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UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS CONCERNING THEIR SHIFT AND DIFFERENTIAL POLYNOMIAL

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Abstract. In this paper, we investigate the uniqueness of meromorphic functions by considering their shift, q-difference and differential polynomial. We obtain some results which extend and generalize the results given by Chao Meng and Gang Liu [9].

1. Introduction and main results. We assume that the reader is well aware of the standard notations and definitions used in the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ etc. The reader can refer ([5], [13], [14]). We shall denote by $S(r,f)$, any quantity which satisfies $S(r,f) = o(T(r,f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure. Throughout the paper a meromorphic function always means a non-constant meromorphic function in the open complex plane and a constant always means a complex valued constant, unless otherwise mentioned.

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Let f and g be two non-constant meromorphic functions in the open complex plane. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^+ \cup \{\infty\}$ the set, $E(a,f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where each a-point of f with multiplicity k is counted k times in the set and the set, $\overline{E}(a,f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where the multiplicities are ignored. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity) and we have $E(a,f) = E(a,g)$; Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity) and we will have $\overline{E}(a,f) = \overline{E}(a,g)$.

We need the following definitions and notations.

Definition 1.1 ([6]). *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, then we say that f and g share the value a with the weight k.*

The definition implies that, if f and g share a value a with the weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a,k) to mean that, f, g share the value a with the weight k. Clearly, if f, g share (a,k), then f, g share (a,p) for any integer p, such that, $0 \leq p < k$. Also we note that, f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.2 ([8]). *For two meromorphic functions f, g and for a, $b \in \mathbb{C} \cup \{\infty\}$ and for a positive integer k,*

(i) $N_{(k}(r,a;f)\,(\overline{N}_{(k}(r,a;f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k,*

(ii) $N(r,a;f|g=b)\,(\overline{N}(r,a;f|g=b))$ *denotes the counting function (reduced counting function) of those a-points of f which are the b-points of g,*

(iii) $N(r,a,f|g\neq b)\,(\overline{N}(r,a;f|g\neq b))$ *denotes the counting function (reduced counting function) of those a-points of f which are not the b-points of g,*

(iv) $N_p(r,a;f) = \overline{N}(r,a;f) + \sum_{k=2}^{p} \overline{N}_{(k}(r,a;f)$,

(*v*) $N_2(r, a; f|g = b)\,(N_2(r, a; f|g \neq b))$ *denotes the counting function of those a-points of f which are (are not) the b-points of g, where an a-point of f with multiplicity m is counted m times if* $m \leq 2$ *and twice if* $m > 2$,

(*vi*) $N_{k)}(r, a; f)\,(\overline{N}_{k)}(r, a; f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k.*

Definition 1.3 ([1]). *Let f and g be two meromorphic functions such that f and g share the value 1 IM. Let* z_0 *be a 1-point of f of order p, and a 1-point of g of order q. We denote, by* $\overline{N}_L\left(r, \dfrac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $q < p$, *by* $\overline{N}_E^{(2}\left(r, \dfrac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $2 \leq q = p$, *by* $\overline{N}_E^{1)}\left(r, \dfrac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $p = q = 1$, *and by* $\overline{N}_{f>2}\left(r, \dfrac{1}{g-1}\right)$ *the counting functions of those 1-points of f and g such that* $p > q = 2$, *each point in these counting functions is counted only once. In the same way, we can define* $\overline{N}_L\left(r, \dfrac{1}{g-1}\right)$, $\overline{N}_E^{(2}\left(r, \dfrac{1}{g-1}\right)$, $\overline{N}_{g>2}\left(r, \dfrac{1}{f-1}\right)$.

Definition 1.4 ([7]). *Let* $n_{0j}, n_{1j}, \ldots, n_{kj}$ *be non-negative integers. The expression,*

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}},$$

is called a differential monomial generated by f of degree $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ *and weight* $\Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij}$. *The sum,*

$$P[f] = \sum_{i=1}^{l} b_j M_j[f],$$

is called a differential polynomial generated by f of degree $\gamma_p = \max\{\gamma_{M_j} : 1 \leq j \leq l\}$ *and weight* $\Gamma_p = \max\{\Gamma_{M_j} : 1 \leq j \leq l\}$, *where* $T(r, b_j) = S(r, f)$ *for* $j = 1, 2, \ldots, l$.

The numbers, $\underline{\gamma}_p = \min\{\gamma_{M_j} : 1 \leq j \leq l\}$ and k (the highest order of the derivative of f in $P[f]$) are called, respectively, the lower degree and order

of $P[f]$. $P[f]$ is said to be homogeneous if $\gamma_p = \underline{\gamma}_p$. Also $P[f]$ is called a quasi differential polynomial generated by f if instead of assuming $T(r, b_j) = S(r, f)$ we just assume that, $m(r, b_j) = S(r, f)$ for the coefficients $b_j (j = 1, 2, \dots, l)$.
We denote, by $\sigma = \max\{\Gamma_{M_j} - \gamma_{M_j} : 1 \le j \le l\} = \max\{n_{1j} + 2n_{2j} + \dots + kn_{kj} : 1 \le j \le l\}$.

Nevanlinna had proved that, if two non-constant entire functions f and g on the complex plane share four distinct finite values (ignoring multiplicity), then $f = g$ and this number four cannot be reduced [11, page 1]. But in 1976, Rubel and Yang [11] considered a special case by taking g as the first derivative of f and obtained the following result.

Theorem A ([11])**.** *If f is a non-constant entire function in the finite complex plane and if f and f' share two distinct values (counting multiplicity), then $f'=f$.*

Thus Rubel and Yang [11] showed that a derivative is worth two values. Since then the study of uniqueness of meromorphic functions sharing values with their derivatives became a subject of much interest.

In 2018, Qi et al. [10] by considering the shifts, studied the value sharing problem related to $f'(z)$ and $f(z + c)$, where c is a finite complex number and they obtained the following result.

Theorem B ([10])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order and $n \ge 9$ be an integer. If $[f'(z)]^n$ and $f^n(z + c)$ share $a(\ne 0)$ and ∞ CM, then $f'(z) = tf(z + c)$, for a constant t, that satisfies $t^n = 1$.*

In 2020, Meng and Liu [9] extended the above result by considering the k^{th} derivative of f and obtained the following results.

Theorem C ([9])**.** *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

(I) $[f^{(k)}(z)]^n$ *and* $f^n(z + c)$ *share* $(1, 2)$, $(\infty, 0)$ *and* $n \ge 2k + 8$*;*

(II) $[f^{(k)}(z)]^n$ *and* $f^n(z + c)$ *share* $(1, 2)$, (∞, ∞) *and* $n \ge 2k + 7$*;*

(III) $[f^{(k)}(z)]^n$ *and* $f^n(z + c)$ *share* $(1, 0)$, $(\infty, 0)$ *and* $n \ge 3k + 14$*;*

then $f^{(k)}(z) = tf(z + c)$, for a constant t, that satisfies $t^n = 1$.

Corollary C ([9])**.** *Let f be a non-constant entire function of finite order and $n \ge 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(z + c)$ share $(1, 2)$, then $f^{(k)}(z) = tf(z + c)$, for a constant t, that satisfies $t^n = 1$.*

Meng and Liu [9] further studied the same problem by replacing the shifts $f(z+c)$ by q-difference, *i.e.*, $f(qz)$ and obtained the following results.

Theorem D ([9]). *Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

(I) $[f^{(k)}(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$, $(\infty,0)$ *and* $n \geq 2k+8$*;*

(II) $[f^{(k)}(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$, (∞,∞) *and* $n \geq 2k+7$*;*

(III) $[f^{(k)}(z)]^n$ *and* $f^n(qz)$ *share* $(1,0)$, $(\infty,0)$ *and* $n \geq 3k+14$*;*

then $f^{(k)}(z) = tf(qz)$, *for a constant* t, *that satisfies* $t^n = 1$.

Corollary D ([9]). *Let f be a non-constant entire function of finite order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, then $f^{(k)}(z) = tf(qz)$, for a constant t, that satisfies $t^n = 1$.*

Question. Since $(f^{(k)})^n$ is nothing, but a differential monomial generated by f, it is natural to ask whether $(f^{(k)})^n$ can be extended to a differential polynomial $P[f]$ generated by f?

In this paper we give an affirmative answer to the above question and extend and improve the above mentioned theorems (A - D) and their corollaries to more generalized form. Thus, by investigating the uniqueness of meromorphic functions of the form $f^n(z+c)$, $f^n(qz)$, and $\{P[f]\}^n$ we obtain the following results.

Theorem 1.1. *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ *and* $f^n(z+c)$ *share* $(1,2)$ *and* ∞ *IM, and* $n \geq 2\gamma_p + 2\sigma + 6$*;*

(II) $\{P[f]\}^n$ *and* $f^n(z+c)$ *share* $(1,2)$ *and* ∞ *CM, and* $n \geq 2\gamma_p + 2\sigma + 5$*;*

(III) $\{P[f]\}^n$ *and* $f^n(z+c)$ *share* $(1,0)$ *and* ∞ *IM, and* $n \geq 3\gamma_p + 3\sigma + 11$*;*

then $P[f] = tf(z+c)$, *for a constant* t, *such that* $t^n = 1$.

Corollary 1.1. *Let f be a non-constant entire function of finite order and $n \geq 2\gamma_p + 3$ be an integer. If $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$, then $\{P[f]\} = tf(z+c)$, for a constant t, such that $t^n = 1$.*

Theorem 1.2. *Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,2)$ *and* ∞ *IM, and* $n \geq 2\gamma_p + 2\sigma + 6$*;*

(II) $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,2)$ *and* ∞ *CM, and* $n \geq 2\gamma_p + 2\sigma + 5$*;*

(III) $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,0)$ *and* ∞ *IM, and* $n \geq 3\gamma_p + 3\sigma + 11$*;*

then $P[f] = tf(qz)$*, for a constant* t*, such that* $t^n = 1$*.*

Corollary 1.2. *Let* f *be a non-constant entire function of zero order and* $n \geq 2\gamma_p + 3$ *be an integer. If* $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,2)$*, then* $\{P[f]\} = tf(qz)$*, for a constant* t*, such that* $t^n = 1$*.*

Remarks. If suppose $P[f] = f^{(k)}$, then we get, $(\gamma_p = \gamma_{M_1} = 1)$, $(\Gamma_p = \Gamma_{M_1} = (k+1))$, $(\sigma = \Gamma_{M_1} - \gamma_{M_1} = k)$ and thus,

(i). Theorem 1.1 reduces to Theorem C,

(ii). Corollary 1.1 reduces to Corollary C,

(iii). Theorem 1.2 reduces to Theorem D,

(iv). Corollary 1.2 reduces to Corollary D.

Example 1.1. Let $f(z) = \sin(z)$, $c = \frac{\pi}{2}$, $n = 16$, $P[f] = ff^{(1)} + f^{(1)} + f^{(1)}f^{(2)}$, then, we get $\underline{\gamma}_p = 1$, $\gamma_p = 2$, $\Gamma_p = 5$ and $\sigma = 3$. Since $\{f(z+c)\}^n$ and $\{P[f]\}^n$ share $1, \infty$ CM and hence they share $(1,2)$ and ∞ CM, and as $n \geq 15$, we get $P[f] = tf(z+c)$, where $t^n = 1$.

Example 1.2. Let $f(z) = \sin^2(z)$, $c = 2\pi$, $n = 16$, $P[f] = f + 4f^{(1)} + 4f^{(2)} + f^{(3)} + f^{(4)}$, then, we get $\underline{\gamma}_p = \gamma_p = 1$, $\Gamma_p = 5$ and $\sigma = 4$. Since $\{f(z+c)\}^n$ and $\{P[f]\}^n$ share $1, \infty$ CM and hence they share $(1,2)$ and ∞ CM, and as $n \geq 15$, we get $P[f] = tf(z+c)$, where $t^n = 1$.

Example 1.3. Let $f(z) = e^z - 1$, $c = 0$, $n = 12$, $P[f] = f^{(2)} - f^{(1)} + f$, then, we get $\underline{\gamma}_p = \gamma_p = 1$, $\Gamma_p = 3$ and $\sigma = 2$. Since $\{f(z+c)\}^n$ and $\{P[f]\}^n$ share $1, \infty$ CM and hence they share $(1,2)$ and ∞ CM, and as $n \geq 11$, we get $P[f] = tf(z+c)$, where $t^n = 1$.

From the above examples it is clear that, the condition for 'n' in the above theorems and their corollaries are sufficient, but not necessary.

2. Lemmas. This section provides all the necessary lemmas used in the sequel.

Let F and G be two non-constant meromorphic functions, we shall denote by Ψ the following function.

$$\Psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 ([2]). *Let F, G be two non-constant meromorphic functions. If F, G share $(1,2)$ and (∞,k), where $0 \le k \le \infty$ and $\Psi \not\equiv 0$, then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G),$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.2 ([12]). *Let f be a non-constant meromorphic function and let $a_1, a_2, \ldots, a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.3 ([4]). *Let f be a meromorphic function of finite order $\rho(f)$ and let c be a non-zero complex constant. Then*

$$T(r,f(z+c)) = T(r,f(z)) + O(r^{\rho(f)-1+\epsilon}) + O(\log r),$$

for an arbitrary $\epsilon > 0$.

Lemma 2.4 ([15]). *Suppose that two non-constant meromorphic functions F and G share 1 and ∞ IM. Let Ψ be given as in (2.1). If $\Psi \not\equiv 0$, then*

$$T(r,F) + T(r,G) \le 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) + 2N_E^{(2}\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G).$$

Lemma 2.5 ([16]). *Let f be a zero order meromorphic function and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$T(r,f(qz)) = (1+O(1))T(r,f(z)),$$
$$N(r,f(qz)) = (1+O(1))T(r,f(z)),$$

on a set of lower logarithmic density 1.

Lemma 2.6 ([3])**.** *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial in f. Then*

$$m\left(r,\frac{P[f]}{f^{\gamma_p}}\right)\le(\gamma_p-\underline{\gamma}_p)m\left(r,\frac{1}{f}\right)+S(r,f),$$

$$m\left(r,\frac{P[f]}{f^{\underline{\gamma}_p}}\right)\le(\gamma_p-\underline{\gamma}_p)m\left(r,f\right)+S(r,f),$$

$$N\left(r,\frac{P[f]}{f^{\gamma_p}}\right)\le(\gamma_p-\underline{\gamma}_p)N\left(r,\frac{1}{f}\right)+\sigma\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f),$$

$$N\left(r,P[f]\right)\le\gamma_pN(r,f)+\sigma\overline{N}(r,f)+S(r,f),$$

$$T(r,P[f])\le\gamma_pT(r,f)+\sigma\overline{N}(r,f)+S(r,f),$$

where $\sigma=\max\{n_{1j}+2n_{2j}+3n_{3j}+\cdots+kn_{kj};\ 1\le j\le l\}$.

Lemma 2.7. *Let f be a non-constant meromorphic function and n be a positive integer, if $F=f^n(z+c)$, where c is a finite complex number and $G=\{P[f]\}^n$, where $P[f]$ is a differential polynomial, then $F\cdot G\ne 1$.*

Proof. On the contrary, suppose $F\cdot G=1$,

$$i.e.,\ \ f^n(z+c)\{P[f]\}^n=1.$$

From the above, it is clear that the function f can't have any zeros and poles. Therefore

$$\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)=\overline{N}(r,f).$$

So by the First fundamental theorem of Nevanlinna and Lemma 2.6, we have,

$$\begin{aligned}(n+n\gamma_p)T(r,f)&=T\left(r,\frac{1}{f^n(z+c)f^{n\gamma_p}}\right)+S(r,f)\\&\le T\left(r,\frac{\{P[f]\}^n}{f^{n\gamma_p}}\right)+S(r,f)\\&\le nT\left(r,\frac{\{P[f]\}}{f^{\gamma_p}}\right)+S(r,f)\\&\le n\left[m\left(r,\frac{P[f]}{f^{\gamma_p}}\right)+N\left(r,\frac{P[f]}{f^{\gamma_p}}\right)\right]+S(r,f)\\&\le n\left[(\gamma_p-\underline{\gamma}_p)T(r,f)\right]+n\sigma\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)\end{aligned}$$

$$(n + n\gamma_p)T(r, f) \le n\left[(\gamma_p - \underline{\gamma}_p)T(r, f)\right] + S(r, f)$$

$$(n + n\underline{\gamma}_p)T(r, f) \le S(r, f),$$

which is a contradiction. Thus $F \cdot G \neq 1$. This completes the proof of the Lemma 2.7. □

In a similar manner we can prove the following Lemma.

Lemma 2.8. *Let f be a non-constant meromorphic function and n be a positive integer, if $F = f^n(qz)$, where c is a finite complex number and $G = \{P[f]\}^n$, where $P[f]$ is a differential polynomial, then $F \cdot G \neq 1$.*

3. Proof of Theorems.

3.1. Proof of Theorem 1.1.

Proof. Let us consider,

$$F = f^n(z + c) \quad \text{and} \quad G = \{P[f]\}^n. \tag{3.1}$$

(I). Suppose $\{P[f]\}^n$ and $f^n(z + c)$ share $(1, 2)$ & ∞ IM, and $n \ge 2\gamma_p + 2\sigma + 6$. Then it follows directly from the assumptions of the theorem, that F and G share $(1, 2)$ & ∞ IM. Let Ψ be defined as in (2.1). It can be easily seen that the possible poles of Ψ occur at,

(i) multiple zeros of F and G

(ii) those 1-points of F and G whose multiplicities are different

(iii) those poles of F and G whose multiplicities are different

(iv) zeros of F' and G' which are not the zeros of $F(F - 1)$ and $G(G - 1)$ respectively.

We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then it follows from Lemma 2.1 that,

$$\begin{aligned} T(r, F) \le\ & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) \\ & + \overline{N}(r, G) + \overline{N_*}(r, \infty; F, G) + S(r, F) + S(r, G). \end{aligned} \tag{3.2}$$

Using Lemma 2.2 and Lemma 2.3, we can write

$$T(r, F) = nT(r, f(z + c)) + S(r, f) \tag{3.3}$$

$$= nT(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + S(r, f). \tag{3.4}$$

We obviously have the following,

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f(z+c)}\right) &\le 2T(r, f(z+c)) + S(r, f) \\ &\le 2T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f), \end{aligned} \tag{3.5}$$

$$\overline{N}(r, F) = \overline{N}(r, f(z+c)) \le T(r, f(z+c)) \le T(r, f) + O(r^{\rho(f)-1+\epsilon}), \tag{3.6}$$

$$\begin{aligned} \overline{N}_*(r, \infty; F, G) \le \overline{N}(r, F) \le T(r, f(z+c)) \\ \le T(r, f) + O(r^{\rho(f)-1+\epsilon}) + S(r, f). \end{aligned} \tag{3.7}$$

Since $\overline{E}(\infty, f^{(k)}) = \overline{E}(\infty, f)$, we have

$$\overline{N}(r, G) = \overline{N}(r, P[f]) = \overline{N}(r, f). \tag{3.8}$$

From Lemma 2.6, we have

$$\begin{aligned} N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{P[f]}\right) &\le 2T(r, P[f]) + S(r, f) \\ &\le 2(\gamma_p + \sigma)T(r, f) + S(r, f). \end{aligned} \tag{3.9}$$

By combining (3.2) to (3.9), we deduce that

$$(n - 2\gamma_p - 2\sigma - 5)T(r, f) \le S(r, f), \tag{3.10}$$

which contradicts that $n \ge 2\gamma_p + 2\sigma + 6$. Thus, we have $\Psi = 0$ and hence

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By integrating the above equation twice on both sides, we get

$$\frac{1}{F-1} = \frac{C}{G-1} + D, \tag{3.11}$$

where $C \neq 0$ and D are constants. From (3.11), we have

$$G = \frac{(D-C)F + (C-D-1)}{DF - (D+1)}. \tag{3.12}$$

Now, we have the following three cases:
Case 1. Suppose that $D \neq 0, -1$. Then, from (3.12), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{D+1}{D}}\right) = \overline{N}(r, G). \tag{3.13}$$

From the Second fundamental theorem, Lemma 2.3 and (3.8), we have

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{D+1}{D}}\right) + S(r,f) \\ &\leq 3T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{3.14}$$

which contradicts $n \geq 2\gamma_p + 2\sigma + 6$.
Case 2. Suppose $D = -1$. From (3.12), we have

$$G = \frac{(C+1)F - C}{F}. \tag{3.15}$$

(i) If $C \neq -1$, from (3.15), we get

$$\overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{3.16}$$

From Second fundamental theorem, Lemma 2.3 and Lemma 2.6, we get

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) + S(r,f) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\ &\leq (\gamma_p + \sigma + 2)T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{3.17}$$

which contradicts with $n \geq 2\gamma_p + 2\sigma + 6$.
(ii) If $C = -1$, then from (3.15), we get $F \cdot G = 1$,
i.e.,

$$f^n(z+c)\{P[f]\}^n = 1, \tag{3.18}$$

which is a contradiction from Lemma 2.7.

Case 3. Suppose that $D = 0$. From (3.12), we have

$$G = CF - (C - 1). \tag{3.19}$$

If $C \neq 1$, from (3.19), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{3.20}$$

Then from the Second fundamental theorem, Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) + S(r,f) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}(r, \frac{1}{f(z+c)}) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\ &\leq (\gamma_p + \sigma + 2)T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{3.21}$$

which again contradicts $n \geq 2\gamma_p + 2\sigma + 6$. Hence, $C = 1$. Thus From (3.19), we have $F = G$, *i.e.*,

$$f^n(z+c) = \{P[f]\}^n.$$

Hence $P[f] = tf(z+c)$, for a constant t, such that $t^n = 1$. This proves (I) of Theorem 1.1.

(II). Suppose $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$ & ∞ CM and $n \geq 2\gamma_p + 2\sigma + 5$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$ & ∞ CM. Let Ψ be defined as in (2.1). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.1, we have

$$\begin{aligned} T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \\ \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{3.22}$$

It is obvious that,

$$\overline{N}_*(r,\infty;F,G) = 0. \tag{3.23}$$

By combining (3.22), (3.4), (3.5), (3.6), (3.8), (3.9), and (3.23), we have

$$(n-2\gamma_p-2\sigma-4)T(r,f)\leq O(r^{\rho(f)-1+\epsilon})+S(r,f), \tag{3.24}$$

which contradicts with, $n\geq 2\gamma_p+2\sigma+5$. Hence, we get $\Psi=0$. Now, by following the steps of the proof of (I) of the Theorem 1.1, we can get the proof of (II) of Theorem 1.1.

(III). Suppose $\{P[f]\}^n$ and $f^n(z+c)$ share $1,\infty$ CM and $n\geq 3\gamma_p+3\sigma+11$. Then it follows directly from the assumptions of the theorem, that F and G share $1,\infty$ CM. Let Ψ be defined as in (2.1). We claim $\Psi=0$, on the contrary if $\Psi\neq 0$, then from Lemma 2.4, we have

$$\begin{aligned}T(r,F)+T(r,G)\leq{}& 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\&+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)\\&+S(r,F)+S(r,G).\end{aligned}\tag{3.25}$$

Since,

$$\begin{aligned}N_E^{1)}\left(r,\frac{1}{F-1}\right)&+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)\\&\leq N\left(r,\frac{1}{G-1}\right)\leq T(r,G)+O(1).\end{aligned}\tag{3.26}$$

Therefore, from (3.25) and (3.26), we get

$$\begin{aligned}T(r,F)\leq{}& 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)\\&+N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).\end{aligned}\tag{3.27}$$

From Lemma 2.2 and Lemma 2.3, we have,

$$T(r,F)=nT(r,f(z+c))+S(r,f)=nT(r,f)+O(r^{\rho(f)-1+\epsilon})+S(r,f). \tag{3.28}$$

We obviously have,

$$
\begin{aligned}
N_L\left(r, \frac{1}{F-1}\right) \le N\left(r, \frac{F}{F'}\right) &\le N\left(r, \frac{F'}{F}\right) + S(r,f) \\
&\le \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,f) \\
&\le \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r,f) \\
&\le 2T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
N_L\left(r, \frac{1}{G-1}\right) \le N\left(r, \frac{G}{G'}\right) &\le N\left(r, \frac{G'}{G}\right) + S(r,f) \\
&\le \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\
&\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\
&\le (\gamma_p + \sigma + 1)T(r,f) + S(r,f). \qquad (3.30)
\end{aligned}
$$

By combining (3.27), (3.5), (3.6), (3.9), (3.28), (3.29), and (3.30), we get

$$(n - 3\gamma_p - 3\sigma - 10)T(r,f) \le O(r^{\rho(f)-1+\epsilon}) + S(r,f), \qquad (3.31)$$

which contradicts with $n \ge 3\gamma_p + 3\sigma + 11$. Now, by following the steps of the proof of (I) of Theorem 1.1, we can get the proof of (III) of Theorem 1.1.

This completes the proof of Theorem 1.1. □

3.2. Proof of Corollary 1.1.

Proof. Suppose $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$ and $n \ge 2\gamma_p + 3$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$. Let Ψ be defined as in (2.1). We claim $\Psi = 0$, on the contrary if $\Psi \ne 0$, then it follows from Lemma 2.1 that

$$
\begin{aligned}
T(r,F) \le N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) &+ \overline{N}(r,F) \\
&+ \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G).
\end{aligned}
$$

Since f is a non-constant entire function, we have $\overline{N}(r,f) = S(r,f)$.
Hence, the above equation reduces to

$$nT(r,f) \le 2T(r,f) + 2\gamma_p T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f).$$

Thus,

$$(n-2\gamma_p-2)T(r,f)\leq S(r,f), \tag{3.32}$$

which contradicts with $n\geq 2\gamma_p+3$. Now, by following the steps of the proof of (I) of Theorem 1.1, we can easily get the proof of Corollary 1.1. □

3.3. Proof of Theorem 1.2.

P r o o f. Let us consider,

$$F=f^n(qz) \text{ and } G=\{P[f]\}^n \tag{3.33}$$

(I). Suppose $\{P[f]\}^n$ and $f^n(qz)$ share $(1,2)$ & ∞ IM and $n\geq 2\gamma_p+2\sigma+6$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ & ∞ IM. We claim $\Psi=0$, on the contrary if $\Psi\neq 0$, then it follows from Lemma 2.1 that,

$$\begin{aligned} T(r,F)\leq{}& N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)\\ &+\overline{N}(r,G)+\overline{N_*}(r,\infty;F,G)+S(r,F)+S(r,G). \end{aligned} \tag{3.34}$$

Using Lemma 2.2 and Lemma 2.5, we can write

$$T(r,F)=nT(r,f(qz))+S(r,f)=nT(r,f)+S(r,f), \tag{3.35}$$

$$\overline{N}(r,F)=\overline{N}(r,f(qz))=\overline{N}(r,f(z))+S(r,f)\leq T(r,f)+S(r,f). \tag{3.36}$$

We obviously have the following,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{f(qz)}\right)\\ &\leq 2T(r,f(qz))+S(r,f)\leq 2T(r,f)+S(r,f), \end{aligned} \tag{3.37}$$

$$\overline{N}_*(r,\infty;F,G)\leq\overline{N}(r,F)\leq T(r,f(qz))\leq T(r,f)+S(r,f). \tag{3.38}$$

Since $\overline{E}(\infty,f^{(k)})=\overline{E}(\infty,f)$, we have

$$\overline{N}(r,G)=\overline{N}(r,P[f])=\overline{N}(r,f). \tag{3.39}$$

From Lemma 2.6, we have

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{P[f]}\right) \le 2T(r,P[f])+S(r,f) \le 2(\gamma_p+\sigma)T(r,f)+S(r,f). \tag{3.40}$$

By combining (3.34) to (3.40), we deduce,

$$(n-2\gamma_p-2\sigma-5)T(r,f)\le S(r,f), \tag{3.41}$$

which contradicts that $n\ge 2\gamma_p+2\sigma+6$. Thus, we have $\Psi=0$ and hence,

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}.$$

By integrating the above equation twice on both sides, we get

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{3.42}$$

where $C\neq 0$ and D are constants. From (3.42), we have

$$G=\frac{(D-C)F+(C-D-1)}{DF-(D+1)}. \tag{3.43}$$

Now, we have the following three cases:

Case 1. Suppose that $D\neq 0,-1$. Then, from (3.43), we have

$$\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)=\overline{N}(r,G). \tag{3.44}$$

From the second fundamental theorem, Lemma 2.5 and (3.8), we have

$$\begin{aligned} nT(r,f)&=T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f)\\ &\le 3T(r,f)+S(r,f), \end{aligned} \tag{3.45}$$

which contradicts with $n\ge 2\gamma_p+2\sigma+6$.

Case 2. Suppose $D=-1$. From (3.43), we have

$$G=\frac{(C+1)F-C}{F}. \tag{3.46}$$

(i) If $C \neq -1$, then from (3.46), we get

$$\overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{3.47}$$

From Second fundamental theorem, Lemma 2.5 and Lemma 2.6, we get

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) + S(r,f) \\ &\leq \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \end{aligned}$$

$$nT(r,f) \leq (\gamma_p + \sigma + 2)T(r,f) + S(r,f), \tag{3.48}$$

which contradicts with $n \geq 2\gamma_p + 2\sigma + 6$.
(ii) If $C = -1$ and from (3.46) we get $F \cdot G = 1$,
i.e.,

$$f^n(qz)\{P[f]\}^n = 1, \tag{3.49}$$

which is a contradiction from Lemma 2.8.
Case 3. Suppose that $D = 0$. From (3.43), we have

$$G = CF - (C - 1). \tag{3.50}$$

If $C \neq 1$, then from (3.46), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{3.51}$$

Then from the Second fundamental theorem, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}(r, \frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) + S(r,f) \\ &\leq \overline{N}(r, f(qz)) + \overline{N}(r, \frac{1}{f(qz)}) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f), \end{aligned}$$

i.e.,

$$nT(r,f) \leq (\gamma_p + \sigma + 2)T(r,f) + S(r,f), \tag{3.52}$$

which again contradicts $n \geq 2\gamma_p + 2\sigma + 6$. Hence, $C = 1$. Thus From (3.19), we have $F = G$,
i.e.,

$$f^n(qz) = \{P[f]\}^n.$$

Hence $P[f] = tf(qz)$, for a constant t, such that $t^n = 1$. This proves (I) of Theorem 1.2.

(II). Suppose $\{P[f]\}^n$ and $f^n(qz)$ share $(1, 2)$ & ∞ CM and $n \geq 2\gamma_p + 2\sigma + 5$. Then it follows directly from the assumptions of the theorem, that F and G share $(1, 2)$ & ∞ CM. Let Ψ be defined as in (2.1). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.1, we have

$$\begin{aligned} T(r,F) \leq{}& N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) \\ &+ \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{3.53}$$

It is obvious that,

$$\overline{N}_*(r,\infty;F,G) = 0. \tag{3.54}$$

By combining (3.53), (3.35), (3.36), (3.37), (3.39), (3.40) and (3.54), we have

$$(n - 2\gamma_p - 2\sigma - 4)T(r,f) \leq S(r,f), \tag{3.55}$$

which contradicts with, $n \geq 2\gamma_p + 2\sigma + 5$. Hence, we get, $\Psi = 0$. Now, by following the steps of the proof of (I) of the Theorem 1.2, we can get the proof of (II) of Theorem 1.2.

(III). Suppose $\{P[f]\}^n$ and $f^n(qz)$ share 1, ∞ CM and $n \geq 3\gamma_p + 3\sigma + l + 10$. Then it follows directly from the assumptions of the theorem, that F and G share 1, ∞ CM. Let Ψ be defined as in (2.1). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.4, we have

$$\begin{aligned} T(r,F) + T(r,G) \leq{}& 3\overline{N}(r,F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right) \\ &+ 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) \\ &+ S(r,F) + S(r,G). \end{aligned} \tag{3.56}$$

Since,

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right)$$

$$\leq N\left(r, \frac{1}{G-1}\right) \leq T(r,G) + O(1). \tag{3.57}$$

Therefore, from (3.56) and (3.57), we get

$$\begin{aligned} T(r,F) \leq 3\overline{N}(r,F) &+ N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2N_L\left(r, \frac{1}{F-1}\right) \\ &+ N_L\left(r, \frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned} \tag{3.58}$$

From Lemma 2.2 and Lemma 2.5, we have

$$T(r,F) = nT(r,f(qz)) + S(r,f) = nT(r,f) + S(r,f). \tag{3.59}$$

We obviously have,

$$\begin{aligned} N_L\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{F}{F'}\right) &\leq N\left(r, \frac{F'}{F}\right) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,f) \\ &\leq \overline{N}(r,f(qz)) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r,f) \\ &\leq 2T(r,f) + S(r,f), \end{aligned} \tag{3.60}$$

$$\begin{aligned} N_L\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{G}{G'}\right) &\leq N\left(r, \frac{G'}{G}\right) + S(r,f) \\ &\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\ &\leq (\gamma_p + \sigma + 1)T(r,f) + S(r,f). \end{aligned} \tag{3.61}$$

By combining (3.58),(3.35), (3.36), (3.37), (3.40), (3.60) and (3.61), we get

$$(n - 3\gamma_p - 3\sigma - 10)T(r,f) \leq S(r,f), \tag{3.62}$$

which contradicts with $n \geq 3\gamma_p + 3\sigma + 11$. Now, by following the steps of the proof of (I) of Theorem 1.2, we can get the proof of (III) of Theorem 1.2.

This completes the proof of Theorem 1.2. □

3.4. Proof of Corollary 1.2.

Proof. Suppose $\{P[f]\}^n$ and $f^n(qz)$ share $(1,2)$ and $n \geq 2\gamma_p + 3$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$. Let Ψ be defined as in (2.1). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then it follows from Lemma 2.1 that

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G).$$

Since f is a non-constant entire function, we have $\overline{N}(r,f) = S(r,f)$. Hence, the above equation reduces to,

$$nT(r,f) \leq 2T(r,f) + 2\gamma_p T(r,f) + S(r,f),$$

Thus,

$$(n - 2\gamma_p - 2)T(r,f) \leq S(r,f), \tag{3.63}$$

which contradicts with $n \geq 2\gamma_p + 3$. Now, by following the steps of the proof of (I) of Theorem 1.2, we will easily get the proof of Corollary 1.2. □

Open questions:

1. Is the condition for 'n', sharp in Theorem 1.1 and Theorem 1.2 ?
2. Can $f(z+c)$ be replaced by $\Delta_c^n f$ in Theorem 1.1 and Theorem 1.2, where $\Delta_c^n f = \sum_{r=0}^{n-1}(-1)^r \binom{n}{r} f(z+(n-r)c)$?

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