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SOME PROPERTIES OF THE GENERALIZED SUBCLASSES OF JANOWSKI TYPE ALPHA-QUASI-CONVEX FUNCTIONS

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ABSTRACT. The present paper is concerned with certain generalized subclasses of alpha-quasi-convex functions in the open unit disc $E = \{z : |z| < 1\}$. We establish some properties such as the coefficient estimates, distortion theorems, growth theorems and radius of quasi convexity for these classes. Various earlier known results will follow as special cases.

1. Introduction. Let \mathcal{A} denote the class of functions f which are analytic in the unit disc $E = \{z : |z| < 1\}$ and further normalized by $f(0) = f'(0) - 1 = 0$ and having the Taylor series expansion of the form

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

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We denote by \mathcal{U} , the class of Schwarzian functions which has the expansion of the form

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

and which are analytic in the unit disc E and satisfying the conditions

$$w(0) = 0, |w(z)| < 1.$$

For two analytic functions f and g in E , we say that f is subordinate to g , if there exists a Schwarzian function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ and is denoted by $f \prec g$. Further, if g is univalent in E , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

By \mathcal{S} , \mathcal{S}^* and \mathcal{K} , we denote the well known classes of functions $f \in \mathcal{A}$, which are respectively univalent, starlike and convex in the unit disc E . It is obvious that $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$.

Kaplan [5], introduced the class \mathcal{C} of close-to-convex functions as follows: A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a convex function $h \in \mathcal{K}$ such that

$$\operatorname{Re} \left(\frac{f'(z)}{h'(z)} \right) > 0, z \in E.$$

Further, Noor [8] introduced the class \mathcal{C}^* of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be quasi-convex if there exists a convex function $h \in \mathcal{K}$ such that

$$\operatorname{Re} \left(\frac{(zf'(z))'}{h'(z)} \right) > 0, z \in E.$$

Every quasi-convex function is convex. Obviously $f(z) \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$.

Mocanu [7], introduced the class of alpha-convex functions which is denoted by \mathcal{M}_α ($0 \leq \alpha \leq 1$). A function $f \in \mathcal{M}_\alpha$ if it satisfying the conditions $f(z)f'(z) \neq 0$ and

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right\} > 0, z \in E.$$

In particular, $\mathcal{M}_0 \equiv \mathcal{S}^*$ and $\mathcal{M}_1 \equiv \mathcal{K}$. Miller et al. [6], proved that all alpha-convex functions are univalent and the class \mathcal{M}_α is a linear combination of the classes \mathcal{S}^* and \mathcal{K} .

Using the idea of of alpha-convex functions, Noor [9] established the class \mathcal{Q}_α ($0 \leq \alpha \leq 1$) of alpha-quasi-convex functions $f \in \mathcal{A}$ and satisfying the condition

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{f'(z)}{h'(z)} + \alpha \frac{(zf'(z))'}{h'(z)} \right\} > 0, h \in \mathcal{K}, z \in E.$$

Obviously, $\mathcal{Q}_0 \equiv \mathcal{C}$ and $\mathcal{Q}_1 \equiv \mathcal{C}^*$. It can be observed that \mathcal{Q}_α forms a linear combination of the classes of close-to-convex functions and quasi-convex functions.

Aouf [2], introduced the class $\mathcal{P}[A, B; \eta]$, ($0 \leq \eta < 1$), which is a subclass of \mathcal{A} that consists of the functions p analytic in E and subordinate to $\frac{1 + [B + (A - B)(1 - \eta)]z}{1 + Bz}$, ($-1 \leq B < A \leq 1$). For $\eta = 0$, the class $\mathcal{P}[A, B; \eta]$ reduces to $\mathcal{P}[A, B]$, the class introduced by Janowski [4].

The various subclasses of close-to-convex functions and quasi-convex functions were extensively studied by many authors such as the classes $\mathcal{C}^*(\alpha, \beta)$, $\mathcal{C}_s^*(\alpha, \beta)$, $\mathcal{C}^*(A, B)$, $\mathcal{C}_s^*(A, B)$, $\mathcal{C}^*(A, B; C, D)$, $\mathcal{C}_s^*(A, B; C, D)$ were studied respectively by Selvaraj and Stelin [13], Selvaraj et al. [14], Xiong and Liu [18], Singh and Singh [16]. Apart from this, various subclasses of alpha-quasi-convex functions such as $\mathcal{Q}_\alpha(1 - 2\beta, -1; 1 - 2\gamma, -1)$, $\mathcal{C}_*(\beta, \gamma)$, $\mathcal{Q}_\alpha(A, B)$, $\mathcal{C}_\lambda^*(\alpha, \beta)$ were investigated respectively by Noor [9], Noor and Al Aboudi [11], Selvaraj and Thirupathi [15] and Selvaraj and Logu [12].

Throughout the paper, to avoid repetition, we make the following assumptions:

$$-1 \leq D \leq B < A \leq C \leq 1, 0 \leq \alpha \leq 1, 0 \leq \eta < 1, z \in E.$$

Getting motivated by the above works, now we are on the stage to introduce the following subclasses of alpha-quasi-convex functions with subordination:

Definition 1. $\mathcal{Q}_\alpha(A, B; C, D; \eta)$ be the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$(1 - \alpha) \frac{f'(z)}{h'(z)} + \alpha \frac{(zf'(z))'}{h'(z)} \prec \frac{1 + [D + (C - D)(1 - \eta)]z}{1 + Dz},$$

where $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{K}(A, B)$.

The following observations are obvious:

- (i) $\mathcal{Q}_\alpha(1, -1; 1, -1; 0) \equiv \mathcal{Q}_\alpha$, the class introduced by Noor [9].
- (ii) For $A = 1 - 2\beta$, $B = -1$, $C = 1 - 2\gamma$, $D = -1$ and $\eta = 0$, the class $\mathcal{Q}_\alpha(A, B; C, D; \eta)$ reduces to $\mathcal{Q}_\alpha(1 - 2\beta, -1; 1 - 2\gamma, -1; 0)$, the class introduced by Noor [9].
- (iii) $\mathcal{Q}_\alpha(1, -1; C, D; 0) \equiv \mathcal{Q}_\alpha(C, D)$, the class studied by Selvaraj and Thirupathi [15].
- (iv) $\mathcal{Q}_1(A, B; C, D; 0) \equiv \mathcal{C}^*(A, B; C, D)$, the class introduced and studied by Singh and Singh [16].

(v) $\mathcal{Q}_1(1, -1; (2\alpha - 1)\beta, \beta; 0) \equiv \mathcal{C}^*(\alpha, \beta)$, the class studied by Selvaraj and Stelin [13].

(vi) $\mathcal{Q}_1(1, -1; C, D; 0) \equiv \mathcal{C}^*(C, D)$, the class studied by Xiong and Liu [18].

Definition 2. Let $\mathcal{Q}_\alpha^*(A, B; C, D; \eta)$ denote the class of functions $f \in \mathcal{A}$ and satisfying the condition that

$$(1 - \alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \prec \frac{1 + [D + (C - D)(1 - \eta)]z}{1 + Dz},$$

where $g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{S}^*(A, B)$.

We have the following observations:

(i) $\mathcal{Q}_\lambda^*(1, -1; (2\alpha - 1)\beta, \beta; 0) \equiv \mathcal{C}_\lambda^*(\alpha, \beta)$, the class studied by Selvaraj and Logu [12].

(ii) $\mathcal{Q}_1^*(A, B; C, D; 0) \equiv \mathcal{C}_s^*(A, B; C, D)$, the class introduced and studied by Singh and Singh [16].

(iii) $\mathcal{Q}_1^*(A, B; 1 - 2\beta, -1; 0) \equiv \mathcal{C}_\beta^*(A, B)$, the class introduced by Noor [10].

In this paper, we study some geometric properties of the classes $\mathcal{Q}_\alpha(A, B; C, D; \eta)$ and $\mathcal{Q}_\alpha^*(A, B; C, D; \eta)$. We investigate the coefficient estimates, distortion theorems, growth theorems and radius of quasi convexity for the functions in these classes. For particular values of the parameters A, B, C, D, α and η , the results already proved by various authors will follow as special cases.

2. Preliminary results.

Lemma 1 ([2]). *If*

$$P(z) = \frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}[C, D; \eta],$$

then

$$|p_n| \leq (C - D)(1 - \eta), n \geq 1.$$

Lemma 2 ([3]). *If $g(z) \in \mathcal{S}^*(A, B)$, then for $A - (n - 1)B \geq (n - 2)$, $n \geq 3$,*

$$|d_n| \leq \frac{1}{(n - 1)!} \prod_{j=2}^n (A - (j - 1)B).$$

Lemma 3 ([3]). If $g(z) \in \mathcal{S}^*(A, B)$, then for $|z| = r < 1$,

$$r(1 - Br)^{\frac{A-B}{B}} \leq |g(z)| \leq r(1 + Br)^{\frac{A-B}{B}}, B \neq 0;$$

$$re^{-Ar} \leq |g(z)| \leq re^{Ar}, B = 0.$$

Lemma 4 ([17]). If $h(z) \in \mathcal{K}(A, B)$, then for $A - (n-1)B \geq (n-2), n \geq 3$,

$$|b_n| \leq \frac{1}{n!} \prod_{j=2}^n (A - (j-1)B).$$

Lemma 5 ([17]). If $h(z) \in \mathcal{K}(A, B)$, then for $|z| = r < 1$,

$$\frac{1}{A} \left[1 - (1 - Br)^{\frac{A}{B}} \right] \leq |h(z)| \leq \frac{1}{A} \left[(1 + Br)^{\frac{A}{B}} - 1 \right], B \neq 0;$$

$$\frac{1}{A} [1 - e^{-Ar}] \leq |h(z)| \leq \frac{1}{A} [e^{Ar} - 1], B = 0.$$

Lemma 6 ([1, 2]). If $P(z) = \frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)}$,

$-1 \leq D < C \leq 1$, $w(z) \in \mathcal{U}$, then for $|z| = r < 1$, we have

$$Re \frac{zP'(z)}{P(z)} \geq \begin{cases} -\frac{(C-D)(1-\eta)r}{(1-[D+(C-D)(1-\eta)]r)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ 2\sqrt{\frac{(1-D)(1-[D+(C-D)(1-\eta)])(1+[D+(C-D)(1-\eta)]r^2)(1+Dr^2)}{(C-D)(1-\eta)(1-r^2)}}, \\ -\frac{(1-[D+(C-D)(1-\eta)]Dr^2)}{(C-D)(1-\eta)(1-r^2)} + \frac{(C+D)-\eta(C-D)}{(C-D)(1-\eta)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\frac{(1 - [D + (C - D)(1 - \eta)])(1 + [D + (C - D)(1 - \eta)]r^2)}{(1 - D)(1 + Dr^2)}}$ and

$$R_2 = \frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr}.$$

3. Properties of the class $\mathcal{Q}_\alpha(A, B; C, D; \eta)$.

Theorem 1. Let $f(z) \in \mathcal{Q}_\alpha(A, B; C, D; \eta)$, then for $A - (n-1)B \geq (n-2), n \geq 2$,

$$(2) \quad |a_n| \leq \frac{1}{[(1 - \alpha)n + \alpha n^2]} \left\{ \frac{1}{(n-1)!} \prod_{j=2}^n (A - (j-1)B) \right.$$

$$\left. + (C - D)(1 - \eta) \left[1 + \sum_{k=2}^{n-1} \frac{1}{(k-1)!} \prod_{j=2}^k (A - (j-1)B) \right] \right\}.$$

The bounds are sharp.

Proof. In Definition 1, using Principle of subordination, we have

$$(3) \quad (1 - \alpha)f'(z) + \alpha(zf'(z))' = h'(z) \left(\frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right),$$

$w(z) \in \mathcal{U}.$

On expanding (3), it yields

$$(4) \quad \begin{aligned} & (1 - \alpha)[1 + 2a_2z + 3a_3z^2 + \cdots + na_nz^{n-1} + \cdots] \\ & + \alpha[1 + 4a_2z + 9a_3z^2 + \cdots + n^2a_nz^{n-1} + \cdots] \\ & = (1 + 2b_2z + 3b_3z^2 + \cdots + nb_nz^{n-1} + \cdots) \\ & \quad \times (1 + p_1z + p_2z^2 + \cdots + p_{n-1}z^{n-1} + \cdots). \end{aligned}$$

Equating the coefficients of z^{n-1} in (4), we have

$$(5) \quad [(1 - \alpha)n + \alpha n^2]a_n = nb_n + (n - 1)p_1b_{n-1} + (n - 2)p_2b_{n-2} \dots + 2p_{n-2}b_2 + p_{n-1}.$$

Applying triangle inequality and using Lemma 1 in (5), it gives

$$(6) \quad \begin{aligned} & [(1 - \alpha)n + \alpha n^2]|a_n| \\ & \leq n|b_n| + (C - D)(1 - \eta) [(n - 1)|b_{n-1}| + (n - 2)|b_{n-2}| + \cdots + 2|b_2| + 1]. \end{aligned}$$

Using Lemma 4 in (6), the result (2) is obvious.

For $n = 2$, equality sign in (2) hold for the functions $f_n(z)$ defined as

$$(7) \quad \begin{aligned} & (1 - \alpha)f'_n(z) + \alpha(zf'_n(z))' \\ & = (1 + B\delta_1z)^{\frac{(A-B)}{B}} \left(\frac{1 + [D + (C - D)(1 - \eta)]\delta_2z^n}{1 + D\delta_2z^n} \right), \\ & \quad B \neq 0, |\delta_1| = 1, |\delta_2| = 1. \end{aligned}$$

□

Remark 1.

(i) For $A = 1, B = -1, C = 1, D = -1, \eta = 0$, Theorem 1 gives the result due to Noor [9].

(ii) For $A = 1 - 2\beta, B = -1, C = 1 - 2\gamma, D = -1, \eta = 0$, Theorem 1 agrees with the result due to Noor [9].

(iii) On putting $A = 1, B = -1, \eta = 0$ in Theorem 1, we can obtain the result proved by Selvaraj and Thirupathi [15].

(iv) For $\alpha = 1, \eta = 0$, Theorem 1 gives the result proved by Singh and Singh [16].

(v) On putting $\alpha = 1, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta, \eta = 0$, Theorem 1 agrees with the result due to Selvaraj and Stelin [13].

(vi) For $\alpha = 1, A = 1, B = -1, \eta = 0$, Theorem 1 gives the result proved by Xiong and Liu [18].

Theorem 2. *If $f(z) \in \mathcal{Q}_\alpha(A, B; C, D; \eta)$, then for $|z| = r, 0 < r < 1$, we have
for $\alpha = 0, B \neq 0$,*

$$(8) \quad \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - Bt)^{\frac{A-B}{B}} dt \\ \leq |f(z)| \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + Bt)^{\frac{A-B}{B}} dt;$$

for $\alpha = 0, B = 0$,

$$(9) \quad \int_0^r \frac{1}{A} \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 + Ae^{-At}) dt \\ \leq |f(z)| \leq \int_0^r \frac{1}{A} \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (Ae^{At} - 1) dt,$$

and for $0 < \alpha \leq 1, B \neq 0$,

$$(10) \quad \frac{1}{\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - Bt)^{\frac{A-B}{B}} dt \right] ds \leq |f(z)| \\ \leq \frac{1}{\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + Bt)^{\frac{A-B}{B}} dt \right] ds;$$

for $0 < \alpha \leq 1, B = 0$,

$$(11) \quad \frac{1}{A\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 + Ae^{-At}) dt \right] ds \leq |f(z)| \\ \leq \frac{1}{A\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (Ae^{At} - 1) dt \right] ds.$$

Estimates are sharp.

Proof. From (3), we have

$$(12) \quad |(1 - \alpha)f'(z) + \alpha(zf'(z))'| = |h'(z)| \left| \frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right| \\ = |h'(z)||P(z)|, w(z) \in \mathcal{U}.$$

Aouf [2], proved that

$$(13) \quad \frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \leq |P(z)| \leq \frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr}.$$

Let $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$.

As $h(z) \in \mathcal{K}(A, B)$, so from Lemma 5, we have

$$(14) \quad \begin{cases} (1 - Br)^{\frac{A-B}{B}} \leq |h'(z)| \leq (1 + Br)^{\frac{A-B}{B}}, & \text{if } B \neq 0; \\ \frac{1}{A} [1 + Ae^{-Ar}] \leq |h'(z)| \leq \frac{1}{A} [Ae^{Ar} - 1], & \text{if } B = 0. \end{cases}$$

Using (13) and (14) in (12), it yields

$$(15) \quad \begin{cases} \left(\frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \right) (1 - Br)^{\frac{A-B}{B}} \leq |F'(z)| \\ \leq \left(\frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr} \right) (1 + Br)^{\frac{A-B}{B}}, & \text{if } B \neq 0; \\ \left(\frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \right) \frac{1}{A} [1 + Ae^{-Ar}] \\ \leq |F'(z)| \leq \left(\frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr} \right) \frac{1}{A} [Ae^{Ar} - 1], & \text{if } B = 0. \end{cases}$$

On integrating, (15) yields

$$(16) \quad \begin{cases} \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - Bt)^{\frac{A-B}{B}} dt \leq |F(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + Bt)^{\frac{A-B}{B}} dt, & \text{if } B \neq 0; \\ \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) \frac{1}{A} [1 + Ae^{-At}] dt \leq |F(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) \frac{1}{A} [Ae^{At} - 1] dt, & \text{if } B = 0. \end{cases}$$

This implies

$$(17) \quad \begin{cases} \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - Bt)^{\frac{A-B}{B}} dt \\ \leq |(1 - \alpha)f(z) + \alpha z f'(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + Bt)^{\frac{A-B}{B}} dt, & \text{if } B \neq 0; \\ \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) \frac{1}{A} [1 + Ae^{-At}] dt \\ \leq |(1 - \alpha)f(z) + \alpha z f'(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) \frac{1}{A} [Ae^{At} - 1] dt, & \text{if } B = 0. \end{cases}$$

For $\alpha = 0$, the results (8) and (9) are obvious from (17).

Also for $0 < \alpha \leq 1$ and on integrating (17), the results (10) and (11) are obvious. Sharpness follows for the function $f_n(z)$ defined in (7). \square

Remark 2.

(i) On putting $A = 1, B = -1, \eta = 0$ in Theorem 2, we can obtain the result proved by Selvaraj and Thirupathi [15].

(ii) For $\alpha = 1, \eta = 0$, Theorem 2 gives the result proved by Singh and Singh [16].

(iii) On putting $\alpha = 1, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta, \eta = 0$, Theorem 2 agrees with the result due to Selvaraj and Stelin [13].

(iv) For $\alpha = 1, A = 1, B = -1, \eta = 0$, Theorem 2 gives the result proved by Xiong and Liu [18].

Theorem 3. Let $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$, where $f(z) \in \mathcal{Q}_\alpha(A, B; C, D; \eta)$, then

$$(18) \quad \operatorname{Re} \frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \frac{1-Ar}{1-Br} - \frac{(C-D)(1-\eta)r}{(1-[D+(C-D)(1-\eta)]r)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ \frac{1-Ar}{1-Br} + \frac{(C+D)-\eta(C-D)}{(C-D)(1-\eta)} \\ + 2 \frac{\sqrt{(1-D)(1-[D+(C-D)(1-\eta)])(1+[D+(C-D)(1-\eta)]r^2)(1+Dr^2)}}{(C-D)(1-\eta)(1-r^2)} \\ - 2 \frac{(1-[D+(C-D)(1-\eta)]Dr^2)}{(C-D)(1-\eta)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 6.

Proof. As $f(z) \in \mathcal{Q}_\alpha(A, B; C, D; \eta)$, we have

$$(1 - \alpha)f'(z) + \alpha(zf'(z))' = h'(z) \left(\frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right) = h'(z)P(z).$$

Here $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$. So on differentiating it logarithmically, we get

$$(19) \quad \frac{(zF'(z))'}{F'(z)} = \frac{(zh'(z))'}{h'(z)} + \frac{zP'(z)}{P(z)}.$$

Now for $h \in \mathcal{K}(A, B)$, we have

$$(20) \quad \operatorname{Re} \left(\frac{(zh'(z))'}{h'(z)} \right) \geq \frac{1 - Ar}{1 - Br}.$$

So using Lemma 6 and inequality (20) in equation (19), the result (18) is obvious. Sharpness follows if we take the function $f_n(z)$ to be same as in (7). \square

Remark 3.

(i) For $\alpha = 1, \eta = 0$, Theorem 3 gives the result proved by Singh and Singh [16].

(ii) On putting $\alpha = 1, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta, \eta = 0$, Theorem 3 agrees with the result due to Selvaraj and Stelin [13].

(iii) For $\alpha = 1, A = 1, B = -1, \eta = 0$, Theorem 3 gives the result proved by Xiong and Liu [18].

4. Properties of the class $\mathcal{Q}_\alpha^*(A, B; C, D; \eta)$.

Theorem 4. Let $f(z) \in \mathcal{Q}_\alpha^*(A, B; C, D; \eta)$, then for $A - (n - 1)B \geq (n - 2), n \geq 2$,

$$(21) \quad |a_n| \leq \frac{1}{[(1 - \alpha)n + \alpha n^2]} \left\{ \frac{n}{(n - 1)!} \prod_{j=2}^n (A - (j - 1)B) \right. \\ \left. + (C - D)(1 - \eta) \left[1 + \sum_{k=2}^{n-1} \frac{k}{(k - 1)!} \prod_{j=2}^k (A - (j - 1)B) \right] \right\}.$$

The results are sharp.

Proof. From Definition 2, using Principle of subordination, we have

$$(22) \quad (1 - \alpha)f'(z) + \alpha(zf'(z))' \\ = g'(z) \left(\frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right), w(z) \in \mathcal{U}.$$

On expanding (22), it yields

$$(23) \quad (1 - \alpha)[1 + 2a_2z + 3a_3z^2 + \cdots + na_nz^{n-1} + \cdots] \\ + \alpha[1 + 4a_2z + 9a_3z^2 + \cdots + n^2a_nz^{n-1} + \cdots] \\ = (1 + 2d_2z + 3d_3z^2 + \cdots + nd_nz^{n-1} + \cdots) \\ \times (1 + p_1z + p_2z^2 + \cdots + p_{n-1}z^{n-1} + \cdots).$$

Equating the coefficients of z^{n-1} in (23), we have

$$(24) \quad [(1 - \alpha)n + \alpha n^2]a_n = nd_n + (n - 1)p_1d_{n-1} + (n - 2)p_2d_{n-2} \cdots + 2p_{n-2}d_2 + p_{n-1}.$$

Applying triangle inequality and Lemma 1 in (24), it gives

$$(25) \quad [(1 - \alpha)n + \alpha n^2]|a_n| \\ \leq n|d_n| + (C - D)(1 - \eta)[(n - 1)|d_{n-1}| + (n - 2)|d_{n-2}| + \cdots + 2|d_2| + 1].$$

Using Lemma 2 in (25), the result (21) is obvious.

For $n = 2$, equality sign in (21) hold for the functions $f_n(z)$ defined as

$$(26) \quad (1 - \alpha)f'_n(z) + \alpha(zf'_n(z))' \\ = (1 + B\delta_1z)^{\frac{(A-B)}{B}} \left(\frac{1 + A\delta_1z^n}{1 + B\delta_1z^n} \right) \left(\frac{1 + [D + (C - D)(1 - \eta)]\delta_2z^n}{1 + D\delta_2z^n} \right), \\ B \neq 0, |\delta_1| = 1, |\delta_2| = 1.$$

□

Remark 4.

(i) For $\alpha = \lambda, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta, \eta = 0$, Theorem 4 agrees with the result due to Selvaraj and Logu [12].

(ii) On putting $\alpha = 1, \eta = 0$, Theorem 4 gives the result proved by Singh and Singh [16].

Theorem 5. *If $f(z) \in \mathcal{Q}_\alpha^*(A, B; C, D; \eta)$, then for $|z| = r, 0 < r < 1$, we have
for $\alpha = 0, B \neq 0$,*

$$(27) \quad \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - At)(1 - Bt)^{\frac{A-2B}{B}} dt \\ \leq |f(z)| \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + At)(1 + Bt)^{\frac{A-2B}{B}} dt;$$

for $\alpha = 0, B = 0$,

$$(28) \quad \int_0^r \frac{1}{A} \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) e^{-At}(1 - At) dt \\ \leq |f(z)| \leq \int_0^r \frac{1}{A} \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) e^{At}(1 + At) dt,$$

and for $0 < \alpha \leq 1, B \neq 0$,

$$(29) \quad \frac{1}{\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - At)(1 - Bt)^{\frac{A-2B}{B}} dt \right] ds \leq |f(z)| \\ \leq \frac{1}{\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + At)(1 + Bt)^{\frac{A-2B}{B}} dt \right] ds;$$

for $0 < \alpha \leq 1, B = 0$,

$$(30) \quad \frac{1}{A\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) e^{-At}(1 - At) dt \right] ds \leq |f(z)| \\ \leq \frac{1}{A\alpha} \int_0^r \left[\frac{1}{s} \int_0^s \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) e^{At}(1 + At) dt \right] ds.$$

Estimates are sharp.

Proof. From (22), we have

$$(31) \quad |(1 - \alpha)f'(z) + \alpha(zf'(z))'| = |g'(z)| \left| \frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right| \\ = |g'(z)| |P(z)|, w(z) \in \mathcal{U}.$$

From (13), we have

$$(32) \quad \frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \leq |P(z)| \leq \frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr}.$$

Let $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$.

As $h(z) \in \mathcal{S}^*(A, B)$, so from Lemma 3, we have

$$(33) \quad \begin{cases} (1 - Ar)(1 - Br)^{\frac{A-2B}{B}} \leq |g'(z)| \leq (1 + Ar)(1 + Br)^{\frac{A-2B}{B}}, & \text{if } B \neq 0; \\ e^{-Ar}[1 - Ar] \leq |g'(z)| \leq e^{Ar}[1 + Ar], & \text{if } B = 0. \end{cases}$$

Using (32) and (33) in (31), it yields

$$(34) \quad \begin{cases} \left(\frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \right) (1 - Ar)(1 - Br)^{\frac{A-2B}{B}} \leq |F'(z)| \\ \leq \left(\frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr} \right) (1 + Ar)(1 + Br)^{\frac{A-2B}{B}}, & \text{if } B \neq 0; \\ \left(\frac{1 - [D + (C - D)(1 - \eta)]r}{1 - Dr} \right) e^{-Ar}[1 - Ar] \\ \leq |F'(z)| \leq \left(\frac{1 + [D + (C - D)(1 - \eta)]r}{1 + Dr} \right) e^{Ar}[1 + Ar], & \text{if } B = 0. \end{cases}$$

On integrating, (34) gives

$$(35) \quad \begin{cases} \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - At)(1 - Bt)^{\frac{A-2B}{B}} dt \leq |F(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + At)(1 + Bt)^{\frac{A-2B}{B}} dt, & \text{if } B \neq 0; \\ \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) e^{-At}[1 - At] dt \leq |F(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) e^{At}[1 + At] dt, & \text{if } B = 0. \end{cases}$$

Therefore, we have

$$(36) \quad \left\{ \begin{array}{l} \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) (1 - At)(1 - Bt)^{\frac{A-2B}{B}} dt \\ \leq |(1 - \alpha)f(z) + \alpha z f'(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) (1 + At)(1 + Bt)^{\frac{A-2B}{B}} dt, \text{ if } B \neq 0; \\ \int_0^r \left(\frac{1 - [D + (C - D)(1 - \eta)]t}{1 - Dt} \right) e^{-At}[1 - At] dt \\ \leq |(1 - \alpha)f(z) + \alpha z f'(z)| \\ \leq \int_0^r \left(\frac{1 + [D + (C - D)(1 - \eta)]t}{1 + Dt} \right) e^{At}[1 + At] dt, \quad \text{if } B = 0. \end{array} \right.$$

For $\alpha = 0$, the results (27) and (28) are obvious from (36).

Also for $0 < \alpha \leq 1$ and on integrating (36), the results (29) and (30) are obvious. Sharpness follows for the function $f_n(z)$ defined in (26). \square

Remark 5.

(i) On putting $A = 1, B = -1, \eta = 0$ in Theorem 5, we obtain the result proved by Selvaraj and Thirupathi [15].

(ii) For $\alpha = 1, \eta = 0$, Theorem 5 gives the result proved by Singh and Singh [16].

(iii) On putting $\alpha = 1, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta, \eta = 0$, Theorem 5 agrees with the result due to Selvaraj and Stelin [13].

(iv) For $\alpha = 1, A = 1, B = -1, \eta = 0$, Theorem 5 gives the result proved by Xiong and Liu [18].

Theorem 6. Let $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$, where $f(z) \in \mathcal{Q}_\alpha^*(A, B; C, D; \eta)$, then

$$(37) \quad \operatorname{Re} \frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \frac{1-Ar}{1-Br} - \frac{(A-B)r}{1-B^2r^2} - \frac{(C-D)(1-\eta)r}{(1-[D+(C-D)(1-\eta)]r)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ \frac{1-Ar}{1-Br} - \frac{(A-B)r}{1-B^2r^2} + \frac{(C+D)-\eta(C-D)}{(C-D)(1-\eta)} \\ + 2 \frac{\sqrt{(1-D)(1-[D+(C-D)(1-\eta)])(1+[D+(C-D)(1-\eta)]r^2)(1+Dr^2)}}{(C-D)(1-\eta)(1-r^2)} \\ - 2 \frac{(1-[D+(C-D)(1-\eta)]Dr^2)}{(C-D)(1-\eta)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 6.

Proof. As $f(z) \in \mathcal{Q}_\alpha^*(A, B; C, D; \eta)$, we have

$$(1 - \alpha)f'(z) + \alpha(zf'(z))' = g'(z) \left(\frac{1 + [D + (C - D)(1 - \eta)]w(z)}{1 + Dw(z)} \right) = g'(z)P(z).$$

Here $F'(z) = (1 - \alpha)f'(z) + \alpha(zf'(z))'$. So on differentiating it logarithmically, we get

$$(38) \quad \frac{(zF'(z))'}{F'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zP'(z)}{P(z)}.$$

Now for $g \in \mathcal{S}^*(A, B)$, we have

$$(39) \quad Re \left(\frac{(zg'(z))'}{g'(z)} \right) \geq \frac{1 - Ar}{1 - Br} - \frac{(A - B)r}{1 - B^2r^2}.$$

So using Lemma 6 and inequality (39) in equation (38), the result (37) is obvious. Sharpness follows for the function $f_n(z)$ to be same as in (26). \square

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