{2}k} \times C_2^{2nk}) \rtimes C_4^{\frac{n}{2}k}) \times C_2^{2nk}, & \text{if } n \text{ is even.} \end{cases}$

Let

$$s = 1 + \sum_{i=0}^{n-1} p_i z^i (x + x^3) + \sum_{i=0}^{n-1} q_i z^i (y + x^3 y) + \sum_{i=0}^{n-1} r_i z^i (xy + x^2 y) \in S$$

and

$$u = 1 + \sum_{j=1}^{n} \sum_{i=1}^{3} (1 + x^i) t_{j_i} z^{j-1}(1+y) \in U,$$

where $p_i, q_i, r_i, t_{j_i} \in F$, then

$$\begin{aligned} s^u &= 1 + \sum_{i=0}^{n-1} p_i z^i (x + x^3) + u^{-1}[\sum_{i=0}^{n-1} q_i z^i (1 + x^3) y + \sum_{i=0}^{n-1} r_i z^i (x + x^2) y] u \\ &= s + \widehat{x} \sum_{j=1}^{n} \sum_{i=0}^{n-1} (q_i + r_i)(t_{j_1} + t_{j_3}) z^{i+j-1} y \in S \end{aligned}$$

Thus U normalizes S. Since $S \cap U = 1$, $|SU| = 2^{6nk} = |V|$. Therefore $V = SU \cong S \rtimes U$. Now $s^u = s$ if and only if $t_{j_1} = t_{j_3}$ for $j = 1, 2, \dots, n$. If $t_{j_1} = t_{j_3}$ for $j = 1, 2, \dots, n$, then $u^2 = 1$. Therefore the elements of order 2 in U act trivially on S. Hence the assertion. □

For $U(F(C_n \times C_2))$, we prove the following:

Theorem 3.2. *Let F be a finite field of characteristic 2 containing 2^k elements and let $G_1 = C_{2^r s} \times C_2$, $r \geq 0$ such that $(2,s) = 1$. Then*

$$U(FG_1) \cong \begin{cases} C_2^{sk} \times C_{2^k-1} \times \prod_{l>1,\ l|s} C_{2^{d_l \cdot k}-1}^{e_l}, & \text{if } r = 0; \\ C_2^{3sk} \times C_{2^k-1} \times \prod_{l>1,\ l|s} C_{2^{d_l k}-1}^{e_l}, & \text{if } r = 1; \\ C_2^{5sk} \times C_4^{sk} \times C_{2^k-1} \times \prod_{l>1,\ l|s} C_{2^{d_l k}-1}^{e_l}, & \text{if } r = 2; \\ C_2^{5.2^{r-2}sk} \times \prod_{t=1}^{r-2} C_{2^{r-t}}^{2^{t-1}sk} \times C_{2^r}^{sk} \times C_{2^k-1} \times \prod_{l>1,\ l|s} C_{2^{d_l k}-1}^{e_l}, & \text{if } r > 2. \end{cases}$$

where $d_l = \mathrm{ord}_l(2^k)$ and $e_l = \dfrac{\phi(l)}{d_l}$.

Proof. Since $(2,s) = 1$, by [6, Chap 3, Page 134] and Theorem 2.2, we have

$$\begin{aligned} FG_1 &\cong (FC_s)(C_2 \times C_{2^r}) \\ &\cong (F \oplus \oplus_{l>1,\ l|s} F_{d_l}^{e_l})(C_2 \times C_{2^r}) \\ &\cong F(C_2 \times C_{2^r}) \oplus \oplus_{l>1,\ l|s} (F_{d_l}(C_2 \times C_{2^r}))^{e_l}. \end{aligned}$$

$d_l = \mathrm{ord}_l(2^k)$ and $e_l = \dfrac{\phi(l)}{d_l}$. Invoking Theorem 2.5, we have the result. □

It should be noted here that if $(2,n) = 1$, then $C_n \times C_2 = C_{2n}$ and so for the structure of $U(F(C_n \times C_2))$ we can apply Theorem 2.3 or we can take $r = 0$ and $s = n$ and use the above theorem.

Now we consider fields of characteristic $p > 2$.

Theorem 3.3. *Let F be a finite field of characteristic $p > 2$ containing $q = p^k$ elements and let $G = C_n \times D_8$ where $n = p^r s$, $r \geq 0$ such that $(p,s) = 1$. If $V = 1 + J(FG)$, then*

$$U(FG)/V \cong C_{q-1}^4 \times GL(2,F) \times \prod_{l>1,\ l|s} \left(C_{q^{d_l}-1}^4 \times GL(2,F_{d_l})\right)^{e_l},$$

where V is a group of exponent p^r and order $p^{2^3 sk(p^r-1)}$. Also, $d_l = \mathrm{ord}_l(q)$ and $e_l = \dfrac{\phi(l)}{d_l}$.

P r o o f. Let $K = \langle z^s \rangle$. Then $G/K \cong H \cong C_s \times D_8$. If $\theta : FG \to FH$ is the canonical ring epimorphism, then by [7, Lemma 8.1.17] and [7, Theorem 7.2.7], $J(FG) = ker(\theta)$, $FG/J(FG) \cong FH$ and $\dim_F J(FG) = 2^3 s(p^r - 1)$. Hence

$$U(FG) \cong V \rtimes U(FH),$$

where $V = 1 + J(FG)$. Clearly, exponent of $V = p^r$ and $|V| = p^{2^3 sk(p^r-1)}$. Now,

$$\begin{aligned} FH &\cong (FC_s)D_8; \\ &\cong (F \oplus \oplus_{l>1,\ l|s} F_{d_l}^{e_l})D_8; \\ &\cong FD_8 \oplus \oplus_{l>1,\ l|s}(F_{d_l}D_8)^{e_l}, \end{aligned}$$

where $d_l = \text{ord}_l(q)$ and $e_l = \dfrac{\phi(l)}{d_l}$. By Theorem 2.1,

$$U(FD_8) \cong C_{q-1}^4 \times GL(2, F).$$

Hence

$$U(FH) \cong C_{q-1}^4 \times GL(2, F) \times \prod_{l>1,\ l|s} \left(C_{q^{d_l}-1}^4 \times GL(2, F_{d_l})\right)^{e_l}. \qquad \square$$

In the above theorem, if $r = 0$, then we have the unit group of the semisimple group algebra FG given by

$$U(FG) \cong C_{q-1}^4 \times GL(2, F) \times \prod_{l>1,\ l|n} C_{q^{d_l}-1}^{4e_l} \times GL(2, F_{d_l})^{e_l}$$

where $d_l = \text{ord}_l(q)$ and $e_l = \dfrac{\phi(l)}{d_l}$.

In [3], Gildea et. al. has been obtained the structure of the unit group of the group algebra $F(C_n \times D_8)$. In this paper, our result is independent of [3] and it was done when we had no knowledge that they have worked in this problem. But our characterization is complete as it is for all characteristic whereas they have done only for characteristic 2.

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