Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **47** (2021), 61–80

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

η-RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS ADMITTING DIFFERENT SEMI-SYMMETRIC STRUCTURES

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Communicated by O. Mushkarov

Abstract. The object of the present paper is to study η-Ricci solitons and Ricci solitons on $(LCS)_n$-manifolds admitting different equivalency group or class of semi-symmetric conditions based on Kundu and Shaikh [29].

1. Introduction. Throughout our manuscript, we denote by Q, S and r the Ricci operator, the Ricci curvature tensor and the scalar curvature, respectively.

Definition 1. *Let T and D be two tensors of type $(0,4)$. A semi-Riemannian (or Riemannian) manifold is said to be of T-semi-symmetric type if $D(U,V)\cdot T=0$, for all U, $V\in\chi(M)$, where $\chi(M)$ denotes the set of all vector fields on the manifold M and $D(U,V)$ acts on T as derivation of tensor algebra.*

2020 *Mathematics Subject Classification:* 53C15, 53C25.

Key words: η-Ricci soliton, $(LCS)_n$-manifold, semi-symmetric structure, η-Einstein manifold.

The above condition is often written as $D\cdot T = 0$. Especially, if we consider $D = T = R$, then the manifold is called semi-symmetric [33]. Details about the semi-symmetry and other conditions of semi-symmetric type are available in ([6, 28, 7, 3, 20, 32]) and also references therein.

In 2013, Kundu and Shaikh [29] investigated the equivalency of various geometric structures. They have established the following conditions:

i) $R\cdot R = 0$, $R\cdot P = 0$, $R\cdot E = 0$, $R\cdot P^* = 0$, $R\cdot \mathcal{M} = 0$, $R\cdot \mathcal{W}_i = 0$ and $R\cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, \ldots, 9$) are equivalent and named such a class by C_1;

ii) $E\cdot R = 0$, $E\cdot P = 0$, $E\cdot E = 0$, $E\cdot P^* = 0$, $E\cdot \mathcal{M} = 0$, $E\cdot \mathcal{W}_i = 0$ and $E\cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, \ldots, 9$) are equivalent and named such a class by C_2;

iii) $R\cdot C = 0$ and $R\cdot K = 0$ are equivalent and named such a class by C_3;

iv) $E\cdot C = 0$ and $E\cdot K = 0$ are equivalent and named such a class by C_4, where the concircular curvature tensor E [35], conformal curvature tensor C [15], conharmonic curvature tensor K [18], projective curvature tensor P [35], M-projective curvature tensor $\mathcal{M}$ [25], $\mathcal{W}_i$-curvature tensor, $i = 1, 2, \ldots, 9$ ([25], [23], [24]) and $\mathcal{W}_i^*$-curvature tensor, $i = 1, 2, \ldots, 9$ [25] are defined respectively by

$$E(U,V) = R(U,V) - \frac{r}{n(n-1)}(U \wedge_g V), \tag{1}$$

$$C(U,V) = R(U,V) - \frac{1}{n-2}[(U \wedge_g QV) + (QU \wedge_g V) + \frac{r}{(n-1)}(U \wedge_g V)], \tag{2}$$

$$K(U,V) = R(U,V) - \frac{1}{n-2}[(U \wedge_g QV) + (QU \wedge_g V)], \tag{3}$$

$$P(U,V) = R(U,V) - \frac{1}{n-1}(U \wedge_S V), \tag{4}$$

$$\mathcal{M}(U,V) = R(U,V) - \frac{1}{2(n-1)}[(U \wedge_g QV) + (QU \wedge_g V)], \tag{5}$$

$$\mathcal{W}_0(U,V) = R(U,V) - \frac{1}{(n-1)}(U \wedge_g QV), \tag{6}$$

$$\mathcal{W}_0^*(U,V) = R(U,V) + \frac{1}{(n-1)}(U \wedge_g QV), \tag{7}$$

$$\mathcal{W}_1(U,V) = R(U,V) - \frac{1}{(n-1)}(U \wedge_S V), \tag{8}$$

$$\mathcal{W}_1^*(U,V) = R(U,V) + \frac{1}{(n-1)}(U \wedge_S V), \tag{9}$$

$$\mathcal{W}_2(U,V) = R(U,V) - \frac{1}{(n-1)}[(QU \wedge_g V) + (U \wedge_g QV) - (U \wedge_S V)], \tag{10}$$

$$\mathcal{W}_2^*(U,V) = R(U,V) + \frac{1}{(n-1)}[(QU \wedge_g V) + (U \wedge_g QV) - (U \wedge_S V)], \tag{11}$$

$$\mathcal{W}_3(U,V) = R(U,V) - \frac{1}{(n-1)}(V \wedge_g QU), \tag{12}$$

$$\mathcal{W}_3^*(U,V) = R(U,V) + \frac{1}{(n-1)}(V \wedge_g QU), \tag{13}$$

$$\mathcal{W}_5(U,V) = R(U,V) - \frac{1}{(n-1)}[(U \wedge_g QV) - (U \wedge_S V)], \tag{14}$$

$$\mathcal{W}_5^*(U,V) = R(U,V) + \frac{1}{(n-1)}[(U \wedge_g QV) - (U \wedge_S V)], \tag{15}$$

$$\mathcal{W}_7(U,V) = R(U,V) + \frac{1}{(n-1)}[(QU \wedge_g V) - (U \wedge_S V)], \tag{16}$$

$$\mathcal{W}_7^*(U,V) = R(U,V) - \frac{1}{(n-1)}[(QU \wedge_g V) - (U \wedge_S V)], \tag{17}$$

$$\mathcal{W}_4(U,V)Z = R(U,V)Z - \frac{1}{(n-1)}[g(U,Z)QV - g(U,V)QZ], \tag{18}$$

$$\mathcal{W}_4^*(U,V)Z = R(U,V)Z + \frac{1}{(n-1)}[g(U,Z)QV - g(U,V)QZ], \tag{19}$$

$$\mathcal{W}_6(U,V)Z = R(U,V)Z - \frac{1}{(n-1)}[S(V,Z)U - g(U,V)QZ], \tag{20}$$

where

$$(U \wedge_B V)\, Z = B(V,Z)U - B(U,Z)V. \tag{21}$$

In 1982, R. S. Hamilton [16] introduced the notion of Ricci flow to investigate a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([22], [21]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}\,(t) = -2R_{ij}. \tag{22}$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, W, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric [30, 17, 14, 4] satisfying

$$(\pounds_W g)\,(U,V) + 2S\,(U,V) + 2\lambda g\,(U,V) = 0, \tag{23}$$

where S is the Ricci tensor, $\pounds_W$ is the Lie derivative operator along the vector field W on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by J. T. Cho and M. Kimura [13]. They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined η-Ricci soliton (g, W, λ, μ), which satisfies the equation

$$(\pounds_W g)\,(U,V) + 2S\,(U,V) + 2\lambda g\,(U,V) + 2\mu\eta\,(U)\,\eta\,(V) = 0, \tag{24}$$

where λ and μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, W, λ, μ) reduces to the notion of Ricci soliton (g, W, λ). Recenty, η-Ricci solitons have been studied by various authors such as A. Singh and S. Kishor [31], A. M. Blaga [10, 11, 9, 8], D. G. Prakasha and B. S. Hadimani [26] and many others.

The present paper is structured as follows. After Introduction, in Section 2, we briefly recall some known results for $(LCS)_n$-manifolds. In Section 3, we discuss about η-Ricci solitons and Ricci solitons on $(LCS)_n$-manifolds and we obtain the relation between the scalars λ and μ, precisely $\lambda - \mu = -(n-1)(\alpha^2 - \rho)$. In the same section we also study η-Ricci solitons and Ricci solitons on $(LCS)_n$-manifolds admitting $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$,

$R\cdot\mathcal{W}_i=0$, $R\cdot\mathcal{W}_i^*=0$, $E\cdot R=0$, $E\cdot P=0$, $E\cdot E=0$, $E\cdot P^*=0$, $E\cdot\mathcal{M}=0$, $E\cdot\mathcal{W}_i=0$, $E\cdot\mathcal{W}_i^*=0$, $R\cdot C=0$, $R\cdot K=0$, $E\cdot C=0$ and $E\cdot K=0$.

2. Properties of $(LCS)_n$-manifolds. Let (M^n,g) be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi)=-1. \tag{25}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(U,\xi)=\eta(U), \tag{26}$$

the equation of the following form holds

$$(\nabla_U\eta)V=\alpha\left[g(U,V)+\eta(U)\eta(V)\right], \tag{27}$$

for all vector fields U,V, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function, which satisfies

$$\nabla_U\alpha=(U\alpha)=d\alpha(U)=\rho\eta(U), \tag{28}$$

ρ being a certain scalar function. If we put

$$\phi U=\frac{1}{\alpha}\nabla_U\xi, \tag{29}$$

then from (27) and (29) we get

$$\phi U=U+\eta(U)\xi, \tag{30}$$

from which it follows that ϕ is a symmetric $(1,1)$-tensor field. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and the $(1,1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold, in brief, $(LCS)_n$-manifold. In an $(LCS)_n$-manifold, the following relations hold [5]:

$$\eta(\xi)=-1,\ \phi\circ\xi=0, \tag{31}$$

$$\eta(\phi U)=0,\ g(\phi U,\phi V)=g(U,V)+\eta(U)\eta(V), \tag{32}$$

$$\eta(R(U,V)Z) = (\alpha^2 - \rho)[g(V,Z)\eta(U) - g(U,Z)\eta(V)], \tag{33}$$

$$R(U,V)\xi = (\alpha^2 - \rho)[\eta(V)U - \eta(U)V], \tag{34}$$

$$R(\xi,U)V = (\alpha^2 - \rho)[g(U,V)\xi - \eta(V)U], \tag{35}$$

$$S(U,\xi) = (n-1)(\alpha^2 - \rho)\eta(U), \tag{36}$$

for any vector fields U, V, Z.

In view of (33), from (1), (2), (3), (4) and (21) one can easily bring out the followings:

$$\begin{aligned} & g(C(U,V)Z,\xi) \\ = \ & \eta(C(U,V)Z) \\ = \ & \left[\frac{r}{(n-1)(n-2)} + (\alpha^2 - \rho) - \frac{(n-1)}{(n-2)}\right][g(V,Z)\eta(U) \\ - \ & g(U,Z)\eta(V)] - \frac{1}{n-2}\left[S(V,Z)\eta(U) - S(U,Z)\eta(V)\right], \end{aligned} \tag{37}$$

$$\begin{aligned} & g(K(U,V)Z,\xi) \\ = \ & \eta(K(U,V)Z) \\ = \ & \left[(\alpha^2 - \rho) - \frac{(n-1)}{(n-2)}\right][g(V,Z)\eta(U) - g(U,Z)\eta(V)] \\ & -\frac{1}{n-2}\left[S(V,Z)\eta(U) - S(U,Z)\eta(V)\right], \end{aligned} \tag{38}$$

$$\begin{aligned} & g(E(U,V)Z,\xi) \\ = \ & \eta(E(U,V)Z) \\ = \ & \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right][g(V,Z)\eta(U) - g(U,Z)\eta(V)], \end{aligned} \tag{39}$$

$$
\begin{aligned}
& g(P(U,V)Z,\xi) \\
= \; & \eta(P(U,V)Z) \\
= \; & (\alpha^2-\rho)[g(V,Z)\eta(U)-g(U,Z)\eta(V)] \\
- \; & \frac{1}{n-1}[S(V,Z)\eta(U)-S(U,Z)\eta(V)]. \qquad (40)
\end{aligned}
$$

Definition 2. *An n-dimensional $(LCS)_n$-manifold is said to be an η-Einstein manifold if the Ricci curvature tensor S is of the form*

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on M^n and η is a 1-form.

In particular, if $b = 0$, then M^n is said to be an *Einstein manifold.*

3. Main results. In this section first we consider an $(LCS)_n$-manifold admitting an η-Ricci soliton (g, ξ, λ, μ). Then, obviously from (24), we get

$$(\pounds_\xi g)(U,V) + 2S(U,V) + 2\lambda g(U,V) + 2\mu\eta(U)\eta(V) = 0. \qquad (41)$$

Now, we express the Lie derivative along ξ on M as follows:

$$
\begin{aligned}
& (\pounds_\xi g)(U,V) \\
= \; & \pounds_\xi (g(U,V)) - g(\pounds_\xi U, V) - g(U, \pounds_\xi V) \\
= \; & \pounds_\xi g(U,V) - g([\xi,U],V) - g(U,[\xi,V]). \qquad (42)
\end{aligned}
$$

Now using (29) in the foregoing equation we obtain

$$(\pounds_\xi g)(U,V) = 2\alpha g(\phi U, V). \qquad (43)$$

By using (43) into (41), we get

$$S(U,V) = -(\alpha+\lambda) g(U,V) - (\alpha+\mu)\eta(U)\eta(V). \qquad (44)$$

Taking $U = V = \xi$ in (44), we obtain

$$\lambda - \mu = -(n-1)(\alpha^2 - \rho). \qquad (45)$$

Thus we can state the following theorems:

Theorem 3. *If (g, ξ, λ, μ) is an η-Ricci soliton on $(LCS)_n$-manifold, then the scalars λ and μ are related by $\lambda - \mu = -(n-1)(\alpha^2 - \rho)$.*

Theorem 4. *If (g, ξ, λ) is a Ricci soliton on an $(LCS)_n$-manifold, then it is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative, respectively.*

Now we consider (g, W, λ, μ) an η-Ricci soliton on an $(LCS)_n$-manifold such that W is pointwise collinear with ξ, i.e. $W = b\xi$, where b is a smooth function. Then from (24), we have

$$\begin{aligned} 0 &= (Ub)\,\eta(V) + (Vb)\,\eta(U) + 2b\alpha\left[g(U,V) + \eta(U)\eta(V)\right] \\ &\quad +2S(U,V) + 2\lambda g(U,V) + 2\mu\eta(U)\,\eta(V). \end{aligned} \tag{46}$$

Setting $V = \xi$ in (46) and using (25), we obtain

$$\begin{aligned} 0 &= -(Ub) + (\xi b)\,\eta(U) + 2(\lambda - \mu)\,\eta(U) \\ &\quad +2(n-1)(\alpha^2 - \rho)\eta(U). \end{aligned} \tag{47}$$

Putting $U = \xi$ in (47) and by the help of (25) we get

$$0 = (\xi b) + (n-1)(\alpha^2 - \rho) + (\lambda - \mu). \tag{48}$$

Using (48) in (47) we obtain

$$0 = -(Ub) + \left[(n-1)(\alpha^2 - \rho) + (\lambda - \mu)\right]\eta(U). \tag{49}$$

Now applying d in (49), we find

$$\left[(n-1)(\alpha^2 - \rho) + (\lambda - \mu)\right] d\eta = 0. \tag{50}$$

Since $d\eta \neq 0$, we get

$$\lambda - \mu = -(n-1)(\alpha^2 - \rho). \tag{51}$$

Substituting (51) in (48), we conclude that b is a constant. Hence it is verified from (46) that

$$S(U,V) = -\left[(b\alpha + \lambda)\,g(U,V) + (b\alpha + \mu)\,\eta(U)\eta(V)\right]. \tag{52}$$

Thus we can state the followings:

Theorem 5. *If (g, W, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold such that W is pointwise collinear with ξ, then W is a constant multiple of ξ and the manifold is an η-Einstein manifold of the form* (52) *with the scalars λ and μ related by $\lambda - \mu = -(n-1)(\alpha^2 - \rho)$.*

In particular, for $\mu = 0$, from (51) we get

$$\lambda = -(n-1)(\alpha^2 - \rho). \tag{53}$$

Theorem 6. *If (g, W, λ) is a Ricci soliton on an $(LCS)_n$-manifold such that W is pointwise collinear with ξ, then it is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative, respectively.*

3.1. η-Ricci solitons on $(LCS)_n$-manifolds admitting the class C_1. Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(R(U,V)\cdot R)(X,Y)Z = 0,$$

which implies

$$\begin{aligned} &g(R(\xi,V)R(X,Y)Z,\xi) - g(R(R(\xi,V)X,Y)Z,\xi) \\ &-g(R(X,R(\xi,V)Y)Z,\xi) - g(R(X,Y)R(\xi,V)Z,\xi) = 0. \end{aligned} \tag{54}$$

Putting $V = X = e_i$ in (54), where $\{e_1, e_2, e_3, \ldots, e_{n-1}, e_n = \xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M^n and taking the summation over i, $1 \le i \le n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi,e_i)R(e_i,Y)Z,\xi) - \sum_{i=1}^{n} g(R(R(\xi,e_i)e_i,Y)Z,\xi) \\ &-\sum_{i=1}^{n} g(R(e_i,R(\xi,e_i)Y)Z,\xi) - \sum_{i=1}^{n} g(R(e_i,Y)R(\xi,e_i)Z,\xi) = 0. \end{aligned} \tag{55}$$

Using (31)–(36), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi,e_i)R(e_i,Y)Z,\xi) \\ &= -(\alpha^2-\rho)\left\{S(Y,Z) + (\alpha^2-\rho)\left[g(Y,Z) + \eta(Y)\eta(Z)\right]\right\}, \end{aligned} \tag{56}$$

$$\sum_{i=1}^{n} g(R(R(\xi,e_i)e_i,Y)Z,\xi) = -(n-1)(\alpha^2-\rho)^2[g(Y,Z) + \eta(Y)\eta(Z)], \tag{57}$$

$$\sum_{i=1}^{n} g(R(e_i,R(\xi,e_i)Y)Z,\xi) = (\alpha^2-\rho)^2[g(Y,Z) + \eta(Y)\eta(Z)], \tag{58}$$

$$\sum_{i=1}^{n} g(R(e_i, Y)R(\xi, e_i)Z, \xi) = (n-1)(\alpha^2 - \rho)^2 \eta(Y)\eta(Z). \tag{59}$$

By virtue of (56), (57), (58) and (59), the equation (55) yields

$$S(Y, Z) = -2(\alpha^2 - \rho)[g(Y, Z) + \eta(Y)\eta(Z)]. \tag{60}$$

In view of (44), (60) takes the form

$$\left[-(\alpha+\lambda) + 2(\alpha^2 - \rho)\right] g(Y, Z) + \left[2(\alpha^2 - \rho) - (\alpha + \mu)\right] \eta(Y)\eta(Z) = 0. \tag{61}$$

Setting $Y = Z = \xi$ in (61), we get

$$\lambda - \mu = 0. \tag{62}$$

Thus, we state the following theorem:

Theorem 7. *If (g, ξ, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_1, then M is an η-Einstein manifold and the scalars λ and μ are related by $\lambda - \mu = 0$.*

Theorem 8. *If (g, ξ, λ) is a Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_1, then it is steady.*

In consequence of (52), (60), we get

$$\left[-(b\alpha+\lambda) + 2(\alpha^2 - \rho)\right] g(Y, Z) + \left[2(\alpha^2 - \rho) - (b\alpha + \mu)\right] \eta(Y)\eta(Z) = 0. \tag{63}$$

Putting $Y = Z = \xi$ in (63), we obtain

$$\lambda - \mu = 0. \tag{64}$$

Thus, we can state the followings:

Theorem 9. *If (g, W, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_1, then the scalars λ and μ are related by $\lambda - \mu = 0$.*

Theorem 10. *If (g, W, λ) is a Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_1, then it is steady.*

3.2. η-Ricci solitons on $(LCS)_n$-manifolds admitting the class C_2. Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(E(U,V)\cdot R)(X,Y)Z=0,$$

which implies

$$\begin{aligned}&g(E(\xi,V)R(X,Y)Z,\xi)-g(R(E(\xi,V)X,Y)Z,\xi)\\ &-g(R(X,E(\xi,V)Y)Z,\xi)-g(R(X,Y)E(\xi,V)Z,\xi)=0.\end{aligned} \tag{65}$$

Setting $V=X=e_i$ in (65) and taking the summation over i, $1\leq i\leq n$, we get

$$\begin{aligned}&\sum_{i=1}^{n}g(E(\xi,e_i)R(e_i,Y)Z,\xi)-\sum_{i=1}^{n}g(R(E(\xi,e_i)e_i,Y)Z,\xi)\\ &-\sum_{i=1}^{n}g(R(e_i,E(\xi,e_i)Y)Z,\xi)-\sum_{i=1}^{n}g(R(e_i,Y)E(\xi,e_i)Z,\xi)=0.\end{aligned} \tag{66}$$

Using (31)-(36) and (39), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}g(E(\xi,e_i)R(e_i,Y)Z,\xi)\\ =&\left[\frac{r}{n(n-1)}-(\alpha^2-\rho)\right]\left\{S(Y,Z)-(\alpha^2-\rho)\left[g(Y,Z)+\eta(Y)\eta(Z)\right]\right\},\end{aligned} \tag{67}$$

$$\begin{aligned}&\sum_{i=1}^{n}g(R(E(\xi,e_i)e_i,Y)Z,\xi)\\ =&(\alpha^2-\rho)\left[\frac{r}{n}-(n-1)(\alpha^2-\rho)\right][g(Y,Z)+\eta(Y)\eta(Z)],\end{aligned} \tag{68}$$

$$\begin{aligned}&\sum_{i=1}^{n}g(R(e_i,E(\xi,e_i)Y)Z,\xi)\\ =&(\alpha^2-\rho)\left[(\alpha^2-\rho)-\frac{r}{n(n-1)}\right][g(Y,Z)+\eta(Y)\eta(Z)],\end{aligned} \tag{69}$$

$$\sum_{i=1}^{n} g(R(e_i, Y)E(\xi, e_i)Z, \xi)$$
$$(70) \qquad = (\alpha^2 - \rho)\left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right](n-1)\eta(Y)\eta(Z).$$

In consequence of (67), (68), (69) and (70), the equation (66) yields

$$(71) \qquad S(Y, Z) = (n-1)(\alpha^2 - \rho)g(Y, Z).$$

In view of (44), (71) takes the form

$$(72) \qquad \left[(n-1)(\alpha^2 - \rho) + (\alpha + \lambda)\right] g(Y, Z) + (\alpha + \mu)\,\eta(Y)\eta(Z) = 0.$$

Replacing Y, Z by ξ in (72), we get

$$(73) \qquad \lambda - \mu = -(n-1)(\alpha^2 - \rho).$$

Thus, we state the following theorems.

Theorem 11. *If (g, ξ, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_2, then M is an Einstein manifold and the scalars λ and μ are related by $\lambda - \mu = -(n-1)(\alpha^2 - \rho)$.*

Theorem 12. *If (g, ξ, λ) is a Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_2, then it is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative, respectively.*

In consequence of (52), (71) takes the form

$$(74) \qquad \left[-(b\alpha + \lambda) - (n-1)(\alpha^2 - \rho)\right] g(Y, Z) - (b\alpha + \mu)\,\eta(Y)\eta(Z) = 0.$$

Setting $Y = Z = \xi$ in (74), we obtain

$$(75) \qquad \lambda - \mu = -(n-1)(\alpha^2 - \rho).$$

Thus, we can state the followings:

Theorem 13. *If (g, W, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_2, then the scalars λ and μ are related by $\lambda - \mu = -(n-1)(\alpha^2 - \rho)$.*

Theorem 14. *If (g, W, λ) is a Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_2, then it is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative, respectively.*

3.3. η-Ricci solitons on $(LCS)_n$-manifolds admitting the class C_3. Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(R(U,V)\cdot C)(X,Y)Z=0,$$

which implies

$$\begin{aligned} & g(R(\xi,V)C(X,Y)Z,\xi)-g(C(R(\xi,V)X,Y)Z,\xi) \\ & -g(C(X,R(\xi,V)Y)Z,\xi)-g(C(X,Y)R(\xi,V)Z,\xi)=0. \end{aligned} \tag{76}$$

Putting $V=X=e_i$ in (76) and taking the summation over i, $1\le i\le n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(R(\xi,e_i)C(e_i,Y)Z,\xi)-\sum_{i=1}^{n} g(C(R(\xi,e_i)e_i,Y)Z,\xi) \\ & -\sum_{i=1}^{n} g(C(e_i,R(\xi,e_i)Y)Z,\xi)-\sum_{i=1}^{n} g(C(e_i,Y)R(\xi,e_i)Z,\xi)=0. \end{aligned} \tag{77}$$

In view of (31)-(36) and (37), we obtain

$$\begin{aligned} & \sum_{i=1}^{n} g(R(\xi,e_i)C(e_i,Y)Z,\xi) \\ = \;& (\alpha^2-\rho)\{[\frac{r}{(n-1)(n-2)}+(\alpha^2-\rho)-\frac{(n-1)}{(n-2)}]\,[g(Y,Z)+\eta(Y)\eta(Z)] \\ & -\frac{1}{n-2}\,[S(Y,Z)+(n-1)\eta(Y)\eta(Z)]\,\}, \end{aligned} \tag{78}$$

$$\begin{aligned} & \sum_{i=1}^{n} g(C(R(\xi,e_i)e_i,Y)Z,\xi) \\ = \;& -(n-1)(\alpha^2-\rho)\left[\frac{r}{(n-1)(n-2)}+(\alpha^2-\rho)-\frac{(n-1)}{(n-2)}\right] \\ & \times[g(Y,Z)+\eta(Y)\eta(Z)]+\frac{(n-1)(\alpha^2-\rho)}{n-2}\,[S(Y,Z)+(n-1)\eta(Y)\eta(Z)]\,, \end{aligned} \tag{79}$$

$$\begin{aligned} & \sum_{i=1}^{n} g(C(e_i,R(\xi,e_i)Y)Z,\xi) \\ = \;& (\alpha^2-\rho)\left[\frac{r}{(n-1)(n-2)}+(\alpha^2-\rho)-\frac{(n-1)}{(n-2)}\right][g(Y,Z)+\eta(Y)\eta(Z)] \end{aligned}$$

$$+\frac{(\alpha^2-\rho)}{n-2}\left[S(Y,Z)+(n-1)\eta(Y)\eta(Z)\right], \tag{80}$$

$$\sum_{i=1}^{n} g(C(e_i,Y)R(\xi,e_i)Z,\xi)=(n-1)(\alpha^2-\rho)(\alpha^2-\rho-1)\frac{(n-1)}{(n-2)}\eta(Y)\eta(Z). \tag{81}$$

By virtue of (78), (79), (80), (81), the equation (77) yields

$$\begin{aligned} &(n+1)S(Y,Z)\\ =\ &\left[r+(\alpha^2-\rho)(n-1)(n-2)-(n-1)^2\right]g(Y,Z)\\ &+[r-(n-1)(\alpha^2-\rho+n+1)]\eta(Y)\eta(Z). \end{aligned} \tag{82}$$

In view of (44), (82) takes the form

$$\begin{aligned} 0 =\ &\left\{-(n+1)(\alpha+\lambda)-\left[r+(\alpha^2-\rho)(n-1)(n-2)-(n-1)^2\right]\right\}g(Y,Z)\\ &-\left\{(n+1)(\alpha+\mu)+\left[r-(n-1)(\alpha^2-\rho+n+1)\right]\right\}\eta(Y)\eta(Z). \end{aligned} \tag{83}$$

Replacing Y, Z by ξ in (83), we get

$$(n+1)(\lambda-\mu)=-(n-1)\left[(n-1)\left(\alpha^2-\rho\right)+2\right]. \tag{84}$$

Thus, we state the following theorems:

Theorem 15. *If (g,ξ,λ,μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_3, then M is an η-Einstein manifold and the scalars λ and μ are related by $(n+1)(\lambda-\mu)=-(n-1)\left[(n-1)\left(\alpha^2-\rho\right)+2\right]$.*

Theorem 16. *If (g,ξ,λ) is a Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_3, then it is shrinking, steady or expanding according to $(n-1)\left(\alpha^2-\rho\right)+2$ being positive, zero or negative, respectively.*

In consequence of (52), (82) takes the form

$$\begin{aligned} &-\left[(n+1)(b\alpha+\lambda)+r+(\alpha^2-\rho)(n-1)(n-2)-(n-1)^2\right]g(Y,Z)\\ =\ &[(n+1)(b\alpha+\mu)+r-(n-1)(\alpha^2-\rho+n+1)]\eta(Y)\eta(Z). \end{aligned} \tag{85}$$

Setting $Y=Z=\xi$ in (85), we get

$$(n+1)(\lambda-\mu)=-(n-1)\left[(n-1)\left(\alpha^2-\rho\right)+2\right].$$

Thus, we state the following theorems:

Theorem 17. *If (g, W, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_3, then the scalars λ and μ are related by* $(n+1)(\lambda-\mu) = -(n-1)\left[(n-1)\left(\alpha^2-\rho\right)+2\right]$.

Theorem 18. *If (g, W, λ) is a Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_3, then M is shrinking, steady or expanding according to $(n-1)\left(\alpha^2-\rho\right)+2$ is positive, zero or negative, respectively.*

3.4. η-Ricci solitons on $(LCS)_n$-manifolds admitting the class C_4. Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(E(U,V)\cdot K)(X,Y)Z = 0,$$

which implies

$$\begin{aligned} & g(E(\xi,V)K(X,Y)Z,\xi) - g(K(E(\xi,V)X,Y)Z,\xi) \\ & -g(K(X,E(\xi,V)Y)Z,\xi) - g(K(X,Y)E(\xi,V)Z,\xi) = 0. \end{aligned} \tag{86}$$

Putting $V = X = e_i$ in (86) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(E(\xi,e_i)K(e_i,Y)Z,\xi) - \sum_{i=1}^{n} g(K(E(\xi,e_i)e_i,Y)Z,\xi) \\ & -\sum_{i=1}^{n} g(K(e_i,E(\xi,e_i)Y)Z,\xi) - \sum_{i=1}^{n} g(K(e_i,Y)E(\xi,e_i)Z,\xi) = 0. \end{aligned} \tag{87}$$

Using (31)-(36) and (38), (39), we obtain

$$\begin{aligned} & \sum_{i=1}^{n} g(E(\xi,e_i)K(e_i,Y)Z,\xi) \\ = \ & \frac{kr}{n-2} g(Y,Z) - k^2\left[g(Y,Z)+\eta(Y)\eta(Z)\right] \\ & -\frac{k}{n-2}\left[S(Y,Z)+(n-1)(\alpha^2-\rho)\eta(Y)\eta(Z)\right], \end{aligned} \tag{88}$$

$$\begin{aligned} & \sum_{i=1}^{n} g(K(E(\xi,e_i)e_i,Y)Z,\xi) \\ = \ & -k^2(n-1)\left[g(Y,Z)+\eta(Y)\eta(Z)\right] \end{aligned}$$

$$+\frac{k(n-1)}{n-2}\left[S(Y,Z)+(n-1)(\alpha^2-\rho)\eta(Y)\eta(Z)\right], \tag{89}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K(e_i,E(\xi,e_i)Y)Z,\xi)\\ =\;& k^2\left[g(Y,Z)+\eta(Y)\eta(Z)\right]\\ &-\frac{k}{n-2}\left[S(Y,Z)+(n-1)(\alpha^2-\rho)\eta(Y)\eta(Z)\right],\end{aligned} \tag{90}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K(e_i,Y)E(\xi,e_i)Z,\xi)\\ =\;& -k^2(n-1)\eta(Y)\eta(Z)-\frac{k}{n-2}\left[(n-1)(\alpha^2-\rho)-r\right]\eta(Y)\eta(Z).\end{aligned} \tag{91}$$

By virtue of (88), (89), (90) and (91), the equation (87) yields

$$\begin{aligned}&\frac{n-1}{(n-2)}S(Y,Z)-\left[\frac{r}{n-2}+k^2(n-3)\right]g(Y,Z)\\ =\;& \left[k^2(n-3)+k(n-1)(1+\alpha^2-\rho)-\frac{kr}{n-2}\right]\eta(Y)\eta(Z),\end{aligned} \tag{92}$$

where

$$k=(\alpha^2-\rho)-\frac{(n-1)}{(n-2)}.$$

In view of (44), (92) takes the form

$$\begin{aligned}&-\left[(n-1)(\alpha+\lambda)+r+k^2(n-2)(n-3)\right]g(Y,Z)\\ =\;& [k^2(n-2)(n-3)+k(n-2)(n-1)(1+\alpha^2-\rho)-kr\\ +\;& (n-1)(\alpha+\mu)]\eta(Y)\eta(Z).\end{aligned} \tag{93}$$

Replacing Y, Z by ξ in (93), we get

$$\lambda-\mu=\left[k(n-2)(1+\alpha^2-\rho)-\frac{r(k+1)}{(n-1)}\right]. \tag{94}$$

Thus, we state the following theorems:

Theorem 19. *If (g, ξ, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_4, then M is an η-Einstein manifold and the scalars λ and μ are related by $\lambda - \mu = k(n-2)(1+\alpha^2-\rho) - \dfrac{r(k+1)}{(n-1)}$.*

Theorem 20. *If (g, ξ, λ) is a Ricci soliton on an $(LCS)_n$-manifold M admitting the class C_3, then it is shrinking, steady or expanding according to $k(n-2)(1+\alpha^2-\rho) - \dfrac{r(k+1)}{(n-1)}$ being negative, zero or positive, respectively.*

In consequence of (52), (92) takes the form

$$\begin{aligned} & -\left[(n-1)(b\alpha+\lambda) + r + k^2(n-3)(n-2)\right] g(Y,Z) \\ = \; & [k^2(n-2)(n-3) + k(n-1)(n-2)(1+\alpha^2-\rho) - kr \\ + \; & (n-1)(b\alpha+\mu)]\eta(Y)\eta(Z). \end{aligned} \tag{95}$$

Setting $Y = Z = \xi$ in (95), we obtain

$$\lambda - \mu = \left[k(n-2)(1+\alpha^2-\rho) - \frac{r(k+1)}{(n-1)}\right]. \tag{96}$$

Thus, we state the following theorems:

Theorem 21. *If (g, W, λ, μ) is an η-Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_4, then the scalars λ and μ are related by $\lambda - \mu = \left[k(n-2)(1+\alpha^2-\rho) - \dfrac{r(k+1)}{(n-1)}\right]$.*

Theorem 22. *If (g, W, λ) is a Ricci soliton on an $(LCS)_n$-manifold M such that W is pointwise collinear with ξ and admits the class C_4, then it is shrinking, steady or expanding according to $k(n-2)(1+\alpha^2-\rho) - \dfrac{r(k+1)}{(n-1)}$ being negative, zero or positive, respectively.*

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Received April 28, 2021