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CLOSED WEAK SUPPLEMENTED PROPERTY IN THE LATTICES

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ABSTRACT. Let L be a lattice with the greatest element 1. Following the concept of closed weak supplemented modules, we define closed weak supplemented filters and we will make an intensive investigate the basic properties and possible structures of these filters.

1. Introduction. The notion of a supplement submodule was introduced in [10] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules U and V of a module M , V is said to be a supplement of U in M or U is said to have a supplement V in M if $U + V = M$ and $U \cap V \ll V$. The module M is called supplemented if every submodule of M has a supplement in M . See [3] and [12] for results and the definitions related to supplements and supplemented modules. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [16]. Supplemented modules are also discussed

in [11]. Recently, several authors have studied different generalizations of supplemented modules. For submodules U and V of a module M , V is said to be weak supplement of U in M or U is said to have a weak supplement V in M if $U + V = M$ and $U \cap V \ll M$. M is called a weak supplemented module if every submodule of M has a weak supplement in M . A module M is called \oplus -supplemented if for every submodule N there is a direct summand K of M which is a supplement of N in M [9]. A closed submodule N of M is a submodule which has no proper essential extension in M . A module M is called closed weak supplemented if for any closed submodule N of M , there is a submodule K of M such that $M = N + K$ and $N \cap K \ll M$ [15, 14]. Recently, the study of the supplemented property in the rings, modules and lattices has become quite popular (see for example [3, 6, 7, 9, 12, 14, 15]).

Let L be a distributive lattice with 1. In the present paper, we are interested in investigating closed weak supplemented filters of L to use other notions of closed weak supplemented, and associate which exist in the literature as laid forth in [15, 14]. Here, we extend several concepts from module theory to lattice theory. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [1, 2, 4, 5, 6, 7]). Supplemented property (resp. w -supplemented property) in the lattices have already been investigated in [6] (resp. [7]). Here is a brief outline of this paper. Among many results in this paper, in Section 2, we characterize closed subfilters of a filter (Proposition 2.4) and some basic properties of closed subfilters are given in the Proposition 2.5, Proposition 2.6 and Proposition 2.7. In section 3, we collect some basic properties concerning closed weak supplemented filters of L . It is shown (Theorem 3.8) that if F is a filter of L such that $F = F_1 \oplus F_2$, then F is closed weak supplemented if and only if F_i is closed weak supplemented for all $1 \leq i \leq 2$. Also, it is proved (Theorem 3.11) that if F is a closed weak supplemented filter of L such that $\frac{F}{\text{Rad}(F)}$ is semisimple, then there exist a semisimple subfilter F_1 and a subfilter F_2 with $\text{Rad}(F_2) \trianglelefteq F_2$ such that $F = F_1 \oplus F_2$. In section 4, we will investigate the relations between closed weak supplemented filters and other filters, such as, extending filters, weak supplemented filters and local filters.

Let us briefly review some definitions and tools that will be used later [2]. By a lattice we mean a poset (L, \leq) in which every couple elements x, y has a g.l.b. (called the meet of x and y , and written $x \wedge y$) and a l.u.b. (called the join of x and y , and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L . Setting $X = L$, we see that any nonvoid

complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be maximal if E is a filter in L with $P \subsetneq E$, then $E = L$. If F is a filter of a lattice L , then the radical of F , denoted by $\text{Rad}(F)$, is the intersection of all maximal subfilters of F .

Let L be a lattice. If A is a subset of L , then the filter generated by A , denoted by $T(A)$, is the intersection of all filters that is containing A . A filter F is called finitely generated if there is a finite subset A of F such that $F = T(A)$. A subfilter G of a filter F of L is called small in F , written $G \ll F$, if, for every subfilter H of F , the equality $T(G \cup H) = F$ implies $H = F$. A subfilter G of F is called essential in F , written $G \leq F$, if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F . Let G be a subfilter of a filter F of L . A subfilter H of F is called a supplement of G in F if $F = T(G \cup H)$ and H is minimal with respect to this property, or equivalently, $F = T(G \cup H)$ and $G \cap H \ll H$. H is said to be a supplement subfilter of F if H is a supplement of some subfilter of F . F is called a supplemented filter if every subfilter of F has a supplemented in F [6]. Let G, H be subfilters of a filter F of L . If $F = T(G \cup H)$ and $G \cap H \ll F$, then H is called a weak supplement of G in F . If every subfilter of F has a weak supplement in F , then F is called a weak supplemented filter. A filter F is called \oplus -supplemented if every subfilter G of F has a supplement K in F which is also a direct summand of F . A filter F of a lattice L is called hollow if $F \neq \{1\}$ and every proper subfilter G of F is small in F . F is called local if it has exactly one maximal subfilter that contains all proper subfilters. A subfilter G of a filter F of L is said to be closed subfilter of F if the only solution of the relation $G \leq H \subseteq F$ is $H = G$ (clearly, $\{1\}$ and F are always closed subfilters of F). A filter F of L is called an extending filter if every subfilter is essential in a direct summand of F .

Proposition 1.1 ([5]). *A non-empty subset F of a lattice L is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

Proposition 1.2 ([6], Lemma 2.4, Theorem 2.6 and Proposition 2.16). *Let F be a filter of a distributive lattice L with 1.*

- (1) *If $A \ll F$ and $C \subseteq A$, then $C \ll F$.*
- (2) *If A, B are subfilters of F with $A \ll B$, then A is a small subfilter*

in subfilters of F that contains the subfilter of B . In particular, $A \ll F$.

(3) If F_1, F_2, \dots, F_n are small subfilters of F , then $T(F_1 \cup F_2 \cup \dots \cup F_n)$ is also small in F .

(4) If A, B, C and D are subfilters of F with $A \ll B$ and $C \ll D$, then $T(A \cup C) \ll T(B \cup D)$.

(5) $\text{Rad}(F) = T(\cup_{G \ll F} G)$.

(6) If $F = G \oplus H$, then $\text{Rad}(F) = \text{Rad}(G) \oplus \text{Rad}(H)$. Moreover, If U is a semisimple filter, then $\text{Rad}(U) = \{1\}$.

Lemma 1.3 ([6], Proposition 2.1). (1) Let A be an arbitrary non-empty subset of L . Then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \dots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.

(2) Let A, B and C be subfilters of a filter F of L . Then $T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C))$. In particular, if $F = T(T(A \cup B) \cup C)$, then $F = T(T(C \cup B) \cup A) = T(T(A \cup C) \cup B)$.

(3) (Modular law) If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.

Lemma 1.4 ([7], Lemma 2.3). Let U, V be subfilters of a filter F of L such that V is a direct summand of F with $U \subseteq V$. Then $U \ll F$ if and only if $U \ll V$.

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice (L, \leq) , we define a relation on L , given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then \sim is an equivalence relation on L , and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order \leq_Q on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. It is straightforward to check that $\left(\frac{L}{F}, \leq_Q\right)$ is a poset. The following notation below will be kept in this paper: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of \leq_Q , $(a \vee b) \wedge F$ is an upper bound for the set X . If $c \wedge F$ is any upper bound of X , then we can easily show that $(a \vee b) \wedge F \leq_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $\left(\frac{L}{F}, \leq_Q\right)$ is a lattice.

Remark 1.5 ([7]). Let G be a subfilter of a filter F of L .

(1) If $a \in F$, then $a \wedge F = F$. By the definition of \leq_Q , it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition \leq_Q , we can easily show that if L is distributive, then $\frac{L}{F}$ is distributive.

(4) $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.

(5) If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L .

(6) If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.

(7) If H and V are filters of L containing G , then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.

(8) If H is a filter of L containing G , then $\frac{T(F \cup H)}{G} = T\left(\frac{H}{G} \cup \frac{F}{G}\right)$.

2. Basic properties of closed subfilters. Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. A lattice L is called semisimple, if for each proper filter F of L , there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$. In this case, we say that F is a direct summand of L , and we write $L = F \oplus G$. A filter F of L is called a semisimple filter, if every subfilter of F is a direct summand. A simple lattice (resp; filter), is a lattice (resp; filter) that has no filters besides the $\{1\}$ and itself [6]. We need the following remark proved in [6, Lemma 2.15].

Remark 2.1. (1) If subfilter H of F is maximal with respect to $G \cap H = \{1\}$, then we say that H is a complement of G in F . Using the maximal principle we readily see that if G is a subfilter of F , then the set of those subfilters of F whose intersection with G is $\{1\}$ contains a maximal element H . Thus every subfilter G of F has a complement in F .

(2) A subfilter G of F is essential if and only if for each $1 \neq x \in F$ there exists an element $a \in L$ such that $1 \neq a \vee x \in G$.

(3) If H is a complement of G in F , then $T(G \cup H) \trianglelefteq F$.

(4) Assume that U_1, V_1, U_2 and V_2 are subfilters of F and let $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $F = V_1 \oplus V_2$. Then $U_1 \oplus U_2 \trianglelefteq V_1 \oplus V_2$ if and only if $U_1 \trianglelefteq V_1$ and $U_2 \trianglelefteq V_2$.

Lemma 2.2. *Every direct summand of a filter F of L is a closed subfilter of F .*

Proof. Let G be a direct summand of F . Then $F = T(G \cup H)$ and $H \cap G = \{1\}$ for some subfilter H of F . Suppose that $G \trianglelefteq G' \subseteq F$. Then $F = T(G' \cup H)$ and since $(G' \cap H) \cap G = G \cap H = \{1\}$, we have $G' \cap H = \{1\}$. Let $x \in G'$. Then $x = (x \vee g) \wedge (x \vee h)$ for some $g \in G$ and $h \in H$. As $x \vee h \in G' \cap H = \{1\}$, we get $x = x \vee g \in G$; so $G' \subseteq G$. Thus G is a closed subfilter. \square

Lemma 2.3. *Let F be a filter of L . Then the following hold:*

- (1) *If G, H are subfilters of F with $G \cap H = \{1\}$, then there exists a complement K of G in F such that $H \subseteq K$.*
- (2) *If G is a subfilter of F , then there exists a subfilter H of F containing G such that $G \trianglelefteq H$ and H is a complement for some subfilter of F .*

Proof. (1) Use Zorn's Lemma.

(2) Let G' be a complement of G in F . By (1), there exists a complement H of G' in F with $G \subseteq H$. Let $K \neq \{1\}$ be a subfilter H . Then $G' \subseteq T(K \cup G')$ gives $T(K \cup G') \cap G \neq \{1\}$. It follows that there are elements $1 \neq g \in G$, $g' \in G'$ and $h \in K$ such that $g = (g \vee g') \wedge (g \vee h)$. Since $g \vee g' \in G \cap G' = \{1\}$, we get $g = g \vee h \in G \cap K$. Thus $G \trianglelefteq H$. \square

Proposition 2.4. *If G is a subfilter of a filter F of L , then the following conditions are equivalent:*

- (1) *G is a closed subfilter of F ;*
- (2) *G is a complement for some subfilter H of F ;*
- (3) *If H is any complement for G in F , then G is a complement for H in F ;*
- (4) *If $G \subseteq K$ are subfilters of F with $K \trianglelefteq F$, then $\frac{K}{G} \trianglelefteq \frac{F}{G}$.*

Proof. (1) \Rightarrow (4) If $\frac{H}{G}$ is a subfilter of $\frac{F}{G}$ such that $\frac{H}{G} \cap \frac{K}{G} = \frac{H \cap K}{G} = \frac{G}{G} = \{1\}$, then $H \cap K = G$. Since $K \trianglelefteq F$, we have $H \cap K \trianglelefteq F \cap H$; so $G \trianglelefteq H$. The assumption that G is closed in F gives us $G = H$; so $\frac{H}{G} = \{1\}$.

(4) \Rightarrow (3) Since $G \cap H = \{1\}$, G can be enlarged to a complement G' for H By Lemma 2.3. By the modular law, $G' \cap T(G \cup H) = T(G \cup (G' \cap H)) = T(G) = G$, whence $\frac{T(G \cup H)}{G} \cap \frac{G'}{G} = \frac{G' \cap T(G \cup H)}{G} = \frac{G}{G} = \{1\}$. According

Remark 2.1, $T(G \cup H) \leq F$, and then from (4) we obtain $\frac{T(G \cup H)}{G} \leq \frac{F}{G}$. Thus $\frac{G'}{G} = \{1\}$, and so $G' = G$ is a complement for H .

(3) \Rightarrow (2) Since every subfilter G of F has a complement H in F , we have G is a complement of H in F by (3).

(2) \Rightarrow (1) Suppose that $G \leq G' \subseteq F$. Since $(G' \cap H) \cap G = G \cap H = \{1\}$, we have $G' \cap H = \{1\}$, and then the maximality of G implies that $G = G'$. Thus G is closed in F . \square

Proposition 2.5. *If F is a filter of L , then F is extending if and only if every closed subfilter is a direct summand of F .*

Proof. Assume that F is a extending filter and let G be a closed subfilter of F . By assumption, there is a direct summand H of F such that $G \leq H$; hence $G = H$. Conversely, assume that G is a subfilter of F . By Proposition 2.4 and Lemma 2.3 (2), there exists a closed filter K of F such that $G \leq K$, as required. \square

Proposition 2.6. *Let A, B and F be filters of L such that $A \subseteq B \subseteq F$. If A is closed in B and B is closed in F , then A is closed in F .*

Proof. Let A' be a complement for A in B , and let B' be a complement for B in F . According Remark 2.1, $T(B \cup B') \leq F$; hence Proposition 2.4 shows that $\frac{T(B \cup B')}{B} \leq \frac{F}{B}$. We claim that $\frac{T(B \cup B')}{A} \leq \frac{F}{A}$. Let $x \wedge A \in \frac{F}{A}$. Then $x \wedge B \in \frac{F}{B}$ gives $B \neq (a \wedge B) \vee_Q (x \wedge B) = (x \vee a) \wedge B \in \frac{T(B \cup B')}{B}$ for some $a \wedge B \in \frac{L}{B}$. It follows that $a \vee x \in T(B \cup B')$, and so $(a \vee x) \wedge A \in \frac{T(B \cup B')}{A}$. If $(a \vee x) \wedge A = A$, then $a \vee x \in A \subseteq B$ which is impossible. Thus $\frac{T(B \cup B')}{A} \leq \frac{F}{A}$ by Remark 2.1 (2). Since $T(B \cup B') = T((A \cup B') \cup B) \subseteq$

$$T(T(A \cup B') \cup B) \subseteq T(T(A \cup B) \cup B') = T(B \cup B'),$$

we get that $T(T(A \cup B') \cup B) = T(B' \cup B)$ which implies that

$$\frac{T(B \cup B')}{A} = \frac{T(T(A \cup B') \cup B)}{A} = T\left(\frac{B}{A} \cup \frac{T(A \cup B')}{A}\right) \leq \frac{F}{A}.$$

Using Remark 2.1 (3) and Proposition 2.4 again, we obtain that $T(A \cup A') \leq B$ and then $\frac{T(A \cup A')}{A} \leq \frac{B}{A}$. According Remark 2.1 (4), it follows that

$$T\left(\frac{T(A \cup A')}{A} \cup \frac{T(A \cup B')}{A}\right) \leq T\left(\frac{B}{A} \cup \frac{T(A \cup B')}{A}\right) \leq \frac{F}{A};$$

hence $T\left(\frac{T(A \cup A')}{A} \cup \frac{T(A \cup B')}{A}\right) = \frac{T(A \cup T(A' \cup B'))}{A} \leq \frac{F}{A}$. Now suppose that we have $A \leq K \subseteq F$. Since $(K \cap T(A' \cup B')) \cap A = A \cap T(A' \cup B') = \{1\}$, we get that $K \cap T(A' \cup B') = \{1\}$. Using the modular law, we find that $K \cap T(A \cup T(A' \cup B')) = T(A \cup (K \cap T(A' \cup B'))) = A$, whence $\frac{K}{A} \cap \frac{T(A \cup T(A' \cup B'))}{A} = \{1\}$. As $\frac{T(A \cup T(A' \cup B'))}{A} \leq \frac{F}{A}$, we obtain $\frac{K}{A} = \{1\}$; hence $K = A$. Therefore A is closed in F . \square

Proposition 2.7. *Let U and G be subfilters of a filter F of L with $G \subseteq U$. If U is closed in F , then $\frac{U}{G}$ is closed in $\frac{F}{G}$.*

Proof. Suppose that $\frac{U}{G} \leq \frac{K}{G} \subseteq \frac{F}{G}$ for some subfilter $\frac{K}{G}$ of $\frac{F}{G}$. For any $a \in K \setminus U$, then $a \notin G$ and $a \wedge G \neq \{\bar{1}\}$, where $\bar{1} = G = 1 \wedge G$. Since $\frac{U}{G} \leq \frac{K}{G}$, there is $b \wedge G \in \frac{L}{G}$ such that $\{\bar{1}\} \neq (a \vee b) \wedge G \in \frac{U}{G}$. Then there is $u \in U \setminus G$ such that $(a \vee b) \wedge G = u \wedge G$; hence $(a \vee b) \wedge g = u \wedge g' \in G$ for some $g, g' \in G$. Thus $1 \neq a \vee b \in U$ since U is a filter. Hence $U \leq K$ by Remark 2.1 and $U = K$. Thus $\frac{U}{G}$ is closed in $\frac{F}{G}$. \square

3. Basic properties of closed weak supplemented filters. In this section we collect some basic properties concerning closed weak supplemented filters of L . Our starting point is the following definition.

Definition 3.1. *A filter F of L is called closed weak supplemented if for any closed subfilter G of F , there is a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H \ll F$.*

Remark 3.2. (1) Clearly, any weak supplemented filter is closed weak supplemented.

(2) By Proposition 2.5, if F is a extending filter, then every closed subfilter G of F is a direct summand, and so $F = T(G \cup H)$ and $G \cap H = \{1\} \ll F$ for some subfilter H of F ; hence any extending filter is closed weak supplemented.

(3) Since local filters are hollow and hollow filters are supplemented; hence closed weak supplemented. So we have the following implications: local \Rightarrow hollow \Rightarrow supplemented \Rightarrow weak supplemented \Rightarrow closed weak supplemented.

(4) Let G be a subfilter of a semisimple filter F . Since G is a direct summand of F , we get G is closed in F by Lemma 2.2. Thus every subfilter of a

semisimple filter of F is closed in F .

Proposition 3.3. *Let F be any filter of L . If any proper closed subfilter $G \neq \{1\}$ of F is maximal in F , then F is closed weak supplemented.*

Proof. By Remark 3.2 (2), it is enough to show that F is extending. Let $G \neq \{1\}$ is a proper closed subfilter that is maximal in F . Clearly, G is not essential in F . It follows that there is an element $f \in F \setminus G$ such that for all $1 \neq a \in L$ with $a \vee f \neq 1$, then $a \vee f \notin G$ which implies that $G \subsetneq T(T(\{f\}) \cup G) \subseteq F$; so $F = T(T(\{f\}) \cup G)$ and $T(\{f\}) \cap G = \{1\}$. Thus $F = G \oplus T(\{f\})$, as needed. \square

Proposition 3.4. *If F is a closed weak supplemented filter of L , then any direct summand of F is closed weak supplemented.*

Proof. Assume that G is any direct summand of F and let H be any closed subfilter of G . Since G is a closed subfilter of F by Lemma 2.2, we get that H is a closed subfilter of F by Proposition 2.6. Then there exists a subfilter K of F such that $F = T(K \cup H)$ and $K \cap H \ll F$. Thus $G = G \cap T(H \cap K) = T(H \cup (G \cap K))$. It is enough to show that $H \cap (G \cap K) = H \cap K \ll G$. By Lemma 1.4, $K \cap H \ll F$ gives $K \cap H \ll G$, as required. \square

Lemma 3.5. *Let G, H be subfilters of a filter F of L such that H is a weak supplement of G in F . If $F = T(U \cup H)$ for some subfilter U of G , then H is a weak supplement of U in F .*

Proof. By assumption, it suffices to show that $U \cap H \ll F$. Let X be a subfilter F such that $F = T(X \cup (U \cap H))$. Thus $F \subseteq T(X \cup (G \cap H)) \subseteq F$; so $F = T(X \cup (G \cap H))$. It follows that $X = F$ since $G \cap H \ll F$. Thus H is a weak supplement U in F . \square

Proposition 3.6. *Let F_1, G be subfilters of a filter F of L such that $T(F_1 \cup G)$ has a weak supplement of X in F and $F_1 \cap T(X \cup G)$ has a weak supplement Y in F_1 . Then $T(X \cup Y)$ is a weak supplement G in F .*

Proof. To simplify our notation let $B = T(X \cup G) \cap F_1 \subseteq T(X \cup G)$. By hypothesis, $T(X \cup T(F_1 \cup G)) = F$, $X \cap T(F_1 \cup G) \ll F$, $T(Y \cup B) = F_1$ and $Y \cap B = Y \cap T(X \cup G) \ll F_1$ (so $Y \cap T(X \cup G) \ll F$ by Proposition 1.2 (2)). By Lemma 1.3, we have $F = T(X \cup T(F_1 \cup G)) = T(F_1 \cup T(X \cup G)) =$

$$T(T(B \cup Y) \cup T(X \cup G)) = T(Y \cup T(B \cup T(X \cup G))) =$$

$T(Y \cup T(X \cup G)) \subseteq T(G \cup T(X \cup Y)) \subseteq F$; hence $F = T(G \cup T(X \cup Y))$. It is enough to show that $T(X \cup Y) \cap G \ll F$. Since $T(G \cup Y) \subseteq T(F_1 \cup G)$ and

$F = T(G \cup T(X \cup Y)) = T(X \cup T(G \cup Y))$, X also is a weak supplement of $T(G \cup Y)$ in F by Lemma 3.5 which implies that $T(G \cup Y) \cap X \ll F$. Now by Proposition 1.2 (4), $T(X \cup Y) \cap G \subseteq T((X \cap T(G \cup Y)) \cup (Y \cap T(X \cup G))) \ll T(F \cup F) = F$; hence $T(X \cup Y) \cap G \ll F$ by Lemma 3.5. \square

Proposition 3.7. *Assume that F is a filter of L and let $F = T(F_1 \cup F_2)$ with each F_i ($i = 1, 2$) closed weak supplemented. Suppose that (1) $F_i \cap T(F_j \cup G)$ is closed in F_i (2) $F_j \cap T(G \cup K)$ is closed in F_j , where K is a weak supplement of $F_i \cap T(F_j \cup G)$ in F_i , $i \neq j$, for any closed subfilter G of F . Then F is closed weak supplemented.*

Proof. Let G be a closed subfilter of F . Then $T(F_1 \cup T(F_2 \cup G)) = T(G \cup T(F_1 \cup F_2)) = F$ has a trivial supplement $\{1\}$ in F . Since $F_1 \cap T(F_2 \cup G)$ is closed in F_1 and F_1 is closed weak supplemented, then there is a subfilter K of F_1 such that $F_1 = T(K \cup (F_1 \cap T(F_2 \cup G)))$ and $K \cap F_1 \cap T(F_2 \cup G) = K \cap T(F_2 \cup G) \ll F_1$. By Proposition 3.6, K is a weak supplement of $T(F_2 \cup G)$ in F ; so $F = T(K \cup T(F_2 \cup G))$. Since $F_2 \cap T(G \cup K)$ is closed in F_2 and F_2 is closed weak supplemented, then $F_2 \cap T(G \cup K)$ has a weak supplement H in F_2 . Again by Proposition 3.6, $T(H \cup K)$ is a weak supplement of G in F . Thus F is closed weak supplemented. \square

Theorem 3.8. *Let F be a filter of L such that $F = F_1 \oplus F_2$. Then F is closed weak supplemented if and only if F_i is closed weak supplemented for all $1 \leq i \leq 2$.*

Proof. We first show that if A, B and C are subfilters of F , then

$$A \cap T(B \cup C) = T((A \cap B) \cup (A \cap C)).$$

Since $(A \cap B) \cup (A \cap C) \subseteq A, T(B \cup C)$, we get $T((A \cap B) \cup (A \cap C)) \subseteq A \cap T(B \cup C)$. For the reverse inclusion, assume that $x \in A \cap T(B \cup C)$. Then $x = x \vee x = x \vee ((x \vee b) \wedge (x \vee c))$ for some elements $b \in B$ and $c \in C$; so $(x \vee b) \wedge (x \vee c) \leq x$. Now $x \vee b \in A \cap B$ and $x \vee c \in A \cap C$ gives $x \in T((A \cap B) \cup (A \cap C))$, and so we have equality. The necessity is clear by Proposition 3.4. Conversely, let G be any closed subfilter of F . To simplify our notation let $G \cap F_1 = D_1$ and $G \cap F_2 = D_2$. Then for each $1 \leq i \leq 2$, D_i is closed in F_i . In fact, suppose that $D_1 \triangleleft K \subseteq F_1$. Since $D_2 \triangleleft D_2$, we have $G = G \cap F = G \cap T(F_1 \cup F_2) = T(D_1 \cup D_2) \triangleleft T(K \cup D_2)$ by Remark 2.1; hence $G = T(D_1 \cup D_2) = T(K \cup D_2)$ since G is closed in F . If $x \in K$ (so $x \in F_1$), then $x = (x \vee d_1) \wedge (x \vee d_2)$ for some $d_1 \in D_1$ and $d_2 \in D_2$. Since $x \vee d_2 \in F_1 \cap F_2 = \{1\}$, we get that $x \in D_1$. So $K = D_1$ and D_1 is closed in F_1 . Similarly, D_2 is closed in F_2 . By assumption, there exist a subfilter K_1 of F_1 and a subfilter K_2 of F_2 such that $F_1 = T(D_1 \cup K_1)$,

$D_1 \cap K_1 = K_1 \cap G \ll F_1$, $F_2 = T(D_2 \cup K_2)$ and $K_2 \cap D_2 = K_2 \cap G \ll F_2$. It follows that $T((K_1 \cap G) \cup (K_2 \cap G)) = G \cap T(K_1 \cup K_2) \ll T(F_1 \cup F_2) = T(F) = F$ by Proposition 1.2 (4) and $F = T(F_1 \cup F_2) = T(T(K_1 \cup D_1) \cup T(K_2 \cup D_2)) = T(T(K_1 \cup K_2) \cup T(D_1 \cup D_2)) = T(T(K_1 \cup K_2) \cup G)$. Thus F is closed weak supplemented. \square

The proof of the following lemma is well-known, but we give the details for convenience.

Lemma 3.9. *If K, H are subfilters of a filter F of L with $H \ll F$, then $\frac{T(H \cup K)}{K} \ll \frac{F}{K}$.*

Proof. Assume that $A = T(H \cup K)$ and let $\frac{F}{K} = T\left(\frac{A}{K} \cup \frac{G}{K}\right) = \frac{T(A \cup G)}{K}$ for some subfilter $\frac{G}{K}$ of $\frac{F}{K}$; so $F = T(T(H \cup K) \cup G) = T(H \cup T(K \cup G)) = T(H \cup G)$. Then $H \ll F$ gives $G = F$. This completes the proof. \square

Theorem 3.10. *Let X be a closed subfilter of a filter F of L . If F is a closed weak supplemented filter, then $\frac{F}{X}$ is closed weak supplemented.*

Proof. Assume that F is a closed weak supplemented filter and let $\frac{U}{X}$ be a closed subfilter of $\frac{F}{X}$. Let $U \leq K \subseteq F$. Then Proposition 2.4 (4) gives $\frac{U}{X} \leq \frac{K}{X}$ which implies that $\frac{U}{X} = \frac{K}{X}$. Thus U is closed in F . By assumption, there is a weak supplement V of U in F . It is enough to show that $\frac{T(X \cup V)}{X}$ is a weak supplement of $\frac{U}{X}$ in $\frac{F}{X}$. Since V is a weak supplement of U in F , we have $F = T(U \cup V)$ and $U \cap V \ll F$. Thus $\frac{T(X \cup (U \cap V))}{X} \ll \frac{F}{X}$ by Lemma 3.9. Since $F = T(U \cup V)$, we have $T\left(\frac{U}{X} \cup \frac{T(X \cup V)}{X}\right) = \frac{T(U \cup T(X \cup V))}{X} = \frac{F}{X}$ by Remark 1.5 (8) and

$$\frac{U}{X} \cap \frac{T(V \cup X)}{X} = \frac{U \cap T(V \cup X)}{X} = \frac{T(X \cup (U \cap V))}{X} \ll \frac{F}{X}.$$

Therefore $\frac{T(X \cup V)}{X}$ is a weak supplement of $\frac{U}{X}$ in $\frac{F}{X}$. Thus $\frac{F}{X}$ is closed weak supplemented. \square

Theorem 3.11. *Let F be a closed weak supplemented filter of L such that $\frac{F}{\text{Rad}(F)}$ is semisimple. Then there exist a semisimple subfilter F_1 and a subfilter F_2 with $\text{Rad}(F_2) \trianglelefteq F_2$ such that $F = F_1 \oplus F_2$.*

Proof. By Remark 2.1 (3) and Proposition 2.4, for $\text{Rad}(F)$, there exists a closed subfilter F_1 of F such that $F_1 \cap \text{Rad}(F) = \{1\}$ and $T(F_1 \cup \text{Rad}(F)) \trianglelefteq F$. We first show that F_1 is semisimple. Let K be any subfilter of F_1 (so $K \cap \text{Rad}(F) = \{1\}$). There is a subfilter $\frac{H}{\text{Rad}(F)}$ of $\frac{F}{\text{Rad}(F)}$ such that

$$\frac{F}{\text{Rad}(F)} = T\left(\frac{H}{\text{Rad}(F)} \cup \frac{T(K \cup \text{Rad}(F))}{\text{Rad}(F)}\right)$$

and $\frac{H}{\text{Rad}(F)} \cap \frac{T(K \cup \text{Rad}(F))}{\text{Rad}(F)} = \{\text{Rad}(F)\}$. An inspection will show that $F = T(H \cup T(K \cup \text{Rad}(F))) = T(H \cup K)$ and $H \cap K \subseteq H \cap T(K \cup \text{Rad}(F)) = \text{Rad}(F)$ (so $H \cap K \subseteq K \cap \text{Rad}(F) = \{1\}$). Thus

$$F_1 = F_1 \cap T(K \cup H) = T(K \cup (F_1 \cap H))$$

and $(F_1 \cap H) \cap K = K \cap H = \{1\}$. Therefore K is a direct summand of F_1 . Since F is closed weak supplemented, there is a subfilter F_2 of F such that $F = T(F_1 \cup F_2)$ and $F_1 \cap F_2 \ll F$ (so $F_1 \cap F_2 \subseteq \text{Rad}(F)$). Since $F_1 \cap F_2$ is a subfilter of both F_1 and $\text{Rad}(F)$, we get $F_1 \cap F_2 = \{1\}$. Thus $F = F_1 \oplus F_2$ and $\text{Rad}(F) = T(\text{Rad}(F_1) \cup \text{Rad}(F_2)) = T(\text{Rad}(F_2) \cup \{1\}) = \text{Rad}(F_2)$ by Proposition 1.2 (6). Since $T(F_1 \cup \text{Rad}(F_2)) \trianglelefteq F = T(F_1 \cup F_2)$, $\text{Rad}(F_2) \trianglelefteq F_2$ by Remark 2.1 (4), as required. \square

4. Further results. In this section, we will investigate the relations between closed weak supplemented filters and other filters, such as, extending filters, weak supplemented filters and local filters. Our starting point is the following definition (see [13]):

Definition 4.1. *A filter F of L is called refinable if for any subfilters G, H of F with $F = T(G \cup H)$, there exists a direct summand K of F with $K \subseteq G$ and $T(K \cup H) = F$.*

Proposition 4.2. *Let F be a refinable filter of L . Then the following are equivalent:*

- (1) F is \oplus -supplemented;
- (2) F is supplemented;
- (3) F is weak supplemented.

Proof. (1) \Rightarrow (2) \Rightarrow (3) are clear.

(3) \Rightarrow (1) If G is a subfilter of F , then $F = T(G \cup H)$ and $G \cap H \ll F$ for some subfilter H of F . Since F is refinable, there exists a direct summand K of F such that K is a subfilter of H and $F = T(G \cup K)$. As $G \cap K \subseteq G \cap H \ll F$; so $G \cap K \ll F$. It follows that $G \cap K \ll K$ by Lemma 1.4. Thus F is \oplus -supplemented. \square

Proposition 4.3. *Let F be a filter of L with $\text{Rad}(F) = \{1\}$. Then the following are equivalent:*

- (1) *F is a closed weak supplemented filter;*
- (2) *F is extending.*

Proof. (1) \Rightarrow (2) Let G be a closed subfilter of F . Then $F = T(G \cup H)$ and $G \cap H \ll F$ for some subfilter H of F and so $G \cap H \subseteq \text{Rad}(F) = \{1\}$ which implies that G is a direct summand of F . Thus F is extending. (2) \Rightarrow (1) follows from the Remark 3.2 (2). \square

For the remainder of this section we will study the relation closed weak supplemented filters and weak supplemented filters. Let F be a filter of L . If every subfilter is closed in F (for example, by Remark 3.2 (4), F is semisimple), then F is closed weak supplemented if and only if F is weak supplemented. For other cases, we have the following:

Theorem 4.4. *Suppose that for any subfilter U of a filter F of L , there is a subfilter K of F , which is a weak supplement of some maximal subfilter N of F such that $T(K \cup U)$ is closed in F . Then F is closed weak supplemented if and only if F is weak supplemented.*

Proof. The necessity is clear. Conversely, assume that F is a closed weak supplemented filter. It is enough to show that U has a weak supplement in F . Let X be a weak supplement of $T(K \cup U)$ in F . To simplify our notation let $X \cap K = A$, $K \cap U = B$ and $X \cap U = C$. Since $T(K \cup U)$ is closed in F , then $F = T(X \cup T(K \cup U))$ and $X \cap T(K \cup U) = T((X \cap K) \cup (X \cap U)) = T(A \cup C) \ll F$. As K is a weak supplement of N in F , we have $F = T(K \cup N)$ and $K \cap N \ll F$. We split the proof into two cases:

Case 1. $K \cap T(X \cup U) = T((K \cap X) \cup (K \cap U)) = T(A \cup B) \subseteq K \cap N \ll F$ (so $T(A \cup B) \ll F$ by Proposition 1.2 (1)). Then $K \cap T(X \cup U) = T(A \cup B) \subseteq T(C \cup T(A \cup B)) = T(T(A \cup C) \cup T(A \cup B)) \ll T(F \cup F) = F$; so $K \cap T(X \cup U) \ll F$ by Proposition 1.2. Since $T(U \cup T(K \cup U)) = T(X \cup T(K \cup U)) = F$, we get $T(K \cup X)$ is a weak supplement of U in F .

Case 2. $K \cap T(X \cup U) \not\subseteq K \cap N$. We first show that $K \cap N$ is a maximal subfilter of K . Let $K \cap N \subsetneq H \subseteq K$. There is an element $x \in H$ such that

$x \notin N$; so $F = T(N \cup T(\{x\}))$ since N is a maximal subfilter. It follows that $F = T(N \cup H)$ which implies that $K = K \cap T(N \cup H) = T(H \cup (K \cap N)) = H$ by Lemma 1.3. Thus $K \cap N$ is a maximal subfilter of K . Therefore, $K \cap N \subsetneq T((K \cap N) \cup (K \cap T(X \cup U))) \subseteq K$; so $K = T((K \cap N) \cup (K \cap T(X \cup U)))$ and since $K \cap N \ll F$, we have $F = T(T(K \cup U) \cup X) = T(T(X \cup U) \cup K) =$

$$T(T(X \cup U) \cup T((K \cap N) \cup (K \cap T(X \cup U)))) =$$

$T((K \cap N) \cup T(T(X \cup U) \cup (K \cap T(X \cup U)))) = T((K \cap N) \cup T(X \cup U)) = T(X \cup U)$. Since $U \cap X \subseteq X \cap T(K \cup U) \ll F$, we get $U \cap X \ll F$. Thus X is a weak supplement of U in F . \square

Lemma 4.5. *Let G be a closed subfilter of a closed weak supplemented filter F of L . Suppose that $H \ll F$. Then there exists a subfilter K of F such that $F = T(K \cup G) = T(T(K \cup G) \cup H)$, $K \cap G \ll F$ and $K \cap T(G \cup H) \ll F$.*

Proof. Since F is closed weak supplemented, there is a subfilter K of F such that $T(G \cup K) = F$ and $K \cap G \ll F$. Moreover, $T(T(K \cup G) \cup H) = T(F \cup H) = T(F) = F$. It remains to show that

$$K \cap T(G \cup H) = T((K \cap H) \cup (K \cap G)) \ll F.$$

Let $F = T(T((K \cap G) \cup (K \cap H)) \cup X)$ for some subfilter X of F which implies that $F = T((K \cap G) \cup T(X \cup (K \cap H)))$. It follows that $F = T(X \cup (K \cap H))$ since $K \cap G \ll F$. Then $F \subseteq T(X \cup H) \subseteq F$; hence $F = T(H \cup X)$. Now $H \ll F$ gives $X = F$. This completes the proof. \square

Theorem 4.6. *Let F be a filter of L . Suppose that for any subfilter G of F , there is a closed subfilter K (depending on G) of F such that $G = T(H \cup K)$ or $K = T(G \cup H)$ for some subfilter $H \ll F$. Then F is closed weak supplemented if and only if F is weak supplemented.*

Proof. Assume that F is a closed weak supplemented filter and let G be a subfilter of F . We split the proof into two cases.

Case 1. Suppose that there exists a closed subfilter K of F such that $G = T(K \cup H)$ for some $H \ll F$. By Lemma 4.5, there exists a subfilter K' of F such that $F = T(T(K \cup K') \cup H) = T(K' \cup T(K \cup H)) = T(K' \cup G)$ and $K' \cap T(K \cup H) = K' \cap G \ll F$; so F is weak supplemented.

Case 2. Suppose that there exists a closed subfilter K of F such that $K = T(G \cup H)$ for some $H \ll F$. Since F is a closed weak supplemented filter, there is a subfilter K' of F such that $F = T(K \cup K')$ and $K \cap K' \ll F$. Thus $F = T(K \cup K') = T(T(G \cup H) \cup K') = T(H \cup T(G \cup K'))$; hence $F = T(G \cup K')$

since $H \ll F$. Since $G \cap K' \subseteq K \cap K' \ll F$, we get $G \cap K' \ll F$. Thus F is weak supplemented. The other implication is clear. \square

Corollary 4.7. *Let F be a refinable filter of L . Suppose that for any subfilter G of F , there is a closed subfilter K (depending on G) of F such that $G = T(H \cup K)$ or $K = T(G \cup H)$ for some subfilter $H \ll F$. Then the following are equivalent:*

- (1) F is \oplus -supplemented;
- (2) F is supplemented;
- (3) F is weak supplemented;
- (4) F is closed weak supplemented.

Proof. Apply Theorem 4.6 and Proposition 4.2. \square

Corollary 4.8. *Let F be a refinable filter of L with $\text{Rad}(F) = \{1\}$. Suppose that for any subfilter G of F , there is a closed subfilter K (depending on G) of F such that $G = T(H \cup K)$ or $K = T(G \cup H)$ for some subfilter $H \ll F$. Then the following are equivalent:*

- (1) F is \oplus -supplemented;
- (2) F is supplemented;
- (3) F is weak supplemented;
- (4) F is closed weak supplemented;
- (5) F is extending.

Proof. Apply Theorem 4.6 and Proposition 4.3. \square

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