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MULTI-STEP HIGH CONVERGENCE ORDER METHODS FOR SOLVING EQUATIONS

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ABSTRACT. The local convergence analysis of iterative methods is important since it indicates the degree of difficulty for choosing initial points. In the present study we introduce generalized multi-step high order methods for solving nonlinear equations. The local convergence analysis is given using hypotheses only on the first derivative which actually appears in the methods in contrast to earlier works using hypotheses on higher derivatives. This way we extend the applicability of these methods. The analysis includes computable radius of convergence as well as error bounds based on Lipschitz-type conditions not given in earlier studies. Numerical examples conclude this study.

1. Introduction. Iterative regularization models are changing the face of the world by offering the scientists and mathematicians the opportunity to examine many real life problems, with a far greater generality and precision.

To make use of the full power of the iterative methods, they must have a firm grip on numerical techniques developed for various mathematical models and their analysis. Application of the iterative schemes is found in any scientific field where real world problems are modeled into mathematical equations.

Iterative schemes/methods are general terminology used for certain class of numerical schemes where the solution procedure starts with an approximate value/function and then apply the method repeatedly to obtain a better approximation. Many mathematical equations are in the form (or get reduced to),

$$(1.1) \qquad\qquad\qquad\qquad F(x) = 0,$$

where $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a Fréchet-differentiable operator, $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces and D is a nonempty open convex subset of $\mathcal{B}_1$. Such equations can be linear or nonlinear in nature and there are various iterative schemes used to obtain the solution. Also, these iterative schemes are useful in solving many optimization problems from different disciplines. Many of these methods are firmly based on various calculus and functional analysis concepts and they can be effectively implemented by taking the advantage of the speed and power of modern computer technologies. In particular three step methods have been introduced in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$ (i a natural number) to solve nonlinear systems [1, 2, 3, 5, 6, 7, 14, 16, 17, 18, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31]. Given $q \in \mathbb{N}$, continuous iteration operators $\varphi_j : D^j \longrightarrow \mathcal{B}_1, (j = 1, 2, \ldots, q)$, we introduce in a Banach space setting q multi-step method defined for each $n = 0, 1, 2, \ldots$ by

$$
(1.2) \qquad
\begin{aligned}
y_n^{(1)} &= \varphi_1(x_n) \\
y_n^{(2)} &= \varphi_2(x_n, y_n^{(1)}) \\
&\vdots \\
y_n^{(q)} &= x_{n+1} = \varphi_q(x_n, y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(q-1)}),
\end{aligned}
$$

where x_0 is an initial point. Usually φ_1 is an iteration operator of convergence order $p \geq 2$. Numerous popular iterative methods are special cases of method (1.2) [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 22, 23, 21, 24, 25, 26, 27, 28, 29, 30, 31] (see Section 3). In particular, for $q = 1$, Theorem 2.1 is a consequence of Corollary 3.7 [21].

The local convergence analysis usually involves Taylor expansions and conditions on higher order derivatives not appearing in the iterative methods considered in the earlier studies [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 22, 23, 21, 24, 25, 26, 27, 28, 29, 30, 31]. Moreover, convergence analysis

using Taylor expansion do not provide computable radius of convergence and error estimates on the distances $\|x_n - x^*\|$. Therefore the initial point is a shot in the dark. These problems limit the usage of these methods. That is why in the present study using only conditions on the first derivative, we address the preceding problems in the more general setting of methods (1.2) and Banach space.

We find computable radii of convergence as well as error bounds on the distances based on Lipschitz-type conditions. The order of convergence is found using computable order of convergence (COC) or approximate computational order of convergence (ACOC) [28] (see Remark 2.2) that do not require usage of higher order derivatives. This way we expand the applicability of three step method (1.2) under weak conditions.

The rest of the study is organized as follows: Section 2 contains the local convergence of method (1.2), where in the concluding Section 3 applications and numerical examples can be found.

2. Local convergence analysis. The local convergence analysis of method (1.2) is based on some parameters and scalar functions that appear in the proof.

Suppose conditions $(\mathcal{A})$ hold:

$(\mathcal{A}_0)$ There exist functions $\psi_j : [0, \lambda_j)^j \longrightarrow [0, \infty), j = 1, 2, \ldots q$, continuous and nondecreasing in each variable such that $\psi_j(0, 0, \ldots, 0) = 0$ and the equation $\psi_j(t, t, \ldots, t) = 1$ has at least one solving in $(0, \lambda_j)$. Denote by ρ_j the smallest such solution, and define a radius of convergence r by

$$(2.1) \qquad\qquad r = \min\{\rho_1, \ldots, \rho_q\}.$$

$(\mathcal{A}_1)$ $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a continuously Fréchet differentiable operator.

$(\mathcal{A}_2)$ There exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$.

$(\mathcal{A}_3)$ For each $j = 1, 2, \ldots, q$ and $u_1, u_2, \ldots u_j \in D \cap \bar{U}(x^*, r)$,

$$(2.2) \quad \|\varphi_j(u_1, \ldots, u_j) - x^*\| \leq \psi_j(\|u_1 - x^*\|, \ldots, \|u_j - x^*\|)\|u_1 - x^*\|.$$

$(\mathcal{A}_4)$ $\bar{U}(x^*, r) \subseteq D$.

We shall adopt the notations $U(x, \mu) = \{y \in \mathcal{B}_1 : \|x - y\| < \mu\}$ and $\bar{U}(x, \mu)\{y \in \mathcal{B}_1 : \|x - y\| \leq \mu\}$ for $x \in \mathcal{B}_1$ and $\mu > 0$.

Next, we present the local convergence analysis of method (1.2) under the conditions "$\mathcal{A}$" and the preceding notation.

Theorem 2.1. *Suppose that the "$\mathcal{A}$" conditions hold. Then sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r)$ by method (1.2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following error bounds hold*

$$(2.3) \qquad \|y_n^{(1)} - x^*\| \leq \psi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r,$$

$$
\begin{aligned}
\|y_n^{(j)} - x^*\| &\leq \psi_j(\|x_n - x^*\|, \|y_n^{(1)} - x^*\|, \ldots, \|y_n^{(j-1)} - x^*\|)\|x_n - x^*\| \\
(2.4) \qquad &\leq \|x_n - x^*\|, \ j = 2, \ldots, q,
\end{aligned}
$$

where the iteration functions are defined previously.

P r o o f.   Using induction, condition $x_0 \in U(x^*, r)$ and conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, we obtain that estimates (2.3) and (2.4) hold for $n = 0$ and $y_0^{(j)} \in U(x^*, r)$. That is we have

$$(2.5) \qquad \|y_0^1 - x^*\| \leq \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,$$

$$
\begin{aligned}
\|y_0^{(j)} - x^*\| &\leq \psi_j(\|x_0 - x^*\|, \|y_0^{(1)} - x^*\|, \ldots, \|y_0^{(j-1)} - x^*\|) \\
&\quad \times \|x_0 - x^*\| \\
&\leq \psi_j(\|x_0 - x^*\|, \|x_0 - x^*\|, \|x_0 - x^*\|, \ldots, \|x_0 - x^*\|) \\
(2.6) \qquad &\quad \times \|x_0 - x^*\| \leq \|x_0 - x^*\|.
\end{aligned}
$$

By simply replacing $x_0, y_0^{(j)}$ by $x_k, y_k^{(j)}$, respectively in the preceding computations, we obtain estimates (2.3) and (2.4). Then, from the estimate

$$(2.7) \qquad \|x_{k+1} - x^*\| = \|y_n^{(q)} - x^*\| \leq c\|x_n - x^*\| < \|x_n - x^*\|,$$

where $c = \psi_q(\|x_0 - x^*\|, \|x_0 - x^*\|, \ldots \|x_0 - x^*\|) \in [0, 1)$, we get $\lim\limits_{k \longrightarrow \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. $\square$

Remark 2.2. We can compute the computational order of convergence (COC) [28] defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results [5, 6, 12, 18, 23, 24, 25, 26, 27, 28, 29, 30, 31]. It is worth noticing that the computation of ξ and ξ_1 uses method (1.2) and does not depend on Theorem 2.1 which simply guarantees convergence to x^*. In particular, the computation of ξ_1 does not even require knowledge of x^*.

3. Applications and numerical examples.

Application 3.1. Let us specialize method (1.2) by setting $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i, q = 1, \varphi_1(x_k) = x_k - F'(x_k)^{-1}F(x_k),$

$$\varphi_2(x_k, y_k^{(1)}) = y_k^{(1)} - \bar{\tau}_k F'(x_k)^{-1}F(y_k^{(1)}) \text{ and}$$

$$(3.1) \qquad \psi_3(x_k, y_k^{(1)}, y_k^{(2)}) = y_k^{(2)} - \alpha_k F'(y_k^{(2)})^{-1}F(y_k^{(2)}).$$

Then, method (1.2) reduces to method (5.3) in [31] defined by

$$(3.2) \qquad \begin{aligned} y_k^{(1)} &= \varphi_1(x_k) \\ y_k^{(2)} &= \varphi_2(x_k, y_k^{(1)}) \\ x_{k+1} &= \varphi_3(x_k, y_k^{(1)}, y_k^{(2)}). \end{aligned}$$

It was shown in [31, Theorem 1, Theorem 5] that if operator F is sufficiently many times differentiable, $F'(x)$ is continuous on $D, F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ then for x_0 sufficiently close to x^* method (1.2) converges with order $p \geq 2$ if and only if $\bar{\tau}_k$ and α_k satisfy certain conditions involving hypotheses on higher derivatives for F. Further special choices of $\bar{\tau}_k$ and α_k are given in the following table leading to other pth order methods.

Application 3.2 Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i, \varphi_1(x_k, y_k^{(1)}) = \varphi_p(x_k, y_k^{(1)})$ and φ_p denotes the iteration function of pth order. Then, again according to Theorem 6 in [31] the method (1.2) has order of convergence $p + 2$ under certain conditions of α_k. As an example, we present the choices given by

$$(3.3) \qquad \alpha_k = \frac{1}{2}(5I - 3F'(x_k)^{-1}F'(y_k)),$$

$$(3.4) \qquad \alpha_k = 3I - 2F'(x_k)^{-1}F'(y_k)$$

$$(3.5) \qquad \alpha_k = F'(x_k)^{-1}F'(y_k)$$

$$(3.6) \qquad \alpha_k = \left(\left(1 - \frac{1}{\alpha}\right)F'(x_k) + \frac{1}{\alpha}F'(y_k)\right)^{-1}F'(x_k)$$

Table 1. Different methods

Methods	Order	$\bar{\tau}_k$	α_k
Cordero et al. [6]	4	$2l - F'(x_k)^{-1}F(y_k)$	
Sharma [23]	4	$3l - 2F'(x_k)^{-1}[y_k, x_k; F]$	
Grau-Senchez et al. [12]	4	$(2[y_k, x_k; F] - F'(x_k)^{-1})F'(x_k)$	
Sharma et al. [25]	4	$r_k = I - \frac{3}{4}(s_k - I) + \frac{9}{8}(s_k - I)^2$ $s_k = F'(x_k)^{-1}F'(y_k)$ $r_k = \frac{1}{2}(-I + \frac{9}{4}F'(y_k)^{-1}F'(x_k)$ $+\frac{3}{4}F'(x_k)^{-1}F'(y_k)$	
Gran-Sanchez et al. [12] Xiao et al. [29]	5	$\tau_k = \frac{1}{2}(I + F'(y_k)^{-1}F'(x_k)$	$F'(y_k)^{-1}F'(x_k)$
Cordero et al. [5]	5	$2(I - F'(x_k)^{-1}F'(y_k))^{-1}$	$F'(y_k)^{-1}F'(x_k)$
Xiao et al. [29]	5	$y_k = x_k - aF'(x_k)^{-1}F'(x_k)$ $\bar{\tau}_k = ((1 - \frac{1}{2a})I$ $+\frac{1}{2a}F'(x_k)^{-1}F'(y_k))^{-1}$	$-I + 2(\frac{1}{2a}F'(y_k)$ $+(1 - \frac{1}{2a})F'(x_k))^{-1}F'(x_k)$
Sharma et al. [24]		$a = \frac{1}{2}, (F'(y_k)^{-1}F'(x_k) - I)\theta_k$	$2F'(y_k)^{-1}F'(x_k) - I$
Sharma et al. [26]	6	$3I - 2F'(x_k)^{-1}[y_k, x_k; F]$	$3I - 2F'(x_k)^{-1}[y_k, x_k; F]$
Xiao et al. [29]	6	$y_k = x_k - aF'(x_k)^{-1}F'(x_k)$ $r_k = \frac{1}{2}(-I + \frac{9}{4}F'(y_k)^{-1})F'(x_k)$ $+\frac{3}{4}F'(x_k))^{-1}F'(y_k)$ $(2[y_k, x_k; F]^{-1} - F'(x_k))^{-1}F'(x_k)$	$(1 - \frac{1}{a})I$ $+\frac{1}{a}F'(y_k)^{-1}F'(x_k)$ $(2[y_k, x_k; F] - F'(x_k))^{-1}F'(x_k)$
Grau-Sanchez et al. [12]	6	$(2[y_k, x_k; F] - F'(x_k)^{-1})F'(x_k)$	$2[y_k, x_k; F]^{-1}F'(x_k) - I$
Cordero et al. [6]	6	$a = \frac{1}{a}, (F'(x_k) - 2F'(y_k))^{-1}$ $(3F'(x_k)\theta_k^{-1} - 4F'(x_k)$	$(F'(x_k) - 2F'(y_k))^{-1}F(x_k)$

$$(3.7) \qquad \alpha_k = \left(\left(1 + \frac{1}{\alpha}\right)F'(y_k) - \frac{1}{a}F'(x_k)\right)^{-1}F'(y_k)$$

(see [3, 4, 31]).

Let us consider the special case of method (1.2) given for $q = 2$, $\bar{t}_k = \alpha_k = 1$ by

$$y_k^{(1)} = \varphi_1(x_k) = x_k - F'(x_k)^{-1}F(x_k)$$
$$x_{k+1} = \varphi_2(x_k, y_k^{(1)}) = y_k^{(1)} - F'(x_k)^{-1}F(y_k^{(1)}).$$

Suppose that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \text{ for each } x \in D$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|)$$

and

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|)$$

for each $x, y \in D_0 := D \cap U(x^*, \rho_0)$, where $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$, $w : [0, \rho_0) \longrightarrow [0, +\infty)$, $v : [0, \rho_0) \longrightarrow [0, +\infty)$ are continuous nondecreasing functions with $w_0(0) = w(0) = 0$ provided that equation $w_0(t) = 1$ has a least positive solution ρ_0.

Then, we have in turn

$$\|x_{k+1} - x^*\| \leq \|F'(y_k)^{-1} \int_0^1 [F'(x^* + \theta(y_k - x^*)) - F'(x^*)](y_k - x^*)d\theta\|$$

$$+ \|F'(y_k)^{-1}(F'(x_k) - F'(y_k))F'(x_k)^{-1}F(y_k)\|$$

$$\leq \frac{\int_0^1 w(\theta\|y_k - x^*\|)d\theta\|y_k - x^*\|}{1 - w_0(\|y_k - x^*\|)}$$

$$+ \frac{(w_0(\|x_k - x^*\|) + w_0(\|y_k - x^*\|))\int_0^1 v(\theta\|y_k - x^*\|)d\theta\|y_k - x^*\|}{(1 - w_0(\|y_k - x^*\|))(1 - w_0(\|x_k - x^*\|))}$$

$$\leq \frac{\int_0^1 w(\theta g_1(\|x_k - x^*\|)\|x_k - x^*\|)d\theta g_1(\|x_k - x^*\|)\|x_k - x^*\|}{1 - w_0(\|x_k - x^*\|)}$$

so we can choose

$$\psi_1(t) = \frac{\int_0^1 w((1 - \theta)t)d\theta}{1 - w_0(t)},$$

$$\psi_2(s, t) = \psi_2(t) = \left[\frac{\int_0^1 w((1 - \theta)\psi_1(t)t)d\theta}{1 - w_0(\psi_1(t)t)} \right.$$

$$\left. + \frac{(w_0(t) + w_0(\psi_1(t)t))\int_0^1 v(\theta\psi_1(t)t)d\theta}{(1 - w_0(t))(1 - w_0(\psi_1(t)t))} \right] \psi_1(t) = 0,$$

$r = \min\{r_1, r_2\}$ respectively, where r_1, r_2 are the smallest positive solutions of equations $\psi_1(t) = 1$ and $\psi_2(t) = 1$. Using the above choice we present the following examples:

Example 3.1. Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x, \; F_2'(y) = (e-1)y+1, \; f_3(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F, F_3)$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ conditions, we get for $\alpha = 1$, $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, v(t) = e^{\frac{1}{e-1}}$. The radii are

$$r_1 = 0.3827, \; r_2 = 0.2523 = r.$$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

(3.8) $$F(\varphi)(x) = \varphi(x) - 5 \int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \;\; \text{for each } \xi \in D.$$

Then, we get for $x^* = 0$, that $w_0(t) = 7.5t, w(t) = 15t$ and $v(t) = 2$. Then the radii are

$$r_1 = 0.0667, \; r = r_2 = 0.0439.$$

Example 3.3. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, and $D = \bar{U}(0,1)$. Consider the equation

(3.9) $$x(s) = \int_0^1 T(s,t)(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8})dt,$$

where the kernel T is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$(3.10) \qquad T(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t, \end{cases}$$

Define operator $F : C[0, 1] \longrightarrow [0, 1]$ by

$$(3.11) \qquad F(x(s)) = \int_0^1 T(s,t)(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8})dt - x(s).$$

Then, we have

$$(3.12) \qquad F'(x)\mu(s) = \mu(s) - \int_0^1 T(s,t)(\frac{3}{4}x(t)^{\frac{1}{2}} + \frac{x(t)}{4})\mu(t)dt.$$

Notice that $x^*(s) = 0$ is a solution of $F(x(s)) = 0$. Using (3.10), we obtain

$$(3.13) \qquad \left\| \int_0^1 T(s,t)dt \right\| \le \frac{1}{8}.$$

Then, by (3.12) and (3.13), we have that

$$(3.14) \qquad \|F'(x) - F'(y)\| \le \frac{1}{32}\left(3\|x - y\|^{\frac{1}{2}} + \|x - y\|\right).$$

We have $w_0(t) = w(t) = \frac{1}{32}(3t^{1/2} + t)$ and $v(t) = 1 + w_0(t)$. Then the radii are

$$r_1 = 19.4772, r_2 = 0.3889,$$

so we choose $r = 1$ since $\bar{U}(x^*, r) \subset D$ [5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

In view of (3.14) earlier results requiring hypotheses on the second Fréchet derivative or higher cannot be used to solve equation $F(x(s)) = 0$.

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