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EXTENDING THE CONVERGENCE REGION OF M-STEP ITERATIVE PROCEDURES

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Abstract. The convergence region of iterative procedures is small in general, and it becomes smaller as m increases. This problem limits the choice of starting points, and consequently the applicability of these methods. The novelty of this work lies in the fact that, we extend the convergence region by using specializations of the Lipschitz constants used before. Further advantages include improved error estimations and uniqueness results. The results are tested favorably to us on examples.

1. Introduction. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ stand for Banach spaces, $\Omega \subset \mathcal{B}_1$ denote a convex and open set. Consider mapping $G : \Omega \longrightarrow \mathcal{B}_2$, to be nonlinear and Fréchet differentiable. One of the most challenging tasks is to find a solution x_* of

$$G(x) = 0. \tag{1.1}$$

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Solving (1.1) is needed, since many problems from various disciplines can be made to look like equation (1.1) by resorting to Mathematical modeling. The solution x_* is rarely obtained in closed or analytical form. Hence, researchers develop iterative procedures that approximate x_* provided that the initial point $x_0 \in \Omega$ is close enough to x_*. A great effort is given recently to develop high convergence order iterative procedures [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 23, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26]. Let m be a natural number.

In this article, we give the semi-local convergence of the $m-$step Traub or Newton-type method defined as

$$
\begin{aligned}
x_n &= y_n^{(0)} \\
y_n^{(1)} &= y_n^{(0)} - G'(y_n^{(0)})^{-1} G(y_n^{(0)}) \\
y_n^{(2)} &= y_n^{(1)} - G'(y_n^{(0)})^{-1} G(y_n^{(1)}) \\
&\cdots \\
x_{n+1} &= y_n^{(m)} = y_n^{(m-1)} - G'(y_n^{(0)})^{-1} G(y_n^{(m-1)}).
\end{aligned} \tag{1.2}
$$

Notice that if $m = 1$, we obtain Newton's method [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 23, 13, 14, 15].

Method (1.2) has convergence order $m + 1$ [1, 2, 20]. But it only uses the first derivative. This is in contrast to third order methods such as Euler, Chebyshev, Halley and other methods that utilize expensive higher order than one derivatives. That explains why many works have appeared dealing with the convergence and efficiency of method (1.2) [1, 2, 21, 26]. A major drawback of these works is the convergence region which is small in general and becomes even smaller as m increases. This problem limits the availability of initial points. Motivated by this concern, we use an extension of our technique introduced in [10] for Newton's method combined with our notion of the restricted convergence region in combination with the center-Lipschitz condition to find a strict subset of Ω also containing the iterates. This techniques gives us at least as small constants as before leading to a larger convergence region as well as to the carrying out of less computations to obtain a predetermined error accuracy. The information on the uniqueness of x_* in a neighborhood of Ω is also improved. Related work can be found in the excellent paper by Donchev et al. [17], where numerous Lipschitz-type conditions are studied in the setting of fixed points. Then, our results can be written in those settings too in a suitable way. The novelty of this work lies in the observation that these developments involve not further conditions because the new parameters constitute specializations of the old ones. Relevant work but for generalized equations has been given in [14].

The rest of the work lays out: In Section 2, the convergence of method (1.2) is studied using majorizing sequences. Some examples are given in Section 3.

2. Semi-local analysis. We base the semi-local convergence analysis of method (1.2) on some scalar majorizing sequences.

Lemma 2.1. *Let ℓ_0, ℓ and $r_0^{(1)}$ be given positive parameters. Denote by γ the unique solution in the interval $(0,1)$ of the polynomial equation*

$$p(t) = 0, \tag{2.1}$$

where

$$p(t) = \ell_0 t^{m+1} - \frac{\ell}{2}(2 - t^m - t^{m-1}). \tag{2.2}$$

Define the scalar sequence $\{t_n\}$ for each $n = 0, 1, 2, \ldots,$ and $i = 1, 2, \ldots,$ $m-1$ by

$$\begin{aligned} t_0 &= 0,\ r_n^{(0)} = t_n, \\ r_{n+1}^{(1)} &= t_{n+1} + \frac{\bar{k}(t_{n+1} - t_n + r_n^{(m-1)} - t_n)(t_{n+1} - r_n^{(m-1)})}{2(1 - \ell_0(t_{n+1} - t_0))}, \qquad (2.3) \\ r_n^{(m)} &= t_{n+1}, r_n^{(i+1)} = r_n^{(i)} + \frac{\bar{k}(r_n^{(i)} - t_n + r_n^{(i-1)} - t_n)(r_n^{(i)} - r_n^{(i-1)})}{2(1 - \ell_0(t_n - t_0))}, \qquad (2.4) \end{aligned}$$

where $\bar{k} = \begin{cases} \ell_0, & n = 0 \\ \ell, & n = 1, 2, 3 \ldots \end{cases}$. *Suppose that*

$$\ell_0 t_2 < 1,\ 0 \leq \frac{\ell(t_2 - t_1 + r_1^{(m-1)} - t_1)}{2(1 - \ell_0 t_2)} \leq \gamma < 1 - \frac{\ell_0(r^{(2)} - r_0^{(1)})}{1 - \ell_0 r_0^1}. \tag{2.5}$$

Then, sequence $\{t_n\}$ is bounded from above by $t_{**} = r_0^{(1)} + \dfrac{r_0^{(2)} - r_0^{(1)}}{1 - \gamma}$, non decreasingly convergent to its unique least upper bound denoted by t_* so that $0 \leq t_* \leq t_{**}$. Moreover, the following items hold

$$r_n^{(i)} - r_n^{(i-1)} \leq \gamma(r_n^{(i-1)} - r_n^{(i-2)}) \leq \cdots \leq \gamma^{mn+i-2}(r_0^{(2)} - r_0^{(1)}) \tag{2.6}$$

$$r_{n+1}^{(1)} - r_n^{(m)} \leq \gamma^{m(n+1)-1}(r_0^{(2)} - r_0^{(1)}), \tag{2.7}$$

and

$$t_n = r_n^{(0)} \leq r_1^{(1)} \leq r_n^{(2)} \leq \cdots \leq r_n^{(m-1)} \leq r_n^{(m)} = t_{n+1}. \tag{2.8}$$

The proof is a nontrivial extension of the one, we gave for Newton's method in [10] *for* $m = 1$.

P r o o f. The intermediate value theorem together with the estimations $p(0) = -\ell < 0$ and $p(1) = \ell_0$ assures that equation $p(t) = 0$ has at least one solution in the interval $(0,1)$. By $p'(t) = (m+1)\ell_0 t^k + \frac{\ell}{2}(tm + m - 1) > 0$ for each $t \in (0,1)$, function p increases on $(0,1)$, so it crosses the $x-$ axis only one time between 0 and 1. We name the unique solution of equation $p(t) = 0$ in $(0,1)$ by γ.

Clearly, if we show estimations (2.6)–(2.8) then sequence $\{t_n\}$ shall be nondecreasing. Evidently, (2.6)–(2.8) by (2.3)–(2.4) hold true, if

$$0 \leq \frac{\ell(r_j^{(i)} - t_j + r_j^{(i-1)} - t_j)}{2(1 - \ell_0 t_j)} \leq \gamma, \tag{2.9}$$

$$0 \leq \frac{\ell(r_j^{(m)} - t_j + r_j^{(m-1)} - t_j)}{2(1 - \ell_0 t_{j+1})} \leq \gamma \tag{2.10}$$

and

$$t_j = r_j^{(0)} \leq r_j^{(1)} \leq r_j^{(2)} \leq \cdots \leq r_j^{(m-1)} \leq r_j^{(m)} = t_{j+1}. \tag{2.11}$$

If $r_0^{(i)} - r_0^{(i-1)} \geq 0$, then it follows from $r_0^{(i+1)} - r_0^{(i)} = \frac{\ell_0}{2}(r_0^{(i)} + r_0^{(i-1)})(r_0^{(i)} - r_0^{(i-1)})$ that $r_0^{(i+1)} - r_0^{(i)} \geq 0$. But $r_0^{(0)} = t_0^{(0)} = t_0 = 0$ and $r_0^{(1)} > 0$, so $r_0^{(i+1)} - r_0^{(i)} \geq 0$. Then (2.11) holds for $j = 0$. Then, by (2.5), estimations (2.9) and (2.10) hold for $m = 1$ by the left hand side inequality in (2.5). Next, assume (2.9)–(2.11) hold for $m = 2, 3, \ldots, n$ Notice that

$$\begin{aligned} 0 \;&\leq\; \frac{\ell(r_{j-1}^{(i)} - t_{j-1} + r_{j-1}^{(i-1)} - t_{j-1})}{2(1 - \ell_0 t_{j-1})} \\ &\leq\; \frac{\ell(r_j^{(i)} - t_j + r_j^{i-1} - t_j)}{2(1 - \ell_0 t_j)} \end{aligned}$$

$$\leq \frac{\ell(r_j^{(m)} - t_j + r_j^{(m-1)} - t_j)}{2(1 - \ell_0 t_{j+1})}. \tag{2.12}$$

Hence, it suffices to show only (2.10). We need to use the induction hypotheses to obtain an upper bound on $r_n^{(i)}$:

$$\begin{aligned}
r_n^{(i)} &\leq r_n^{(i-1)} + \gamma^{mn+i-2}(r_0^{(2)} - r_0^{(1)}) \\
&\leq r_n^{(i-2)} + \gamma^{mn+i-3}(r_0^{(2)} - r_0^{(1)}) + \gamma^{mn+i-2}(r_0^{(2)} - r_0^{(1)}) \\
&\vdots \\
&\leq r_n^{(1)} + (\gamma^{mn} + \cdots + \gamma^{mn+i-2})(r_0^{(2)} - r_0^{(1)}) \\
&\leq r_{n-1}^{(m)} + (\gamma^{mn-1} + \cdots + \gamma^{mn+i-2})(r_0^{(2)} - r_0^{(1)}) \\
&\vdots \\
&\leq r_0^{(2)} + \gamma(r_0^{(2)} - r_0^{(1)}) + \cdots + \gamma^{mn+i-2}(r_0^{(2)} - r_0^{(1)}) \\
&= r_0^{(2)} - (r_0^{(2)} - r_0^{(1)}) + (r_0^{(2)} - r_0^{(1)}) \\
&\quad + \gamma(r_0^{(2)} - r_0^{(1)}) + \cdots + \gamma^{mn+i-2}(r_0^{(2)} - r_0^{(1)}) \\
&= r_0^{(1)} + (1 + \gamma + \cdots + \gamma^{mn+i-2})(r_0^{(2)} - r_0^{(1)}) \\
&= r_0^{(1)} + \frac{1 - \gamma^{mn+i-1}}{1 - \gamma}(r_0^{(2)} - r_0^{(1)}) \\
&\leq r_0^{(1)} + \frac{r_0^{(2)} - r_0^{(1)}}{1 - \gamma} = t_{**}. \qquad (2.13)
\end{aligned}$$

Evidently (2.10) holds, if

$$\frac{\ell(r_0^{(2)} - r_0^{(1)})[(\gamma^{mj-1} + \cdots + \gamma^{mj+m-2}) + (\gamma^{mj-1} + \cdots + \gamma^{mj+m-3})]}{2(1 - \ell_0(r_0^{(1)} + \frac{1-\gamma^{mj+m-1}}{1-\gamma}(r_0^{(2)} - r_0^{(1)})]} \leq \gamma \tag{2.14}$$

or

$$\frac{\ell(r_0^{(2)} - r_0^{(1)})[\frac{1-\gamma^{mj+m-1}}{1-\gamma} - \frac{1-\gamma^{m(j-1)+m-1}}{1-\gamma} + \frac{1-\gamma^{mj+m-2}}{1-\gamma} - \frac{1-\gamma^{m(j-1)+m-1}}{1-\gamma}]}{2(1 - \ell_0(r_0^{(1)} + \frac{1-\gamma^{mj+m-1}}{1-\gamma}(r_0^{(2)} - r_0^{(1)})]} \leq \gamma \tag{2.15}$$

or

$$\frac{\ell(r_0^{(2)}-r_0^{(1)})\gamma^{mj-2}(2-\gamma^m-\gamma^{m-1})\frac{1}{1-\gamma}}{2(1-\ell_0(r_0^{(1)}+\frac{1-\gamma^{mj+m-1}}{1-\gamma}(r_0^{(2)}-r_0^{(1)})]}\leq 1 \tag{2.16}$$

or

$$\frac{\ell(r_0^{(2)}-r_0^{(1)})\gamma^{mj-2}(2-\gamma^m-\gamma^{m-1})}{2[(1-\ell_0 r_0^{(1)})(1-\gamma)-\ell_0(1-\gamma^{mj+m-1})(r_0^{(2)}-r_0^{(1)})]}\leq 1. \tag{2.17}$$

It is helpful to define recurrent polynomials defined on $(0,1)$ by

$$\begin{aligned} h_j(t) &= \frac{\ell}{2}(r_0^{(2)}-r_0^{(1)})t^{mj-2}(2-t^m-t^{m-1}) \\ & \quad \ell_0(1-t^{mj+m-1})(r_0^{(2)}-r_0^{(1)})-(1-\ell_0 r_0^{(1)})(1-t). \end{aligned} \tag{2.18}$$

Then, instead of (2.17), we can prove

$$h_j(\gamma)\leq 0. \tag{2.19}$$

To achieve this we need a computation relating two consecutive polynomials:

$$\begin{aligned} h_{j+1}(t) &= \frac{\ell}{2}(r_0^{(2)}-r_0^{(1)})t^{m(j+1)-2}(2-t^m-t^{m-1}) \\ &\quad -(1-\ell_0 r_0^{(1)})(1-t)+\ell_0(1-t^{m(j+1)+m-1})-h_j(t)+h_j(t) \\ &= \frac{\ell}{2}(r_0^{(2)}-r_0^{(1)})t^{mj-2}(2-t^m-t^{m-1}) \\ &\quad -\frac{\ell}{2}(r_0^{(2)}-r_0^{(1)})t^{mj-2}(2-t^m-t^{m-1}) \\ &\quad +\ell_0(1-t^{m(j+1)+m-1})(r_0^{(2)}-r_0^{(1)}) \\ &\quad -\ell_0(1-t^{mj+j-1})(r_0^{(2)}-r_0^{(1)})+h_j(t) \\ &= h_j(t)+p(t)(1-t^m)t^{mj-2}(r_0^{(2)}-r_0^{(1)}). \end{aligned} \tag{2.20}$$

Notice that by $p(\gamma)=0$, and (2.18)

$$\begin{aligned} h_{j+1}(\gamma) &= h_j(\gamma)=h_\infty(\gamma):=\lim_{j\longrightarrow\infty}h_j(\gamma) \\ &= \ell_0(r_0^{(2)}-r_0^{(1)})-(1-\ell_0 r_0^{(1)})(1-\gamma). \end{aligned} \tag{2.21}$$

Then, (2.19) holds, if

$$h_j(\gamma) \leq 0, \tag{2.22}$$

which is true by the left hand side inequality in condition (2.5). The induction for (2.9)–(2.11) is completed. □

Define $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) =: \{T : \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \text{ is a bounded linear operator}\}$, and $S(x, \rho) := \{y \in \mathcal{B}_1 : \|x - y\| < \rho, \rho > 0\}$, and $\bar{S}(x, \rho) = \{y \in \mathcal{B}_1 : \|x - y\| \leq \rho\}$.

Let us consider conditions (B) to be used together with the previous notations in the analysis of method (1.2).

(b1) $G : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is differentiable in the Fréchet sense. There exist $x_0 \in \Omega$, $r_0^{(1)} > 0$ such that $G'(x_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$, and

$$\|G'(x_0)^{-1} G(x_0)\| \leq r_0^{(1)}.$$

(b2) For all $x \in \Omega$, and some $\ell_0 > 0$

$$\|G'(x_0)^{-1}(G'(x_0) - G'(x))\| \leq \ell_0 \|x_0 - x\|.$$

Let $S_0 = \Omega \cap S(x_0, \frac{1}{\ell_0})$.

(b3) For all $x, y \in S_0$, and some $\ell > 0$

$$\|G'(x_0)^{-1}(G'(y) - G'(x))\| \leq \ell \|y - x\|.$$

(b4) Hypotheses of Lemma 2.1 hold.

(b5) $\bar{S}(x_0, t_*) \subseteq \Omega$, where t_* is given in Lemma 2.1.

(b6) There exists $u \geq t_*$ such that $\ell_0(t_* + u) < 2$. Let $S_1 = \Omega \cap \bar{S}(x_0, u)$.

Theorem 2.2. *Assume conditions (B) hold. Then, the following hold* $\{x_n\} \subset \bar{S}(x_0, t_*)$, $\lim_{n \longrightarrow \infty} x_n = x_* \in \bar{S}(x_0, t_*)$,

$$\|x_* - x_n\| \leq t_* - t_n, \tag{2.23}$$

and x_* is the only solution of equation $F(x) = 0$, in the set S_1.

The proof is an extension of the corresponding one given by us in [10] for Newton's method ($m = 1$).

Proof. Mathematical induction is used to show items

(Q1) $\|y_n^{(1)} - x_n\| \leq r_n^{(1)} - t_n$,

(Qi) $\|y_n^{(i)} - y_n^{(i-1)}\| \leq r_n^{(i)} - r_n^{(i-1)}$ for all $i = 2, 3, \dots, m-1$.

(Qk) $\|x_{n+1} - y_n^{(i-1)}\| \leq t_{n+1} - r_n^{(m-1)}$.

If $n = 0$, by (b1) $\|y_0^{(1)} - x_0\| = \|G'(x_0)^{-1}G(x_0)\| \leq r_0^{(1)} \leq t_*$, so $y_0^{(1)} \in \bar{S}(x_0, t_*)$, and (Q1) holds for $n = 0$. By the i^{th} step of method (1.2), $i = 2, 3 \dots, m-1$, (b2), we have in turn that

$$\begin{aligned}
\|y_1^{(1)} - x_1\| &= \|(G'(x_1)^{-1}G'(x_0)(G'(x_0)^{-1}G(x_1))\| \\
&\leq \frac{\|G'(x_0)^{-1}(G(x_1) - G(y_0^{(m-1)})) - G'(x_0)(x_1 - y_0^{(m-1)})\|}{1 - \ell_0\|x_1 - x_0\|} \\
&\leq \frac{\ell_0(\|x_1 - x_0\| + \|y_0^{(m-1)} - x_0\|)\|x_1 - y_0^{(m-1)}\|}{2(1 - \ell_0\|x_1 - x_0\|)} \\
&\leq \frac{\ell_0(t_1 - t_0 + r_0^{(m-1)} - t_0)(t_1 - r_0^{(m-1)})}{2(1 - \ell_0(t_1 - t_0))} \\
&= r_1^{(1)} - t_1,
\end{aligned}$$

and

$$\|y_1^{(1)} - x_0\| \leq \|y_1^{(1)} - x_1\| + \|x_1 - x_0\| \leq r_1^{(1)} - t_1 + t_1 - t_0 = r_1^{(1)} \leq t_*,$$

so $y_0^{(1)} \in \bar{S}(x_0, t_*)$, and $(Q1)$ holds. Next, using (b3), we obtain by method (1.2)

$$\begin{aligned}
\|y_1^{(i)} - y_1^{(i-1)}\| &= \|(G'(x_1)^{-1}G'(x_0)(G'(x_0)^{-1}G(y_1^{(i-1)}))\| \\
&\leq \frac{\|G'(x_0)^{-1}(G(y_1^{(i-1)}) - G(y_1^{(i-2)})) - G'(x_1)(y_1^{(i-1)} - y_0^{(i-2)})\|}{2(1 - \ell_0\|x_1 - x_0\|)} \\
&\leq \frac{\ell(r_1^{(i-1)} - t_1 + r_1^{(i-2)} - t_1)(r_1^{(i-1)} - r_1^{(i-2)})}{2(1 - \ell_0(t_1 - t_0))} \\
&= r_1^{(i)} - r_1^{(i-1)},
\end{aligned}$$

and

$$
\begin{aligned}
\|y_0^{(i)} - y_0^{(i-1)}\| &= \|G'(x_0)^{-1}(G(y_0^{(i-1)}) - G(y_0^{(i-2)}) + G(y_0^{(i-2)})\| \\
&= \|G'(x_0)^{-1}[G(y_0^{(i-2)} - G(y_0^{(i-2)}) - G'(x_0)(y_0^{(i-1)} - y_0^{(i-2)})]\| \\
&\leq \int_0^1 \|G'(x_0)^{-1}(G'(y_0^{(i-2)} + \tau(y_0^{(i-1)} - y_0^{(i-2)}) \\
&\quad -G'(x_0)(y_0^{(i-1)} - y_0^{(i-2)}))d\tau\| \\
&\leq \frac{\ell_0}{2}(\|y_0^{(i-1)} - x_0\| + \|y_0^{(i-2)} - x_0\|)\|y_0^{(i-1)} - y_0^{(i-2)}\| \\
&\leq \frac{\ell_0}{2}(r_0^{(i-1)} - t_0 + r_0^{(i-2)} - t_0)(r_0^{(i-1)} - r_0^{(i-2)}) \\
&= r_0^{(i)} - r_0^{(i-1)}, \qquad (2.24)
\end{aligned}
$$

where, we also used the estimate

$$G'(x_n)(y_n^{(i)} - y_n^{(i-1)}) = -G(y_n^{(i-1)}). \qquad (2.25)$$

Then, we get

$$
\begin{aligned}
\|y_0^{(i)} - x_0\| &\leq \|y_0^{(i)} - y_0^{(i-1)}\| + \|y_0^{(i-1)} - y_0^{(i-2)}\| \\
&\quad + \cdots + \|y_0^{(1)} - x_0\| \\
&\leq r_0^{(i)} - r_0^{(i-1)} + r_0^{(i-1)} - r_0^{(i-2)} + \cdots + r_0^{(1)} - r_0^{(0)} \\
&= r_0^{(i)} \leq t_*, \qquad (2.26)
\end{aligned}
$$

so $y_0^{(i)} \in \bar{S}(x_0, t_*)$. Letting $i = m$ in (2.24 and (2.26) we have that

$$\|y_0^{(m)} - y_0^{(m-1)}\| \leq t_1 - r_0^{(m-1)}, \qquad (2.27)$$

and

$$\|x_1 - x_0\| \leq t_1 \leq t_*,$$

respectively, so $x_1 \in \bar{S}(x_0, t_*)$ and Q_k holds for $k = 0$. Next, we show Q_i for $i = 1, 2, \ldots, m$ when, $n = 1$ by induction on i. Let $v \in S_0$. Then, by (b2), we get that

$$\|G'(x_0)^{-1}(G'(v) - G'(x_0))\| \leq \ell_0\|v - x_0\| < \ell_0\frac{1}{\ell_0} = 1, \qquad (2.28)$$

so $G'(v)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|G'(v)^{-1}G'(x_0)\| \leq \frac{1}{1-\ell_0\|v-x_0\|}, \tag{2.29}$$

by the Banach lemma for invertible mappings [5, 8, 23, 13, 21]. In particular, (2.29) holds for $v = x_1$. Then, by (b2), (2.3) and method (1.2) (first step of the k^{th} step), we obtain in turn that

$$\begin{aligned} \|y_1^{(i)} - x_)\| &\leq \|y_1^{(i)} - y_1^{(i-1)}\| + \|y_1^{(i-1)} - x_0\| \\ &\leq r_1^{(i)} - r_1^{(i-1)} + r_1^{(i-1)} - t_0 = r_1^{(i)} \leq t_*, \end{aligned}$$

so $Q_1^{(i)} \in \bar{S}(x_0, t_*)$, and (Q_i) holds. By using $x_n, y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)}$ instead of $x_1, y_1^{(1)}, y_1^{(2)}, \ldots, y_1^{(m)}$, respectively items (Qi), $i = 1, 2, \ldots, m$ hold. In view of

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|y_n^{(m)} - y_n^{(0)}\| \\ &\leq \sum_{q=0}^{m-1} \|y_n^{(m-q)} - y_n^{(m-q-1)}\| \\ &\leq \sum_{q=0}^{m-1} (r_n^{(m-q)} - r_n^{(m-q-1)}) \\ &= r_n^{(m)} - r_n^{(0)}. \end{aligned} \tag{2.30}$$

That is $\{x_n\}$ is a complete sequence in a Banach space $\mathcal{B}_1$. Hence, there exists $x_* \in \bar{S}(x_0, t_*)$ such that $\lim_{n \longrightarrow \infty} x_n = x_*$. Then, we have by the estimation $\|G'(x_0)^{-1}G(y_n^{(i)})\| \leq \ell(\|x_n^{(i)} - x_n\| + \|y_n^{(i-1)} - x_n\|)\|y_n^{(i)} - y_n^{(i-1)}\| \longrightarrow 0$ as $n \longrightarrow \infty$, that $G(x_*) = 0$. The uniqueness proof is given in [10]. □

Remark 2.3.

(a) The point t_{**} can be used in Theorem 2.2 instead of t_*, if one wants a closed form point.

(b) As already noted in [10], we use initial data until the computation of t_2. This way, we relate the semi-local convergence condition (2.5) to $r_0^{(2)} - r_0^{(1)}$ (which is smaller) instead of $r_0^{(1)} - r_0^{(0)}$ which is traditionally done. It turns out that smaller Lipschitz constants can replace the ones in conditions (B). Indeed, define set $\Omega_2 = \Omega \cap S(x_1, \frac{1}{\ell_0} - r_0^{(1)})$ provided that $\ell_0 r_0^{(1)} < 1$. Clearly

$$\Omega_2 \subseteq \Omega_1, \tag{2.31}$$

and can replace it in Theorem 2.2. But then, by (2.31) the new Lipschitz constant corresponding to ℓ denoted by λ will be such that

$$\lambda \leq \ell. \tag{2.32}$$

Then, λ can also replace ℓ in all estimations, and the resulting analysis will be even finer (see also the numerical example).

(c) The condition in [26] using $r_0^{(1)} - r_0^{(0)}$ in (2.5) instead of $r_0^{(2)} - r_0^{(1)}$ is given by

$$0 \leq \frac{\bar{\ell}(t_1 + r_0^{(m-1)})}{2(1 - \ell_0 t_1)} \leq \gamma < 1 - \ell_0 r_0^{(1)}, \tag{2.33}$$

where $\bar{\ell}$ is the Lipschitz constant on Ω not on S_0 (see (b3)).

(d) The ratio of convergence is γ given in (2.2). According to the proof of Theorem 2.2 it can also be replaced by the upper bound $\bar{\gamma}$ which is computable in advance and given by $\bar{\gamma} = \dfrac{\gamma t_{**}}{1 - \ell_0 t_{**}}$.

3. Numerical examples.

Example 3.1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\Omega = \bar{U}(x_0, 1-\xi)$, $x_0 = 1$ and $\xi \in \left[0, \frac{1}{2}\right)$. Define function $\mathcal{H}$ on Ω by

$$\mathcal{H}(x) = x^3 - \xi.$$

Then, we get by (2.1)-(2.5) and (2.33) that for
(i) $m = 1$, $r_0^{(1)} = \frac{1}{3}(1-\xi)$, $\ell_0 = 3 - \xi, \ell = 2\left(1 + \frac{1}{\ell_0}\right), \bar{\ell} = 2(2-\xi)$. Then conditions of Lemma 2.1 are satisfied for $I_1 = [0.42289, 0.5)$, $I_2 = [0.43289, 0.5)$.
(ii)For $m = 2$, the conditions of Lemma 2.1 are satisfied for $\xi \in I_3 = [0.496599, 0.5)$ but the earlier conditions [1, 2, 21, 26] are satisfied for $\xi \in I_4 = [0.49669, 0.5)$, and $I_4 \subset I_3$. Hence, in both cases $m = 1, 2$ we find infinite many values of ξ for which convergence is not guaranteed under the old condition (2.33). That is the usage of method (1.2) is extended.

4. Conclusion. Our idea of the convergence region in connection to the center Lipschitz condition were utilized to provide a local as well as a semilocal convergence analysis of method (1.2). Due to the fact that we locate a region at least as small as in earlier works [1, 2, 21, 26] containing the iterates, the new Lipschitz parameters are also at least as small. This technique leads to a finer convergence analysis (see Remark 2.3 and the numerical example). The novelty of the paper not only lies in the introduction of the new idea but also obtained using special cases of Lipschitz parameters appearing in [1, 2, 21, 26]. Hence, no additional work to [1, 2, 21, 26] is needed to arrive at these developments. This idea can be used to extend the applicability of other iterative methods appearing [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 23, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26] along the same lines.

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