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A DIOPHANTINE PROBLEM CONCERNING THIRD ORDER MATRICES

Ajai Choudhry

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Abstract. In this paper we find a third order unimodular matrix, none of whose entries is 1 or -1, such that when each entry of the matrix is replaced by its cube, the resulting matrix is also unimodular. Further, we find third order square integer matrices (a_{ij}), none of the integers a_{ij} being 1 or -1, such that $\det(a_{ij}) = k$ and $\det(a_{ij}^3) = k^3$, where k is a nonzero integer.

1. Introduction. This paper is concerned with the problem of finding a 3×3 integer matrix (a_{ij}), with no $a_{ij} = \pm 1$, such that $\det(a_{ij}) = 1$, and further, when each entry of the matrix A is replaced by its cube, then also the determinant is 1, that is, $\det(a_{ij}^3) = 1$. We also consider the more general problem of finding a matrix (a_{ij}), none of the a_{ij} being 0 or ± 1, such that $\det(a_{ij}) = k$ and $\det(a_{ij}^3) = k^3$, where k is a nonzero integer.

It is pertinent to recall that Molnar [7] had posed the problem of finding an $n \times n$ integer matrix (a_{ij}), with no $a_{ij} = \pm 1$, such that $\det(a_{ij}) = 1$ and also

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$\det(a_{ij}^2) = 1$. Following Molnar [7], the problem comes from algebraic topology. The requirement $a_{ij} \neq \pm 1$ is a topological restriction but it also eliminates the trivial cases from number theoretical point of view. Several authors found solutions of the problem when $n = 3$, including also the case when additionally $a_{ij} \neq 0$ [3, 4, 5]. In fact, Dănescu, Vâjâitu, and Zaharescu [2] solved Molnar's problem for matrices of arbitrary order. Guy restricted the problem to 3×3 matrices in his book, "Unsolved problems in number theory" [6, Problem F28, pp. 265–266], but imposed the additional condition that all the entries a_{ij} should also be nonzero. He concluded his discussion by asking, "Will the problem extend to cubes?". This question has, until now, remained completely unanswered.

If $A = (a_{ij})$ is any $n \times n$ matrix, we will write $A^{(3)}$ to denote the matrix (a_{ij}^3) obtained by replacing each entry of the matrix A by its cube[1]. We obtain in this paper a 3×3 matrix $A = (a_{ij})$ whose entries are univariate polynomials with integer coefficients, with no $a_{ij} = \pm 1$, and such that both $\det A$ and $\det(A^{(3)})$ are equal to 1. We also obtain a parametric solution of the more general problem of finding a 3×3 integer matrix (a_{ij}), none of the a_{ij} being 0 or ± 1, and such that $\det(a_{ij}) = k$ and $\det(a_{ij}^3) = k^3$, where k is a nonzero integer.

2. Unimodular matrices that remain unimodular when each entry is replaced by its cube. If M is any $n \times n$ matrix such that both $\det M = 1$ and $\det(M^{(3)}) = 1$, then several other matrices satisfying these conditions can readily be derived from the matrix M. In Section 2.1 we give a lemma that lists out such matrices, and in Section 2.2 we obtain third order matrices satisfying such conditions.

2.1. A general lemma. We will denote the transpose of a matrix M by M^T. Further, we will write E_{ij} to denote the elementary matrix obtained by interchanging the i-th and j-th rows of the identity matrix, and $E_i(\alpha)$ to denote the elementary matrix obtained by multiplying the i-th row of the identity matrix by α.

Lemma 2.1. *If M is any $n \times n$ integer matrix with the property that both $\det M = 1$ and $\det(M^{(3)}) = 1$, then the following integer matrices, derived from the matrix M, also have this property:*

(*i*) *the matrix M^T;*

(*ii*) *the matrices $M_1 = E_{i_1}(-1)E_{i_2}(-1)M$ and $M_2 = ME_{i_1}(-1)E_{i_2}(-1)$ where $i_1, i_2 \in \{1, \ldots, n\}$ such that $i_1 \neq i_2$;*

[1]This notation is adapted from the notation used by Dănescu et al. [2].

(*iii*) *the matrices* $M_3 = E_{i_1i_2}E_{j_1j_2}M, M_4 = ME_{i_1i_2}E_{j_1j_2}$ *and* $M_5 = E_{i_1i_2}ME_{j_1j_2}$ *where* $i_1 \neq i_2$, $j_1 \neq j_2$ *and* $i_1, i_2, j_1, j_2 \in \{1, \ldots, n\}$;

(*iv*) *the matrix* $M([i,j],\alpha) = E_i(\alpha)ME_j(\alpha^{-1})$ *where* $i, j \in \{1, \ldots, n\}$ *and* α *is a nonzero rational number so chosen that the entries of the matrix* $M([i,j],\alpha)$ *are all integers.*

Proof. Clearly, $\det(M^T) = 1$, and $(M^T)^{(3)} = (M^{(3)})^T$, hence $\det((M^T)^{(3)}) = \det((M^{(3)})^T) = \det(M^{(3)}) = 1$, which proves the first part of the lemma. To prove *(ii)*, we note that $\det(E_{i_1}(-1)) = \det(E_{i_2}(-1)) = -1$, hence $\det M_1 = \det M$, and by the definition of $M_1^{(3)}$, it follows that $M_1^{(3)} = E_{i_1}(-1)E_{i_2}(-1)M^{(3)}$, hence $\det(M_1^{(3)}) = \det(M^{(3)}) = 1$. This proves the result for the matrix M_1. The proofs for the other matrices listed at (*ii*) and (*iii*) above are similar and are accordingly omitted.

Finally, regarding the last matrix $M([i,j],\alpha)$, it is readily seen that $\det(M([i,j],\alpha)) = \det M = 1$. Further, on multiplying the entries of the i-th row of the matrix $M^{(3)}$ by α^3 and then multiplying the entries of the j-th column by α^{-3}, we get the matrix $(M([i,j],\alpha))^{(3)}$. It follows that $\det(M([i,j],\alpha)^{(3)}) = \det(M^{(3)}) = 1$. □

2.2. Third order unimodular matrices. We will now obtain third order square integer matrices $A = (a_{ij})$, with no $a_{ij} = \pm 1$, such that both $\det(a_{ij})$ and $\det(a_{ij}^3)$ are equal to 1. We have to solve two simultaneous equations in nine independent variables. A fair amount of computer search yielded essentially only one such matrix, namely,

$$A_1 = \begin{bmatrix} 7 & 11 & 2 \\ 13 & 20 & 3 \\ 2 & 3 & 0 \end{bmatrix}, \tag{2.1}$$

which satisfies the conditions $\det A_1 = 1$ and $\det(A_1^{(3)}) = 1$.

We now give a theorem that gives a parametric solution of the problem.

Theorem 2.2. *The matrix* A *defined by*

$$A = \begin{bmatrix} (16t+1)(2592t^2+288t+7) & (18t+1)(24t+1)(144t+11) & 2 \\ (12t+1)(5184t^2+540t+13) & (72t+5)(1296t^2+153t+4) & 3 \\ 2 & 3 & 0 \end{bmatrix}, \tag{2.2}$$

where t *is an arbitrary parameter, satisfies the conditions* $\det A = 1$ *and* $\det(A^{(3)}) = 1$.

Proof. We begin with the 3×3 matrix $B = (b_{ij})$ where we take

$$b_{13} = b_{23} = b_{31} = b_{32} = 1, \quad b_{33} = 0, \tag{2.3}$$

so that the matrix B may be written as follows:

$$B = \begin{bmatrix} b_{11} & b_{12} & 1 \\ b_{21} & b_{22} & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2.4}$$

We then get,

$$\begin{aligned} \det B &= -b_{11} + b_{12} + b_{21} - b_{22}, \\ \det(B^{(3)}) &= -b_{11}^3 + b_{12}^3 + b_{21}^3 - b_{22}^3. \end{aligned} \tag{2.5}$$

We note that a parametric solution of the simultaneous diophantine equations,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 &= 0, \end{aligned} \tag{2.6}$$

given by Choudhry [1, p. 316], is as follows:

$$\begin{aligned} x_1 &= pq(r^2 - s^2) + q^2r^2, \\ x_2 &= -(p^2s(r+s) - q^2rs), \\ x_3 &= p^2r(r+s) + pqr^2 - q^2rs, \\ x_4 &= -(p^2r(r+s) + pq(r^2 - s^2)), \\ x_5 &= p^2s(r+s) - pqr^2 - q^2r^2, \end{aligned} \tag{2.7}$$

where p, q, r, and s are arbitrary parameters.

With the values of $x_i, i = 1, \ldots, 5$, defined by (2.7), we take

$$b_{11} = x_2, \quad b_{12} = -x_3, \quad b_{21} = -x_4, \quad b_{22} = x_5, \tag{2.8}$$

when we get,

$$\begin{aligned} \det B &= x_1 \\ \det(B^{(3)}) &= x_1^3. \end{aligned} \tag{2.9}$$

We now choose the parameters p, q, r, s, as follows:

$$p = 36t + 3, \quad q = -1, \quad r = 144t + 11, \quad s = -144t - 9, \tag{2.10}$$

where t is an arbitrary parameter, when we get $x_1 = 1$. The entries of the matrix B may now be written in terms of the parameter t. We rename this matrix as C, and write it explicitly as follows:

$$C = \begin{bmatrix} 9(16t+1)(2592t^2+288t+7) & 6(18t+1)(24t+1)(144t+11) & 1 \\ 6(12t+1)(5184t^2+540t+13) & 4(72t+5)(1296t^2+153t+4) & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Since $x_1 = 1$, it follows from (2.9) that the matrix C satisfies the conditions $\det C = 1$ and $\det(C^{(3)}) = 1$.

Now on starting with the matrix C, and using the last matrix listed in Lemma 2.1 four times, in succession, we obtain four matrices $C_i, i = 1, \ldots, 4$, as follows:

$$\begin{aligned} C_1 &= C([1,3], 1/3), \qquad & C_2 &= C_1([2,3], 1/2), \\ C_3 &= C_2([3,1], 3), \qquad & C_4 &= C([3,2], 2). \end{aligned} \tag{2.11}$$

In view of Lemma 2.1, each of the matrices $C_i, i = 1, \ldots, 4$, satisfies the conditions $\det C_i = 1$ and $\det(C_i^{(3)}) = 1$. In fact, the matrix C_4 is the matrix A mentioned in the theorem. It follows that $\det A = 1$ and $\det(A^{(3)}) = 1$. □

When $t = 0$, the matrix A, defined by (2.2), reduces to the matrix A_1 given by (2.1). As a second numerical example, when $t = 1$, we get the matrix

$$A_2 = \begin{bmatrix} 49079 & 73625 & 2 \\ 74581 & 111881 & 3 \\ 2 & 3 & 0 \end{bmatrix},$$

which satisfies the conditions $\det A_2 = 1$ and $\det(A_2^{(3)}) = 1$.

We note that one of the entries of the matrix A given by Theorem 2.2 is always zero. While it would be interesting to find a 3×3 integer matrix A, none of whose entries is 0 or ± 1, such that both $\det A$ and $\det(A^{(3)})$ are equal to 1, we could not find such an example.

3. A more general problem. We will now find third order square integer matrices A such that $\det A = k$ and $\det(A^{(3)}) = k^3$, where $k \neq 1$ is a nonzero integer.

In fact, in Section 2.2, we have already obtained a solution to this problem in terms of four arbitrary parameters p, q, r, s, with $k = pq(r^2 - s^2) + q^2 r^2$, since the matrix B, whose entries are defined by (2.3) and (2.8), satisfies the conditions (2.9) where the value of x_1 is given by (2.7). We note, however, that one entry of the matrix B is always 0.

A computer search for 3×3 integer matrices, none of the entries being 0 or ± 1, such that $\det A = k$ and $\det(A^{(3)}) = k^3$, where k is an integer < 10, yielded just one such example, namely the matrix,

$$M = \begin{bmatrix} -5 & 4 & 10 \\ 5 & 3 & 11 \\ 3 & 2 & 7 \end{bmatrix}, \tag{3.1}$$

such that $\det M = 7$ and $\det(M^{(3)}) = 7^3$. The following theorem gives a more general solution of the problem with the entries of the matrix A being given in terms of polynomials in six arbitrary integer parameters.

Theorem 3.1. *If the polynomial $\phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is defined by*

(3.2)
$$\begin{aligned}\phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) &= -(\alpha_2\beta_3 + \alpha_3\beta_2)\alpha_1^8\alpha_2^2\alpha_3^2\beta_2^4\beta_3^4 - (\alpha_2^4\beta_3^4 - \alpha_2^3\alpha_3\beta_2\beta_3^3 \\ &+ \alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 - \alpha_2\alpha_3^3\beta_2^3\beta_3 + \alpha_3^4\beta_2^4)(\alpha_2\beta_3 + \alpha_3\beta_2)^2\alpha_1^7\beta_1\beta_2\beta_3 - (\alpha_2\beta_3 + \alpha_3\beta_2) \\ &\times (\alpha_2^4\beta_3^4 + \alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 + \alpha_3^4\beta_2^4)\alpha_1^6\alpha_2\alpha_3\beta_1^2\beta_2\beta_3 - 2\alpha_1^5\alpha_2^4\alpha_3^4\beta_1^3\beta_2^3\beta_3^3 \\ &- (\alpha_2\beta_3 + \alpha_3\beta_2)(\alpha_2^4\beta_3^4 - 2\alpha_2^3\alpha_3\beta_2\beta_3^3 + \alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 - 2\alpha_2\alpha_3^3\beta_2^3\beta_3 + \alpha_3^4\beta_2^4)\alpha_1^4\alpha_2^2\alpha_3^2\beta_1^4 \\ &+ 2(\alpha_2^2\beta_3^2 + \alpha_2\alpha_3\beta_2\beta_3 + \alpha_3^2\beta_2^2)\alpha_1^3\alpha_2^4\alpha_3^4\beta_1^5\beta_2\beta_3 + (\alpha_2\beta_3 + \alpha_3\beta_2)(\alpha_2^2\beta_3^2 + \alpha_3^2\beta_2^2) \\ &\times \alpha_1^2\alpha_2^4\alpha_3^4\beta_1^6 + (\alpha_2^2\beta_3^2 + \alpha_3^2\beta_2^2)\alpha_1\alpha_2^5\alpha_3^5\beta_1^7,\end{aligned}$$

with $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, being arbitrary integer parameters, the matrix A defined by

$$A = \begin{bmatrix} \phi(p,q,r,u,v,w) & \phi(q,r,p,v,w,u) & \phi(r,p,q,w,u,v) \\ p & q & r \\ u & v & w \end{bmatrix}, \tag{3.3}$$

satisfies the conditions,

$$\det A = k, \quad \text{and} \quad \det(A^{(3)}) = k^3, \tag{3.4}$$

where

$$\begin{aligned}k &= pqr(pv - qu)(pw - ru)(qw - rv)(p^2v^2 + pquv + q^2u^2) \\ &\times (p^2w^2 + pruw + r^2u^2)(q^2w^2 + qrvw + r^2v^2)(pqw + prv + qru).\end{aligned} \tag{3.5}$$

P r o o f. We begin with the matrix A defined by

$$A = \begin{bmatrix} x & y & z \\ p & q & r \\ u & v & w \end{bmatrix}, \tag{3.6}$$

when Eqs. (3.4) may be written as follows:

$$(qw - rv)x + (ru - pw)y + (pv - qu)z = k, \tag{3.7}$$

$$(q^3w^3 - r^3v^3)x^3 + (r^3u^3 - p^3w^3)y^3 + (p^3v^3 - q^3u^3)z^3 = k^3. \tag{3.8}$$

On eliminating k from Eqs. (3.7) and (3.8), we get,

$$\begin{aligned}(q^3w^3 - r^3v^3)x^3 + (r^3u^3 - p^3w^3)y^3 + (p^3v^3 - q^3u^3)z^3 \\ - ((qw - rv)x + (ru - pw)y + (pv - qu)z)^3 = 0.\end{aligned} \tag{3.9}$$

We note that when $(x, y, z) = (p, q, r)$, both $\det A$ and $\det (A^{(3)})$ vanish, and hence $(x, y, z) = (p, q, r)$ is a solution of Eq. (3.9). Similarly, $(x, y, z) = (u, v, w)$ is also a solution of Eq. (3.9).

Equation (3.9) is a homogeneous cubic equation in the variables x, y and z, and accordingly, we may consider it as an elliptic curve in the projective plane $\mathbb{P}^2$ with two known points on the curve being $P_1 = (p, q, r)$ and $P_2 = (u, v, w)$. If we draw a line joining the points P_1 and P_2 to intersect the elliptic curve (3.9) in a third rational point, say (x_1, y_1, z_1), and take $(x, y, z) = (x_1, y_1, z_1)$, the left-hand side of both Eqs. (3.7) and (3.8) becomes 0, and we do not get a nonzero value of k as desired. Accordingly, we draw a tangent at the point P_1 to intersect the elliptic curve in a point P_3 whose coordinates are as follows:

$$x = \phi(p, q, r, u, v, w),\ y = \phi(q, r, p, v, w, u),\ z = \phi(r, p, q, w, u, v), \tag{3.10}$$

where $\phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is defined by (3.2).

The values of x, y, z, given by (3.10) satisfy Eq. (3.9), and further, the value of k obtained from Eq. (3.7) is given by (3.5). On substituting the values of x, y, z, given by (3.10) in (3.6), we get the matrix A, defined by (3.3), that satisfies the conditions (3.4) with the value of k being given by (3.5) as stated in the theorem. □

We note that in any numerical example of the matrix A, if the greatest common divisor g of the entries of any row (or column) of the matrix A is > 1, then g is also a factor of $\det A$, and we can divide the entries of that row (or column) by g to get a numerically smaller example of a matrix satisfying the specified conditions. For instance, when $(p, q, r, u, v, w) = (2, -3, 3, 3, -2, 4)$, we get a matrix which, on factoring out the greatest common divisor of the entries of the first row, yields the matrix,

$$A = \begin{bmatrix} -57797 & -109147 & -22789 \\ 2 & -3 & 3 \\ 3 & -2 & 4 \end{bmatrix},$$

for which $\det A = 123690$ and $\det(A^{(3)}) = 123690^3$.

In Theorem 3.1 $k \neq \pm 1$ and all a_{ij} can be chosen different from 0. It would be interesting to find such examples also for $k = 1$.

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Ajai Choudhry
13/4 A Clay Square,
Lucknow – 226001, India
e-mail: ajaic203@yahoo.com

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