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ON SOME UNIQUE FIXED POINT THEOREMS WITH RATIONAL EXPRESSIONS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. The aim of this paper is to establish some fixed point results of a self mapping satisfying generalized rational contraction conditions in a metric space endowed with a partial order. These results generalized and widen the result of Sharma and Yuel from metrical space to partially ordered metric space and also a few well known results in an ordered metric space. Some examples are illustrated to support the usability of the presented results.

1. Introduction. First the concept of fixed point theory was introduced by H.Poincare in 1886. Later, M.Frechet in 1906 has given the fixed point theorem in terms of taking distance between the points and the corresponding images of the function at those points in a metric space. Later in 1922, Banach proved a fixed point theorem for contraction mapping in complete metric space. This principle plays an important role in many branches of mathematics. It is a very

popular tool for solving many existence problems in nonlinear analysis. Besides, this famous classical theorem gave an iteration process through which we can obtain better approximation to the fixed point. Banach's fixed point theorem has rendered a key role in solving systems of linear algebraic equations involving iteration process. Iteration procedures have been used in nearly every branch of applied mathematics, convergence proof and also in estimating the process of errors, very often by an application of Banach's fixed point theorem. A lot of generalizations and extensions of this principle have been done by several authors in a metric space, some of which are in [14, 15, 19, 30, 40, 42].

The extended Banach contraction principle in partially ordered sets was first initiated by Wolk [41] and later Monjardet [24]. The existence of fixed points in partially ordered metric spaces with some applications to matrix equations were investigated by Ran and Reurings [29]. Later, Nieto et al.[25, 26, 27] extended the result of [29] and applied their results to ordinary differential equations. There after several authors have been reported the results on fixed point, common fixed point and coupled fixed points for the mappings in various ordered metric species with different topological properties, the readers may refer to [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 16, 20, 21, 22, 23, 28, 31, 32, 43]. Recently, Kalyani et al. [17, 18] and Seshagiri Rao et al. [33, 34, 35, 36, 37, 38] have explored some theorems on fixed point of a monotone mapping, coincidence points and coupled fixed point of mappings in partially ordered metric spaces.

In this paper, we provided some fixed point results for a self-mapping satisfying generalized contraction conditions of rational type in complete partially ordered metric spaces. Our results generalized and extended the main result of Sharma and Yuel [39] in partially ordered metric spaces and also broaden the results of Muhammad Arshad et al. [7]. A few examples are presented to support the results obtained.

2. Preliminaries.
We use frequently the following definitions in our present study.

Definition 1 ([37]). *The triple $(X, d, \preceq)$ is called partially ordered metric spaces, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2 ([37]). *If (X, d) is a complete metric space, then the triple $(X, d, \preceq)$ is called complete partially ordered metric space.*

Definition 3 ([7]). *A partially ordered metric space $(X, d, \preceq)$ is called an ordered complete(OC) if for every convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following condition holds: either*

(i) *if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x$ implies $x \preceq x_n$, for all $n \in \mathbb{N}$, that is, $x = \inf\{x_n\}$, or*

(ii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preceq x$, for all $n \in \mathbb{N}$, that is, $x = \sup\{x_n\}$.*

Definition 4 ([37]). *Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be a non-decreasing mapping, if for every $x, y \in X$ with $x \preceq y$ implies that $Tx \preceq Ty$.*

3. Main results. We start this section with the following definition.

Definition 5. *Let $(X, d, \preceq)$ be a partially ordered metric space. A self-mapping T on X is called an almost Sharma and Yuel contraction if it satisfies the following condition:*

(1)
$$\begin{aligned} d(Tx, Ty) \leq {} & \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta[d(x, Tx) + d(y, Ty)] \\ & + \gamma[d(x, Ty) + d(y, Tx)] + \delta d(x, y) \\ & + L \min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\}, \end{aligned}$$

for any distinct $x, y \in X$ with $x \preceq y$, where $L \geq 0$ and there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$.

Theorem 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that a self-mapping T on X is an almost Sharma and Yuel contraction, continuous and non-decreasing. If there exists certain $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a unique fixed point in X.*

P r o o f.   Let $x_0 \in X$ be an arbitrary and define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq Tx_0$ and T is non-decreasing then by induction we obtain that

(2)
$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Now,

$$\begin{aligned} d(x_{n+1}, x_n) = {} & d(Tx_n, Tx_{n-1}) \\ \leq {} & \alpha \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} + \beta[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] \\ & + \gamma[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] + \delta d(x_n, x_{n-1}) \\ & + L \min\{d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\}, \end{aligned}$$

which implies that

$$d(x_{n+1}, x_n) \leq \left(\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) d(x_n, x_{n-1}) \leq \ldots \leq \left(\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right)^n d(x_1, x_0).$$

Furthermore, the triangle inequality of a metric d for $m \geq n$, we have

$$\begin{aligned}
d(x_n, x_m) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq \left(k^n + k^{n+1} + \cdots + k^{m-1} \right) d(x_0, Tx_0) \\
&\leq \frac{k^n}{1 - k} d(x_1, x_0),
\end{aligned}$$

(3)

where $k = \dfrac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} < 1$. Letting $n \to +\infty$ in the equation (3), we obtain that $d(x_n, x_m) = 0$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence and the completeness of X implies that $x_n \to z$ for some $z \in X$.

Moreover, the continuity of T suggest that

$$Tz = T \left(\lim_{n \to +\infty} x_n \right) = \lim_{n \to +\infty} Tx_n = \lim_{n \to +\infty} x_{n+1} = z.$$

Thus, $z \in X$ is a fixed point of T. Now for the uniqueness of a fixed point, let $w \neq z$ in X be another fixed point of T, then

$$\begin{aligned}
d(w, z) &= d(Tw, Tz) \\
&\leq \alpha \frac{d(w, Tw) d(z, Tz)}{d(w, z)} + \beta [d(w, Tw) + d(z, Tz)] \\
&\quad + \gamma [d(w, Tz) + d(z, Tw)] + \delta d(w, z) \\
&\quad + L \, \min\{d(w, Tz), d(z, Tw), d(w, Tw), d(z, Tz)\} \\
&= (2\gamma + \delta) \, d(w, z) < d(w, z), \text{ since } 2\gamma + \delta < 1,
\end{aligned}$$

which is a contradiction. Hence, $w = z$ which shows that $z \in X$ is a unique fixed point of T. $\square$

Example 1. Define a metric $d : X \times X \to R$, where $X = [0, 1]$ by

$$d(x, y) = |x - y|.$$

Let us define a self-mapping T on X as

$$Tx = \frac{x}{15},$$

then T has a fixed point in X.

P r o o f.   It is obvious that (X, d) is a complete metric space and the mapping T is continuously non-decreasing in X. Assume that $x \preceq y$ means $x \leq y$ then

$$d(Tx, Ty) = \frac{1}{15}|x - y| \leq \frac{1}{5}\, d(x, y).$$

Consequently,

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta[d(x, Tx) + d(y, Ty)]$$
$$+ \gamma[d(x, Ty) + d(y, Tx)] + \frac{1}{5}\, d(x, y)$$
$$+ L \min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\}.$$

From the hypotheses, we have that $0 \leq d(x, Tx) \leq \frac{14}{15}$, $0 \leq d(y, Ty) \leq \frac{14}{15}$, $0 \leq d(x, Ty) \leq \frac{14}{15}$ and $0 \leq d(y, Tx) \leq 1$. Thus, on taking $\delta = \frac{1}{5}$ and for any values of $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$ and for any $L \geq 0$, all the conditions of Theorem 1 are fulfilled. As a result, T has a fixed point $0 \in X$. $\square$

Definition 6. *Let $(X, d, \preceq)$ be a partially ordered metric space. A self-mapping T on X is called Sharma and Yuel contraction if it satisfies the following contraction condition:*

(4)
$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta[d(x, Tx) + d(y, Ty)]$$
$$+ \gamma[d(x, Ty) + d(y, Tx)] + \delta d(x, y),$$

for any distinct $x, y \in X$ with $x \preceq y$ and there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$.

Corollary 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that a self-map T on X is Sharma and Yuel contraction, continuous and non-decreasing. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.*

P r o o f.   Set $L = 0$ in Theorem 1. $\square$

Example 2. Let us define a metric $d : X \times X \to R$, where $X = [0, 1]$ by

$$d(x, y) = |x - y|.$$

Define a self-mapping T on X by

$$Tx = \frac{x}{5},$$

then T has a fixed point in X.

P r o o f.   We know that $(X, d, \preceq)$ is a complete partially ordered metric space, where $x \preceq y$ means $x \leq y$ and T is non-decreasing and continuous.

Now,

$$d(Tx, Ty) = \frac{1}{5}|x - y| \leq \frac{1}{3}\ d(x, y).$$

Therefore,

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta[d(x, Tx) + d(y, Ty)]$$
$$+ \gamma[d(x, Ty) + d(y, Tx)] + \frac{1}{3}\ d(x, y).$$

Also from the hypothesis, we have $0 \leq d(x, Tx) \leq \frac{4}{5}$, $0 \leq d(y, Ty) \leq \frac{4}{5}$, $0 \leq d(x, Ty) \leq \frac{4}{5}$ and $0 \leq d(y, Tx) \leq 1$. Therefore, for taking $\delta = \frac{1}{3}$ and for any values of $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$, all the conditions of Corollary 1 are satisfied. Hence, $0 \in X$ is a fixed point of T. $\square$

In the following theorem, we establish the existence of a unique fixed point of a map T by assuming some iteration of T is continuous.

Theorem 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that a self-map T is non-decreasing and an almost Sharma and Yuel contraction. If there exists certain $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the operator T^p is continuous for some positive integer p, then T has a unique fixed point in X.*

P r o o f.   From Theorem 1, we have a sequence $\{x_n\} \subseteq X$ which converges to some $z \in X$. Therefore, its subsequence $x_{n_k}(n_k = kp)$ also converges to the same z. Hence,

$$T^p z = T^p \left(\lim_{n \to +\infty} x_{n_k} \right) = \lim_{n \to \infty} x_{n_{k+1}} = z.$$

This shows that z is a fixed point of T^p. Now, we have to prove that $Tz = z$. Let m be the smallest positive integer such that $T^m z = z$ and $T^q \neq z$ ($q =$

$1, 2, 3, \ldots, m - 1$). If for some $m > 1$ then

$$
\begin{aligned}
d(Tz, z) &= d(Tz, T^m z) \\
&\leq \alpha \frac{d(z, Tz)d(T^{m-1}z, T^m z)}{d(z, T^{m-1}z)} + \beta[d(z, Tz) + d(T^{m-1}z, T^m z)] \\
&\quad + \gamma[d(z, T^m z) + d(T^{m-1}z, Tz)] + \delta d(z, T^{m-1}z) \\
&\quad + L \min\{d(z, T^m z), d(T^{m-1}z, Tz), d(z, Tz), d(T^{m-1}z, T^m z)\},
\end{aligned}
$$

which intern implies that

$$
d(z, Tz) \leq \left(\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right) \ d(z, T^{m-1}z).
$$

Regarding (1), we have

$$
\begin{aligned}
d(z, T^{m-1}z) &= d(T^m z, T^{m-1}z) \\
&\leq \alpha \frac{d(T^{m-1}z, T^m z).d(T^{m-2}z, T^{m-1}z)}{d(T^{m-1}z, T^{m-2}z)} \\
&\quad + \beta[d(T^{m-1}z, T^m z) + d(T^{m-2}z, T^{m-1}z)] \\
&\quad + \gamma[d(T^{m-1}z, T^{m-1}z) + d(T^{m-2}z, T^m z)] + \delta d(T^{m-1}z, T^{m-2}z) \\
&\quad + L \min\{d(T^{m-1}z, T^{m-1}z), d(T^{m-2}z, T^m z), d(T^{m-1}z, T^m z), \\
&\qquad d(T^{m-2}z, T^{m-1}z)\}.
\end{aligned}
$$

Inductively, we obtain that

$$
d(z, T^{m-1}z) = d(T^m z, T^{m-1}z) \leq k d(T^{m-1}z, T^{m-2}z) \leq \cdots \leq k^{m-1} d(Tz, z),
$$

where $k = \dfrac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} < 1$. Therefore

$$
d(Tz, z) \leq k^m d(Tz, z) < d(Tz, z),
$$

which is a contradiction. Hence, $Tz = z$. Furthermore, the uniqueness of z follows as in Theorem 1. $\square$

Corollary 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that a self-mapping T on X is non-decreasing and satisfies the Sharma and Yuel contraction. If certain $x_0 \in X$ such that $x_0 \preceq Tx_0$ and the operator T^p is continuous for some positive integer p, then T has a unique fixed point in X.*

Proof. Set $L = 0$ in Theorem 2. $\square$

Theorem 3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and T be a non-decreasing self mapping on X. Assume for some positive integer m, T satisfies the following condition*

(5)
$$
\begin{aligned}
d(T^m x, T^m y) \leq {} & \alpha \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} + \beta[d(x, T^m x) + d(y, T^m y)] \\
& + \gamma[d(x, T^m y) + d(y, T^m x)] + \delta d(x, y) \\
& + L \min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\},
\end{aligned}
$$

for all distinct $x, y \in X$ with $x \preceq y$ and $L \geq 0$, and where $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq T^m x_0$ and T^m is continuous, then T has a unique fixed point in X.

Proof. Due to Theorem 1, we conclude that T^m has a unique fixed point, say $z \in X$. Now,

$$
Tz = T\left(T^m z\right) = T^m\left(Tz\right).
$$

Therefore, Tz is also a fixed point of T^m. As a result of Theorem 1, z is a unique fixed point of T^m. Consequently, we have $z = Tz$. Hence, z is a unique fixed point of T. $\square$

Corollary 3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and T be a non-decreasing self map on X. Suppose for some positive integer m, T satisfies the condition*

(6)
$$
\begin{aligned}
d(T^m x, T^m y) \leq {} & \alpha \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} + \beta[d(x, T^m x) + d(y, T^m y)] \\
& + \gamma[d(x, T^m y) + d(y, T^m x)] + \delta d(x, y),
\end{aligned}
$$

for all distinct $x, y \in X$ with $x \preceq y$ and there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq T^m x_0$ and T^m is continuous, then T has a unique fixed point in X.

Proof. Set $L = 0$ in Theorem 3. $\square$

Now, we give the following example.

Example 3. Let $X = [0, 1]$ with the usual metric and usual order $\leq$.

Define an operator $T : X \to X$ as follows:

$$
Tx =
\begin{cases}
0, & \text{if } x \in \left[0, \dfrac{1}{4}\right], \\[2ex]
\dfrac{1}{4}, & \text{if } x \in \left(\dfrac{1}{4}, 1\right].
\end{cases}
$$

It can be easily seen that T is discontinuous and does not satisfy the condition (1) for any $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $0 \le \alpha + 2(\beta + \gamma) + \delta < 1$ for $x = \dfrac{1}{4}, y = 1$. But $T^2(x) = 0$ for all $x \in [0, 1]$. It can be verified that T^2 satisfies all the conditions of Theorem 3 and hence, $0 \in X$ is a unique fixed point of T^2.

In particular, there is an example where Theorem 1 (or Corollary 1) can be applied and not be valid in a complete metric space.

Example 4. Let $X = \{(0, 1), (1, 0), (1, 1)\}$ and let the partial order relation on X be $R = \{(x, x) : x \in X\}$. Observe that the elements only in X are comparable to themselves. Apart from, (X, d) is a complete metric space with the Euclidean distance (d) while with regards $\le$ is a partially ordered set.

Define a map $T : X \to X$ by

$$T(0, 1) = (1, 0), \ T(1, 0) = (0, 1), \ T(1, 1) = (1, 1),$$

is a non-decreasing, continuous and $(1, 1) \le T(1, 1) = (1, 1)$ for $(1, 1) \in X$ and satisfy condition (1) (or(4)). As a result $(1, 1)$ is a fixed point of T.

Besides, for $x = (0, 1)$, $y = (1, 0)$ in X, we have

$$d(Tx, Ty) = \sqrt{2}, \ d(x, Ty) = 0, \ d(y, Tx) = 0, \ d(x, Tx) = \sqrt{2}, \ d(y, Ty) = \sqrt{2},$$

then

$$
\begin{aligned}
d(Tx, Ty) = \sqrt{2} &\le \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta[d(x, Tx) + d(y, Ty)] \\
&\quad + \gamma[d(x, Ty) + d(y, Tx)] + \delta d(x, y) \\
&\le \alpha \cdot \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{2}} + \beta \cdot [\sqrt{2} + \sqrt{2}]. + \delta \cdot \sqrt{2} \\
&= (\alpha + 2\beta + \delta) \cdot \sqrt{2},
\end{aligned}
$$

which implies that, $\alpha + 2\beta + \delta \ge 1$. Accordingly, this example is not valid in the case of usual complete metrical space. Also, notice here that T has a unique fixed point $(1, 1) \in X$.

4. Further results.

Theorem 4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let T be a non-decreasing continuous self mapping defined on X. Assume that for all distinct $x, y \in X$ with $y \preceq x$ and let $A = d(y, Tx) + d(x, Ty)$, the self mapping T satisfies the following contraction condition*

$$
(7) \quad d(Tx, Ty) \leq \begin{cases} \lambda d(x, y) + \theta \left[d(x, Tx) + d(y, Ty) \right] \\ + \eta \left[d(x, Ty) + d(y, Tx) \right] \\ + \mu \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, & \text{if } A \neq 0 \\ 0, & \text{if } A = 0, \end{cases}
$$

where $\lambda, \theta, \eta, \mu$ are non-negative reals with $0 \leq \lambda + 2(\theta + \eta) + \mu < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

P r o o f.   Suppose for some $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $x_0 = Tx_0$, then the proof is finished. Assume that $x_0 \prec Tx_0$. Since T is a non-decreasing and then by induction we obtain that

$$
(8) \qquad x_0 \prec Tx_0 \preceq T^2 x_0 \preceq \cdots \preceq T^n x_0 \preceq T^{n+1} x_0 \preceq \cdots .
$$

Put $x_{n+1} = Tx_n$. If for some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, as result we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Thus, x_{n_0} is a fixed point of T and hence the result. Assume that $x_n \neq x_{n+1}$ for $n \in \mathbb{N}$. Since x_n and x_{n-1} are comparable for $n \in \mathbb{N}$ due to (8), then we have the following two cases.
Case 1: If $A = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) \neq 0$, then from (7) we have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\leq \lambda d(x_n, x_{n-1}) + \theta \left[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}) \right] \\
&\quad + \eta \left[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) \right] \\
&\quad + \mu \frac{d(x_n, Tx_n)d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})},
\end{aligned}
$$

which intern implies that,

$$
\begin{aligned}
d(x_{n+1}, x_n) &\leq \lambda d(x_n, x_{n-1}) + \theta \left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n) \right] \\
&\quad + \eta \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \\
&\quad + \mu \frac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}.
\end{aligned}
$$

Consequently, we have

$$d(x_{n+1}, x_n) \leq h^n d(x_1, x_0),$$

where $h = \dfrac{\lambda + \theta + \eta + \mu}{1 - \theta - \eta} < 1$. Moreover, from the triangular inequality of a metric d for $m \geq n$, we have

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)$$
$$\leq \frac{h^n}{1 - h} d(x_1, x_0),$$

as $m, n \to +\infty$, $d(x_m, x_n) \to 0$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence and converges to some $z \in X$. Further, the continuity of T implies that

$$Tz = T \left(\lim_{n \to +\infty} x_n \right) = \lim_{n \to +\infty} Tx_n = \lim_{n \to +\infty} x_{n+1} = z.$$

Hence, z is a fixed point of T in X.

Case 2: If $A = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) = 0$, then $d(x_{n+1}, x_n) = 0$. This implies that $x_n = x_{n+1}$, a contradiction. Hence, there exists a fixed point z of T in X. $\square$

Example 5. Let us define a self mapping T on $X = [0, 1]$ with usual metric and usual order $\leq$ by

$$Tx = \frac{2x + 4}{9(x^2 + x + \frac{10}{9})},$$

then T has a fixed point in X.

Proof. The definition of a map T, it is continuous and non-decreasing in $X = [0, 1]$. Now for $x \leq y$,

$$d(Tx, Ty) = \frac{1}{9} \left| \frac{2x + 4}{x^2 + x + \frac{10}{9}} - \frac{2y + 4}{y^2 + y + \frac{10}{9}} \right|$$
$$= \frac{1}{9} \left| \frac{2xy(y - x) + 4(x + y)(y - x) + 4(y - x) - \frac{20}{9}(y - x)}{(x^2 + x + \frac{10}{9})(y^2 + y + \frac{10}{9})} \right|$$
$$= \left| \frac{2xy + 4(x + y) + \frac{16}{9}}{9(x^2 + x + \frac{10}{9})(y^2 + y + \frac{10}{9})} \right| |y - x|$$
$$\leq \frac{9}{35} |y - x|,$$

holds for all $x, y \in X$. As we know that $x_0 = 0 \in X$ such that $x_0 = 0 \le Tx_0$. For $\lambda = \dfrac{9}{35}$ and all possible values of $\theta, \eta, \mu \in [0, 1)$ such that $0 \le \lambda + 2(\theta + \eta) + \mu < 1$, all the conditions of Theorem 4 are fulfilled. Hence T has a fixed point $\dfrac{1}{3}$ in X. $\square$

We may extract the continuity of T in Theorem 4, we have the following result.

Theorem 5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and T be a non-decreasing continuous self-mapping on X. Suppose that T satisfies the following condition for all $x, y \in X$ with $y \preceq x$ and $A = d(y, Tx) + d(x, Ty)$:*

$$(9) \quad d(Tx, Ty) \le \begin{cases} \begin{aligned} & \lambda d(x, y) + \theta \left[d(x, Tx) + d(y, Ty) \right] \\ & + \eta \left[d(x, Ty) + d(y, Tx) \right] \\ & + \mu \frac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, \quad \text{if } A \ne 0 \\ & 0, \hspace{6.2cm} \text{if } A = 0, \end{aligned} \end{cases}$$

where $\lambda, \theta, \eta, \mu \in [0, 1]$ such that $0 < \lambda + 2(\theta + \eta) + \mu < 1$. And also assume that X is an ordered complete(OC). If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

P r o o f.   We only have to check that $z = Tz$. Since $\{x_n\} \subset X$ is a non-decreasing sequence and $x_n \to z \in X$, then $z = \sup\{x_n\}$ for all $n \in \mathbb{N}$. Since T is a non-decreasing mapping, then $Tx_n \preceq Tz$ for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \preceq Tz$ for all $n \in \mathbb{N}$. Moreover, as $x_0 \prec x_1 \preceq Tz$ and $z = \sup\{x_n\}$, we get $z \preceq Tz$.

Suppose that $z \prec Tz$. Using a similar arguments as that in the proof of Theorem 4 for $x_0 \preceq Tx_0$, we obtain that $T^n z$ is a non-decreasing sequence and $\lim\limits_{n \to +\infty} T^n z = y$ for certain $y \in X$. Again, using the property of ordered completeness of X, we have that $y = sup\{T^n z\}$. Moreover, from $x_0 \preceq z$, we get $x_n = T^n x_0 \preceq T^n z$, for $n \ge 1$ and $x_n \prec T^n z$, for $n \ge 1$ because $x_n \preceq z \prec Tz \preceq T^n z$, for $n \ge 1$ as x_n and $T^n z$ are comparable and distinct for $n \ge 1$.
Case 1: If $d(T^n z, Tx_n) + d(x_n, T^{n+1} z) \ne 0$, then (9) follows that

$$\begin{aligned} d(x_{n+1}, T^{n+1} z) &= d(Tx_n, T(T^n z)) \\ &\le \lambda d(x_n, T^n z) + \theta \left[d(x_n, x_{n+1}) + d(T^n z, T^{n+1} z) \right] \\ &\quad + \eta \left[d(x_n, T^{n+1} z) + d(T^n z, x_{n+1}) \right] \\ &\quad + \mu \frac{d(x_n, x_{n+1})d(x_n, T^{n+1} z) + d(T^n z, x_{n+1})d(T^n z, T^{n+1} z)}{d(T^n z, x_{n+1}) + d(x_n, T^{n+1} z)}. \end{aligned}$$

Making $n \to +\infty$ in the above inequality, we obtain that

$$d(z, y) \leq (\lambda + 2\eta)\, d(z, y),$$

as $\lambda + 2\eta < 1$, $d(z, y) = 0$, thus $z = y$. Particularly, $z = y = sup\{T^n z\}$ and consequently, $Tz \preceq z$, which is a contradiction. Hence, we conclude that $Tz = z$.
Case 2: If $d(T^n z, T x_n) + d(x_n, T^{n+1} z) = 0$, then $d(x_{n+1}, T^{n+1} z) = 0$. Taking the limit as $n \to +\infty$, we get $d(z, y) = 0$. Then $z = y = \sup\{T^n z\}$, which implies that $Tz \preceq z$, a contradiction. Thus $Tz = z$. $\square$

Now, we prove the sufficient condition for the uniqueness of a fixed point that exists in Theorems 4 & 5 using the fact that for any $y, z \in X$, there exists $x \in X$ which is comparable to y and z.

Theorem 6. *In addition to the above condition in Theorems 4 and 5, one can obtains the uniqueness of a fixed point of T.*

P r o o f.   Suppose there exists $y, z \in X$ are the two fixed points of T. Now, we distinguish the following two cases.
Case 1: Suppose y and z are comparable and distinct. Now, we have the following two subcases:
(i). If $d(z, Ty) + d(y, Tz) \neq 0$ then from contradiction condition, we have

$$
\begin{aligned}
d(y, z) &= d(Ty, Tz) \\
&\leq \lambda d(y, z) + \theta\,[d(y, Ty) + d(z, Tz)] + \eta\,[d(y, Tz) + d(z, Ty)] \\
&\quad + \mu \frac{d(y, Ty)d(y, Tz) + d(z, Ty)d(z, Tz)}{d(z, Ty) + d(y, Tz)} \\
&\leq \lambda d(y, z) + \theta\,[d(y, y) + d(z, z)] + \eta\,[d(y, z) + d(z, y)] \\
&\quad + \mu \frac{d(y, y)d(y, z) + d(z, y)d(z, z)}{d(z, y) + d(y, z)} \\
&\leq (\lambda + 2\eta)\, d(y, z),
\end{aligned}
$$

as $\lambda + 2\eta < 1$, so by the last inequality, we have a contradiction. Thus $y = z$.
(ii). If $d(z, Ty) + d(y, Tz) = 0$, then $d(y, z) = 0$, a contradiction. Thus $y = z$.
Case 2: If y and z are not comparable, then from the hypotheses there exists $x \in X$ comparable to y and z. Monotonicity implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0, 1, 2, \ldots$.

If there exists $n_0 \geq 1$ such that $T^{n_0} x = y$, then as y is fixed point, the sequence $\{T^n x : n \geq n_0\}$ is constant, and consequently $\lim\limits_{n \to +\infty} T^n x = y$. On the other hand, if $T^n x \neq y$ for $n \geq 1$. Now we have two subcases as follows:

(i). If $d(T^{n-1}y, T^n x) + d(T^{n-1}x, T^n y) \neq 0$, then using the contractive condition, we obtain for $n \geq 2$,

$$d(T^n x, y) = d(T^n x, T^n y)$$
$$\leq \lambda d(T^{n-1}x, y) + \theta \left[d(T^{n-1}x, T^n x) + d(y, y) \right] + \eta \left[d(T^{n-1}x, y) + d(y, T^n x) \right]$$
$$+ \mu \frac{d(T^{n-1}x, T^n x)d(T^{n-1}x, y) + d(y, T^n x)d(y, y)}{d(T^n x, y) + d(y, T^{n-1}x)}$$
$$\leq \lambda d(T^{n-1}x, y) + \theta \left[d(T^{n-1}x, y) + d(y, T^n x) \right]$$
$$+ \eta \left[d(T^{n-1}x, y) + d(y, T^n x) \right] + \mu d(T^{n-1}x, y).$$

This implies that

$$d(T^n x, y) \leq \left(\frac{\lambda + \theta + \eta + \mu}{1 - \theta - \eta} \right) d(T^{n-1}x, y).$$

By induction, we get

$$d(T^n x, y) \leq \left(\frac{\lambda + \theta + \eta + \mu}{1 - \theta - \eta} \right)^n d(x, y).$$

Taking limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n \to +\infty} T^n x = y,$$

as $\lambda + 2(\theta + \eta) + \mu < 1$. Using a similar argument, we can prove that

$$\lim_{n \to \infty} T^n x = z.$$

Now, the uniqueness of the limit gives that $y = z$.

(ii). If $d(T^{n-1}y, T^n x) + d(T^{n-1}x, T^n y) = 0$, then $d(T^n x, y) = 0$. Thus

$$\lim_{n \to \infty} T^n x = y.$$

Using a similar argument, we can prove that

$$\lim_{n \to \infty} T^n x = z.$$

Now, the uniqueness of the limit gives that $y = z$. This completes the proof. □

We can obtain some consequences of the Theorems 4,5 & 6 by putting $\lambda = 0$, $\lambda = \theta = 0$ and $\lambda = \eta = 0$ in Section 4.

Remark 1.   (i) If $\theta = \eta = \mu = 0$ in Theorems 4, 5 & 6 then we obtain Theorems 2.1, 2.2 & 2.3 of [25].

(ii). Theorems 4, 5 & 6 reduces to Theorems 15, 17 & 18 of [7] when $\theta = \eta = 0$.

(iii). If $\lambda = \theta = \eta = 0$ in Section 4 then we obtain Theorem 20 of [7].

5. Applications. In this section, we state some applications of integral type contraction for the main results.

Corollary 4. *Let $(X, d, \preceq)$ be a T-orbitally complete partially ordered metric space and let T be a non-decreasing self-mapping defined on X. Suppose that a self mapping T satisfies the following condition:*

(10)
$$\int_0^{d(Tx,Ty)} ds \leq \alpha \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} ds + \beta \int_0^{d(x,Tx)+d(y,Ty)} ds + \gamma \int_0^{d(x,Ty)+d(y,Tx)} ds$$
$$+ \delta \int_0^{d(x,y)} ds + L \int_0^{\min\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}} ds,$$

for all distinct $x, y \in X$ with $x \preceq y$ and for some $\alpha, \beta, \gamma, \delta \in [0,1)$ with $0 < \alpha + 2(\beta + \gamma) + \delta < 1$, where $L \geq 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has at least one fixed point in X.

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