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ON ALGEBROID FUNCTIONS THAT SHARE ONE FINITE VALUE WITH THEIR DERIVATIVE ON ANNULI

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Abstract. In this paper, we discuss the algebroid functions $W(z)$ and $W'(z)$ that share the value 1 CM (counting multiplicities) and share one finite value DM (different multiplicities) with derivative on annuli.

1. Introduction. The value distribution theory of meromorphic functions was extended to the corresponding theory of algebroid functions by Ullrich E [47] and Valiron G [48] around 1930, and some important results on uniqueness for algebroid functions have been obtained (see [16, 41, 43]). It is well known to us that Valiron obtained a famaous $4\nu+1$-valued theorem. Al-Khaladi proved some interesting results on uniqueness of meromorphic functions that share one value with their derivative and one finite value DM (different multiplicities) with first derivatives. In 2013, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna theory for meromorphic functions on annuli (see [32, 33]). In 2009, Cao and Yi [34] investigated the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Yang Tan [37], Yang Tan and Yue Wang [36]

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proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli and also others have proved several results for algebroid functions on annuli [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 45, 46, 51, 50, 52]. Thus it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [35] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A} = \mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

2. Basic notations and definitions. We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [14, 15, 44, 49]).

Let $A_v(z), A_{v-1}(z), \ldots, A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0. \tag{2.1}$$

Then irreducible equation (2.1) defines a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

Let $W(z)$ be a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, we use the following notations

$$m(r, W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r, W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t, W)}{t}dt, \qquad N_2(r, W) = \frac{1}{\nu}\int_1^r \frac{n_2(t, W)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t, \frac{1}{W-a}\right)}{t}dt, \ \ \overline{N}_2\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r\frac{\overline{n}_2\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r,W)=m(r,W)+m\left(\frac{1}{r},W\right)-2m(1,W),\ \ N_0(r,W)=N_1(r,W)+N_2(r,W),$$

$$\overline{N}_0\left(r,\frac{1}{W-a}\right)=\overline{N}_1\left(r,\frac{1}{W-a}\right)+\overline{N}_2\left(r,\frac{1}{W-a}\right),$$

where $w_j(z)(j=1,2,\dots,\nu)$ is one valued branch of $W(z)$, $n_1(t,W)$ is the counting function of poles of the function $W(z)$ in $\{z:t<|z|\le 1\}$ and $n_2(t,W)$ is the counting function of poles of the function $W(z)$ in $\{z:1<|z|\le t\}$ (both counting multiplicity). $\overline{n}_1\left(t,\frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z:t<|z|\le 1\}$ and $\overline{n}_2\left(t,\frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z:1<|z|\le t\}$ (both ignoring multiplicity). $\overline{n}_1^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_1^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ (or $>k$) in $\{z:t<|z|\le 1\}$, each point count only once; $\overline{n}_2^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_2^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ (or $>k$) in $\{z:1<|z|\le t\}$, each point count only once, respectively.

Let $W(z)$ be a ν-valued algebroid function which determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, when $a\in\mathbb{C}$, $n_0\left(r,\frac{1}{W-a}\right)=n_0\left(r,\frac{1}{\psi(z,a)}\right)$, $N_0\left(r,\frac{1}{W-a}\right)=\frac{1}{\nu}N_0\left(r,\frac{1}{\psi(z,a)}\right)$. In particular, when $a=0$, $N_0\left(r,\frac{1}{W}\right)=\frac{1}{\nu}N_0\left(r,\frac{1}{A_0}\right)$. When $a=\infty$, $N_0\left(r,W\right)=\frac{1}{\nu}N_0\left(r,\frac{1}{A_v}\right)$; where $n_0\left(r,\frac{1}{W-a}\right)$ and $n_0\left(r,\frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z)-a$ and $\psi(z,a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, respectively.

Let $W(z)$ and $M(z)$ be ν-valued algebroid functions which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ share the finite value a IM (ignoring multiplicities), if $W(z)-a$ and $M(z)-a$ have the same zeros on annuli. If $W(z)-a$ and $M(z)-a$ have the same zeros with the same multiplicities, we say that $W(z)$ and $M(z)$ share the value a CM (counting multiplicities) on annuli. If

$W(z)-a$ and $M(z)-a$ have the same zeros with different multiplicities, we say that $W(z)$ and $M(z)$ share the value a DM (different multiplicities) on annuli.

Next, let k be a positive integer, we denote by $N_0^{k)}\left(r,\dfrac{1}{W-a}\right)$ is the counting function of zeros of $W(z)-a$ with multiplicity $\leq k$ and $N_0^{(k+1}\left(r,\dfrac{1}{W-a}\right)$ is the counting function of zeros of $W(z)-a$ with multiplicity $>k$. Definitions of the terms $N_0^{k)}$ and $N_0^{(k+1}$ can be similarly formulated. Finally $N_0^2\left(r,\dfrac{1}{W}\right)$ denotes the counting function of zeros of W where a zero of multiplicity k is counted with multiplicity $min\{k,2\}$.

Definition 2.1 ([36]). *Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\dfrac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, the function*

$$T_0(r,W)=m_0(r,W)+N_0(r,W),\qquad 1\leq r<R_0$$

is called Nevanlinna characteristic of $W(z)$.

3. Some lemmas.

Lemma 3.1 ([36] The first fundamental theorem on annuli). *Let $W(z)$ be ν-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\dfrac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, $a\in\mathbb{C}$*

$$m_0(r,a)+N_0(r,a)=T_0\left(r,W\right)+O(1).$$

Lemma 3.2 ([36] The second fundamental theorem on annuli). *Let $W(z)$ be v-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\dfrac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, a_k $(k=1,2,\ldots,p)$ are p distinct complex numbers (finite or infinite), then we have*

$$(p-2v)T_0\left(r,W\right)\leq\sum_{k=1}^{p}N_0\left(r,\frac{1}{W-a_k}\right)-N_1(r,W)+S_0(r,W)\tag{3.1}$$

$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, and

$$S_0(r,W)=m_0\left(r,\frac{W'}{W}\right)+\sum_{j=1}^{p}m_0\left(r,\frac{W'}{W-a_k}\right)+O(1).$$

The remainder of the second fundamental theorem is the following formula

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O(\log r),$$

outside a set of finite linear measure, if $r \to R_0 = +\infty$, *while*

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O\left(\log \frac{1}{R_0 - r}\right),$$

outside a set E *of* r *such that* $\int_E \frac{dr}{R_0 - r} < +\infty$, *when* $r \to R_0 < +\infty$.

Remark 3.1 ([36])**.** The second fundamental theorem on annuli has other forms, as the following:

$$(3.2)\quad (p-1)T_0\left(r,W\right) \leq N_0\left(r,W\right) + \sum_{k=1}^{p} N_0\left(r, \frac{1}{W-a_k}\right) - N_1(r) + Q_1(r,W),$$

$$N_1(r,W) = 2N_0(r,W) - N_0(r,W') + N_0\left(r, \frac{1}{W'}\right),$$

$$Q_1(r,W) = \sum_{k=0}^{p} m_0\left(r, \frac{W'}{W-a_k}\right) + O(1), a_0 = 0.$$

We notice that the following formula is true,

$$(3.3)\qquad \sum_{k=1}^{p} N_0\left(r, \frac{1}{W-a_k}\right) - N_1(r,W) \leq \sum_{k=1}^{p} \overline{N}_0\left(r, \frac{1}{W-a_k}\right).$$

$\overline{N}_0\left(r, \frac{1}{W-a_k}\right)$ is the reduced counting function of zeros(ignoring multiplicity). Then the second fundamental theorem can be rewritten as the following

$$(3.4)\qquad (p-2v)T_0\left(r,W\right) \leq \sum_{k=1}^{p} \overline{N}_0\left(r, \frac{1}{W-a_k}\right) + S_0(r,W).$$

Lemma 3.3 ([36])**.** *Let* $W(z)$ *be* ν*-valued algebroid function which is determined by* (2.1) *on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, *if the following conditions are satisfied*

$$\varliminf_{r\to\infty} \frac{T_0\left(r,W\right)}{\log r} < \infty, \quad R_0 = +\infty,$$

$$\varlimsup_{r\to R_0^-}\frac{T_0\,(r,W)}{\log\frac{1}{(R_0-r)}}<\infty,\quad R_0<+\infty,$$

then $W(z)$ is an algebraic function.

For the proof of our theorem we need the following lemmas

Lemma 3.4. *Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ such that $W'(z)$ is not constant and which satisfies $\overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0^{(2}(r,W)=S_0(r,W)$. Then either*

$$m_0\left(r,\frac{1}{W'-1}\right)=S_0(r,W)\ \text{and}\ N_0^{(2}\left(r,\frac{1}{W''}\right)\le N_0\left(r,\frac{1}{W'}\right)+S_0(r,W)\tag{3.5}$$

or

$$f(z)=\frac{c_2}{z+c_1}+c_3,\tag{3.6}$$

where c_1,$c_2(\neq 0)$ and c_3 are constants.

Proof. We consider the following function

$$G=\left(\frac{W''}{W'}\right)^2-2\nu\left(\frac{W''}{W'}\right)'.\tag{3.7}$$

Therefore

$$m_0(r,W)=S_0(r,W).\tag{3.8}$$

If z_∞ is a simple pole of W on annuli, then from (3.7) we see that G is holomorphic at z_∞. Thus

$$N_0(r,G)\le 2\nu\overline{N}_0^{(2}(r,W)+2\nu\overline{N}_0\left(r,\frac{1}{W'}\right)=S_0(r,W).\tag{3.9}$$

By hypothesis, combining (3.8) and (3.9) we find that

$$T_0(r,W)=S_0(r,W).\tag{3.10}$$

It follows from (3.7) that if z_0 is a zero of W'' of multiplicity $p(\ge 2)$ on annuli and $W'(z_0)\neq 0$, then

$$G(z)=O((z-z_0)^{p-1}).\tag{3.11}$$

If $G(z) \equiv 0$, we have from (3.7) that

$$2\nu \left(\frac{W''}{W'}\right)^{-2} \left(\frac{W''}{W'}\right)' = 1.$$

By integrating three times we conclude (3.6). We next suppose $G(z) \not\equiv 0$. By (3.11) and (3.10), we obtain

$$N_0^{(2}\left(r, \frac{1}{W''}\right) \leq 2\nu N_0\left(r, \frac{1}{G}\right) + N_0\left(r, \frac{1}{W'}\right) = N_0\left(r, \frac{1}{W'}\right) + S_0(r, W).$$

We rewrite (3.7) in the form

$$\frac{1}{W'-1} = \frac{1}{G}\left(\frac{W''}{W'-1} - \frac{W''}{W'}\right)\left((2\nu+1)\frac{W''}{W'} - 2\nu\frac{W'''}{W''}\right).$$

Obviously

$$m_0\left(r, \frac{1}{W'-1}\right) = S_0(r, W),$$

from the fundamental estimate and (3.10), so our lemma is proved. □

Lemma 3.5. *Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ satisfying $N_0\left(r, \frac{1}{W'}\right) + \overline{N}_0^{(2}(r, W) = S_0(r, W)$. If W and W' share the value 1 CM on annuli, then*

$$N_0^{(2}\left(r, \frac{1}{W''}\right) + m_0\left(r, \frac{1}{W'-1}\right) = S_0(r, W) \tag{3.12}$$

P r o o f. This is easy since if $W(z)$ and $W'(z)$ share 1 CM on annuli, then W' is not constant and (3.6) does not hold. Thus (3.12) follows from (3.5) and $N_0\left(r, \frac{1}{W'}\right) = s_0(r, W)$. □

Lemma 3.6. *Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ such that $W'(z)$ is not constant. If $W(z) = 1$ and $W'(z) = 1$ on annuli, then either*

$$N_0^{1)}\left(r, \frac{1}{W'-1}\right) \leq m_0(r, W) + N_0\left(r, \frac{1}{W'}\right) + S_0(r, W)$$

or $W(z)$ satisfies the identity (3.5).

Proof. We set

$$H = \frac{W''(W-1)}{W'(W-1)}. \tag{3.13}$$

Then it is clear that

$$m_0(r,H) \le m_0(r,W) + S_0(r,W). \tag{3.14}$$

From (3.13), we know that if z_∞ is a pole of W of multiplicity $p(\ge 1)$ on annuli, then

$$H(z_\infty) \ne \infty. \tag{3.15}$$

Let z_1 be a zero of $W'-1$ of multiplicity $q \ge 1$ on annuli. Since $W'(z) = 1$ implies that $W(z) = 1$ by assumption, we must have z_1 is a simple zero of $W-1$ on annuli. By simple calculation on the local expansion we see that

$$H(z_1) = q. \tag{3.16}$$

From (3.13), (3.15) and (3.16) it can be seen that the poles of H can only occure at the zeros of W' on annuli. That is,

$$N_0(r,H) \le \overline{N}_0\left(r, \frac{1}{W'}\right). \tag{3.17}$$

Further, if $H \not\equiv 1$, it follows from (3.16), (3.14) and (3.17) that

$$\begin{aligned} N_0^{1)}\left(r, \frac{1}{W'-1}\right) &\le N_0\left(r, \frac{1}{H-1}\right) \le T_0(r,H) + O(1) \\ &\le m_0(r,H) + N_0(r,H) + O(1) \\ &\le m_0(r,W) + \overline{N}_0\left(r, \frac{1}{W'}\right) + S_0(r,W) \end{aligned}$$

Finally, $H \equiv 1$, then

$$\frac{W''}{W'-1} = \frac{W'}{W-1}.$$

By integration, we get (3.5). □

Lemma 3.7. *Let $W(z)$ be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$ such that $W^{(k)}(z)$ is not constant and k is a positive integer. Then either*

$$(W^{(k+1)})^{k+1} = c(f^{(k)} - \lambda)^{k+2} \tag{3.18}$$

for some non zero constant c, or

$$kN_0^{1)}(r,W) \le \overline{N}^{(2}(r,W) + N_0^{1)}\left(r, \frac{1}{W^{(k)}-\lambda}\right) + \overline{N}_0\left(r, \frac{1}{W^{(k+1)}}\right) + S_0(r,W), \tag{3.19}$$

where λ is a constant.

Proof. Let

$$\Psi = (k+1)\frac{W^{(k+2)}}{W^{(k+1)}} - (k+2\nu)\frac{W^{(k+1)}}{W^{(k)}-\lambda}. \tag{3.20}$$

Suppose that z_∞ is a simple pole of W on annuli. Then an elementary calculation gives that $\Psi(z) = O((z-z_\infty)^k)$, which proves that z_∞ is a zero of Ψ of multiplicity k on annuli. Thus, if (3.18) is not true, i.e $\Psi \not\equiv 0$, then

$$kN_0^{1)}(r,W) \le N_0\left(r, \frac{1}{\Psi}\right) \le T_0(r,\Psi) + O(1). \tag{3.21}$$

Note that Ψ can only have simple poles at zeros of $W^{(k+1)}$ or $W^{(k)}-\lambda$ or multiple poles of W. Thus we deduce from (3.20) that

$$N_0(r,\Psi) \le \overline{N}_0^{2)}(r,W) + N_0^{1)}\left(r, \frac{1}{W^{(k)}-\lambda}\right) + \overline{N}_0\left(r, \frac{1}{W^{(k+1)}}\right). \tag{3.22}$$

Again from (3.20) we obtain $m_0(r,\Psi) = S_0(r,W)$. Combining (3.22) and (3.21), we get (3.19). □

Lemma 3.8. *Let $W(z)$ be ν-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$ such that $W^{(k)}(z)$ is not constant. Then either W is as in* (3.6) *or*

$$N_0^{1)}(r,W) \le \overline{N}^{(2}(r,W) + N_0^{1)}\left(r, \frac{1}{W'}\right) + \overline{N}_0\left(r, \frac{1}{W''}\right) + S_0(r,W). \tag{3.23}$$

Proof. Applying Lemma to $k=1$ and $\lambda=0$, we get either that (3.23) holds, or

$$\left(\frac{W''}{W'}\right)^2 = cW' \tag{3.24}$$

for some non zero constant c. Differentiating (3.24), we find that

$$2\nu W''\left(\frac{W''}{W'}\right)' = cW''W'. \tag{3.25}$$

If $W'' \equiv 0$, then W' is a constant. Therefore, $W'' \not\equiv 0$ and so by (3.25), we have $2\nu\left(\frac{W''}{W'}\right)' = cW'$. Combining this with (3.24) we get

$$2\nu\left(\frac{W''}{W'}\right)^{-2} = 1.$$

By integrating three times we conclude (3.6). □

4. Main results. Now, the main theorems of this paper are listed as follows

Theorem 4.1. *Let $W(z)$ be a ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ satisfying $N_0\left(r, \frac{1}{W'}\right) = S_0(r, W)$. Suppose that $W(z)$ and $W'(z)$ share the value 1 CM on annuli. Then*

$$W' - 1 = c(W - 1) \tag{4.1}$$

for some nonzero constants c.

Proof. The proof is by contradiction. Assume that (4.1) is not true. Therefore,

$$T_0(r, W) \leq 2\nu\overline{N}_0(r, W) + S_0(r, W). \tag{4.2}$$

We see from (4.2) that

$$N_0(r, W) = \overline{N}_0^{1)}(r, W) + S_0(r, W). \tag{4.3}$$

Let

$$F = 2\nu\left(\frac{W'}{W - 1} - \frac{W''}{W' - 1}\right) + \frac{W''}{W'}. \tag{4.4}$$

Therefore

$$m_0(r, F) = S_0(r, W). \tag{4.5}$$

If z_∞ is a simple pole of W on annuli, then from (4.4) we see that F will be holomorphic at z_∞. From this, (4.3), the hypothesis of Theorem 4.1 and (4.5) we can deduce that

$$T_0(r,F) = S_0(r,W). \tag{4.6}$$

Suppose that z_0 is a simple zero of W'' which is not a zero of W' on annuli. Since W and W' share 1 CM on annuli, this gives all zeros of $W'-1$ are simple on annuli. Hence $W'(z_0) \neq 1$, and thus

$$F(z_0) = 2\nu \left(\frac{W'(z_0)}{W(z_0)-1} \right) \tag{4.7}$$

from (4.4). Differentiating (4.4) and using $W''(z_0) = 0$, we arrive at

$$F'(z_0) = -2\nu \left(\left(\frac{W'(z_0)}{W(z_0)-1} \right)^2 + \frac{W'''(z_0)}{W'(z_0)-1} \right) + \frac{W'''(z_0)}{W'(z_0)}. \tag{4.8}$$

Combining (4.7) and (4.8) we obtain

$$-2\nu F'(z_0) = F^2(z_0) + \frac{4\nu W'''(z_0)}{W'(z_0)-1} - 2\nu \frac{W'''(z_0)}{W'(z_0)}. \tag{4.9}$$

On the other hand, by (3.7) we find that

$$G(z_0) = -2\nu \frac{W'''(z_0)}{W'(z_0)} \neq 0. \tag{4.10}$$

Substituting (4.10) into (4.9) gives

$$W'(z_0)[F^2(z_0) + 2\nu F'(z_0) - G(z_0)] = F^2(z_0) + 2\nu F'(z_0) + G(z_0). \tag{4.11}$$

If $F^2(z_0) + 2\nu F'(z_0) - G(z_0) = 0$, then from (4.11) we get $G(z_0) = 0$ which contradicts (4.10). Therefore $F^2(z_0) + 2\nu F'(z_0) - G(z_0) \neq 0$, and (4.11) reads

$$W'(z_0) = \frac{F^2(z_0) + 2\nu F'(z_0) + G(z_0)}{F^2(z_0) + 2\nu F'(z_0) - G(z_0)} = a(z_0).$$

By (4.6) and (3.10), it is easy to see that

$$T_0(r,a) = S_0(r,W). \tag{4.12}$$

This means that we the following property, if z_0 is a simple zero of W'' on annuli and $W'(z_0) \neq 0$, then

$$W'(z_0) = a(z_0). \tag{4.13}$$

Let z_1 be a zero of $W - 1$ on annuli. Then the Taylor expansion of W about z_1 on annuli is

$$W(z) - 1 = (z - z_1) + a_2(z - z_1)^2 + a_3(z - z_1)^3 + \cdots, a_2 \neq 0. \tag{4.14}$$

It follows from (4.4) and (4.3) that

$$F(z_1) = 4\nu a_2 - (2\nu + 1)\frac{a_3}{a_2} \quad \textit{and} \quad G(z_1) = 12\nu(a_2^2 - a_3).$$

That is

$$2\nu W''^2(z_1) - F(z_1)W''(z_1) - W'''(z_1) = 0$$

and

$$(2\nu + 1)W''^2(z_1) - 2\nu W'''(z_1) - G(z_1) = 0$$

and eliminating $W''^2(z_1)$ from the last two equations we obtain

$$W'''(z_1) - F(z_1)W''(z_1) + 2\nu G(z_1) = 0. \tag{4.15}$$

Now considering the following equation

$$J = \frac{W''' - (2\nu + 1)FW'' + 2\nu GW'}{W'(W' - 1)}. \tag{4.16}$$

From this, we know that if z_∞ is a simple pole of W on annuli, then J is holomorphic at z_∞ on annuli. Thus we deduce from (4.16), the hypothesis of Theorem 4.1, (4.15), (4.3), (4.6) and (3.10) that

$$N_0(r, J) = S_0(r, W). \tag{4.17}$$

Also, from (4.16), (4.6), (3.10) and Lemma 2, we conclude $m_0(r, J) = S_0(r, W)$. Combining this with (4.17) yields

$$T_0(r, J) = S_0(r, J). \tag{4.18}$$

Eliminating W''' between (4.16) and (3.7) leads to

$$2\nu JW'^2(W' - 1) = (2\nu + 1)W''^2 + (2\nu + 1)GW'^2 - 6\nu FW''W'. \tag{4.19}$$

From (4.18) and (4.13) we get

$$J(z_0) = \frac{(2\nu+1)G(z_0)}{2\nu[a(z_0)-1]}.$$

If $J \not\equiv \dfrac{(2\nu+1)G}{2\nu(a-1)}$, by $N_0\left(r, \dfrac{1}{W'}\right) = S_0(r,W)$, Lemma 2, (4.18), (4.12) and (3.10) we have

$$(4.20)\quad N_0\left(r, \frac{1}{W''}\right) \le N_0\left(r, \frac{1}{J - \frac{(2\nu+1)G}{2\nu(a-1)}}\right) + N_0\left(r, \frac{1}{W'}\right) + N_0^{(2}\left(r, \frac{1}{W''}\right) = S_0(r,W).$$

Thus (4.20), (3.6), Lemma 5, the assumptions of the theorem, and (4.2) imply that

$$T_0(r,W') = S_0(r,W).$$

Hence

$$\begin{aligned} T_0(r,W) &\le N_0\left(r, \frac{1}{W-1}\right) + m_0\left(r, \frac{1}{W-1}\right) \\ &\le N_0\left(r, \frac{1}{W'-1}\right) + m_0\left(r, \frac{1}{W'}\right) + S_0(r,W) \\ &\le 2\nu T_0(r,W') + S_0(r,W) = S_0(r,W). \end{aligned}$$

which is a contradiction. Therefore $J \equiv \dfrac{(2\nu+1)G}{2\nu(a-1)}$ and (4.19) becomes

$$(4.21)\qquad GW'^2(W'-a) = W''(W''-2\nu FW')(a-1).$$

Differentiating (4.21) and then using (4.13) we obtain

$$(4.22)\qquad a'(z_0)a(z_0)G(z_0) = 2\nu[a(z_0-1)]F(z_0)W'''(z_0),$$

we also note from (4.10) and (4.12) that

$$W'''(z_0) = -\frac{1}{2\nu}G(z_0)a(z_0).$$

Now substitute this back into (4.22) and get

$$\frac{a'(z_0)}{a(z_0)-1} = -F(z_0),$$

since $a(z)G(z_0) \neq 0$. In the following we shall treat two cases (i) $\dfrac{a'}{a-1} \not\equiv -F$ and (ii) $\dfrac{a'}{a-1} \equiv -F$ separately

Case (i). $\dfrac{a'}{a-1} \not\equiv -F$. Similarly as the above discussion, we will arrive at the same contradiction.

Case (ii) $\dfrac{a'}{a-1} \equiv -F$. By integrating, we get

$$\frac{1}{(W-1)^2} = \left(\frac{a-1}{c}\right)\left(\frac{1}{W'-1} + \frac{1}{(W'-1)^2}\right). \tag{4.23}$$

where c is a non zero constant. Using (4.23) together with (4.12) and Lemma 2 we find that $m_{0(r,\frac{1}{W-1})} = S_0(r,W)$. Finally, from this, the assumption of Theorem 4.1 and Lemma 3 it can be deduce that

$$\begin{aligned} T_0(r,W) &= N_0\left(r,\frac{1}{W-1}\right) + S_0(r,W) = N_0\left(r,\frac{1}{W''-1}\right) + S_0(r,W) \\ &= N_0^{1)}\left(r,\frac{1}{W-1}\right) + S_0(r,W) \leq m_0\left(r,W\right) + S_0(r,W). \end{aligned}$$

This implies that $N_0(r,W) = S_0(r,W)$, which when combined with (4.2) gives $T_0(r,W') = S_0(r,W)$ which implies the contradiction $T_0(r,W) = S_0(r,W)$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let $W(z)$ be a ν-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$* $(1 < R_0 \leq +\infty)$. *Suppose that $W(z)$ and $W'(z)$ share the value $a(\neq 0,\infty)$ DM on annuli. Then either*

$$W(z) = \frac{a[1+b+(b-1)ce^{2blz}]}{1-ce^{2blz}}, \tag{4.24}$$

where $b,\ c,\ l$ are nonzero constants and $b^2l=-1$, or

$$T_0(r,W') \leq 12\nu\overline{N}_0\left(r,\frac{1}{W'}\right) + S_0(r,W) \tag{4.25}$$

and

$$T_0(r,W) \leq (10\nu+1)\overline{N}_0^2\left(r,\frac{1}{W}\right) + S_0(r,W) \tag{4.26}$$

P r o o f. Suppose that $a = 1$ (the general case follows by considering $\frac{1}{a}W$ instead of $W(z)$). We consider the following function

$$\psi = \frac{2\nu W'}{W-1} - \frac{(2\nu+1)W''}{2\nu(W'-1)} + \frac{W'''}{W''} - \frac{W''}{W'}. \tag{4.27}$$

There for

$$m_0(r,\psi) = S_0(r,W). \tag{4.28}$$

Since W and W' share 1 DM on annuli, all zeros of $W-1$ are simple and all zeros of $W'-1$ with multiplicities not less than two on annuli. And so

$$N_0\left(r,\frac{1}{W-1}\right) = N_0^{1)}\left(r,\frac{1}{W-1}\right) \tag{4.29}$$

and

$$N_0\left(r,\frac{1}{W-1}\right) = \overline{N}_0\left(r,\frac{1}{W'-1}\right) = \overline{N}_0^{(2}\left(r,\frac{1}{W'-1}\right). \tag{4.30}$$

Suppose that z_2 is a zero of $W'-1$ with multiplicity 2 on annuli. Since W and W' share 1 DM on aanuli, we see from (4.29) and (4.30) that

$$\psi(z_2) = 0. \tag{4.31}$$

If z_∞ is a simple pole of W on annuli, then the elementary calculation gives that

$$\psi(z_\infty) = O(1). \tag{4.32}$$

From (4.29) and (4.32) we obtain that the poles of ψ can occur at zeros of W' on annuli, or zeros of W'' which are not zeros of $W'(W'-1)$ on annuli, zeros of $W'-1$ with multiplicities not less than three and multiple poles of W on annuli. Thus

$$N_0(r,\psi) \le \overline{N}_0\left(r,\frac{1}{W'}\right) + \overline{N}_0^{(3}\left(r,\frac{1}{W'-1}\right)\overline{N}_0^{(2}(r,W) + \overline{N}_0^{0}\left(r,\frac{1}{W''}\right), \tag{4.33}$$

where $\overline{N}_0\left(r,\frac{1}{W''}\right)$ denotes the counting function corresponding to the zeros of W'' that are not zeros of $W'(W'-1)$ on annuli, each zero in this counting function is counted only once.

We distinguish the following two cases

Case 1. If $\psi \equiv 0$. Then by integrating two sides of (4.27) we obtain

$$\frac{(W-1)^4}{(W'-1)^3} = c\left(\frac{W'}{W''}\right)^2, \tag{4.34}$$

where c is a non zero constant. If z_q is a zero of $W'-1$ with multiplicity $q(\geq 3)$ on annuli, then from (4.29) and (4.34) we see that

$$O((z-z_q)^{2-q}) = c,$$

this implies that $q = 2$, a contradiction.

Therefore

$$N_0\left(r, \frac{1}{W'-1}\right) = 0. \tag{4.35}$$

Also if z_p is a pole of W with multiplicity $p(\geq 2)$ on annuli, then from (4.34) we find that

$$O((z-z_p)^{1-p}) = c.$$

Hence $p = 1$, a contradiction.

Therefore

$$N_0^{(2}(r, W) = 0. \tag{4.36}$$

It follows that from W and W' share 1 DM on annuli, (4.29), (4.30)), (4.35) and (4.36) that

$$\frac{W'-1}{(W-1)^2} = e^{\alpha}, \tag{4.37}$$

where α is an entire function. Combining (4.34) and (4.37) we get

$$\left(\frac{W''}{W'}\right)\left(\frac{W''}{W'-1} - \frac{W''}{W'}\right) = ce^{2\alpha}. \tag{4.38}$$

Consequently,

$$T_0(r, e^{\alpha}) = S_0(r, W). \tag{4.39}$$

Also we know from (4.38) that

$$\overline{N}_0\left(r, \frac{1}{W'}\right) = S_0(r, W). \tag{4.40}$$

Suppose that z_1 is a simple zero of $W-1$ on annuli. Then by (4.30) and (4.35) we have

$$W(z)-1=(z-z_1)+a_3(z-z_1)^3+\cdots, a_3\neq 0 \tag{4.41}$$

Substituting (4.41) into (4.34) and (4.37) we find that

$$3a_3c=4\nu \quad \text{and} \quad 3a_3=e^{\alpha(z_1)},$$

which implies

$$e^{\alpha(z_1)}=\frac{4\nu}{c}. \tag{4.42}$$

If $e^{\alpha}\not\equiv\frac{4\nu}{c}$, then we obtain from (4.29) and (4.39) that

$$N_0\left(r,\frac{1}{W-1}\right)\leq N_0\left(r,\frac{1}{e^{\alpha}-\frac{4\nu}{c}}\right)\leq T_0(r,e^{\alpha})+O(1)=S_0(r,W). \tag{4.43}$$

From (4.30), (4.40), (4.43) and the second fundamental theorem for algebroid function on annuli, we have

$$\begin{aligned} T_0(r,W') &\leq \overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0\left(r,\frac{1}{W'-1}\right)+\overline{N}_0(r,W)+S_0(r,W)\\ &\leq \overline{N}_0(r,W)+S_0(r,W). \end{aligned} \tag{4.44}$$

Since

$$\begin{aligned} T_0(r,W') &= m_0(r,W')+N_0(r,W')\\ &= m_0(r,W')+N_0(r,W)+\overline{N}_0(r,W), \end{aligned} \tag{4.45}$$

it follows from (5.21) and (5.22) that

$$m_0(r,W')+N_0(r,W)=S_0(r,W),$$

and so

$$T_0(r,W)=S_0(r,w).$$

From this , (4.40) and (4.37) we get

$$T_0(r,W)=S_0(r,w),$$

which is impossible.

Therefore $e^{\alpha} \equiv \dfrac{4\nu}{c}$. Together with (5.14) we arrive at the conclusion (4.24).

Case 2. If $\psi \not\equiv 0$. Then from (4.31), (4.28) and (4.33) we conclude that

$$\begin{aligned}\overline{N}_0^{(2}\left(r,\frac{1}{W'-1}\right)-\overline{N}_0^{(3}\left(r,\frac{1}{W'-1}\right) &\leq N_0\left(r,\frac{1}{\psi}\right)\leq T_0(r,\psi)+O(1)\\ &\leq N_0(r,\psi)+m_0(r,\psi)+O(1)\\ &\leq \overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0^{(3}\left(r,\frac{1}{W'-1}\right)\\ &\quad+\overline{N}_0^{(2}(r,W)+\overline{N}_0\left(r,\frac{1}{W''}\right)+S_0(r,W).\end{aligned}\tag{4.46}$$

Since $N_0(r,W') = N_0(r,W) + \overline{N}_0(r,W)$, from the second fundamental theorem for algebroid functions for W' on annuli

$$T_0(r,W') \leq \overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0\left(r,\frac{1}{W'-1}\right)+\overline{N}_0(r,W)+\overline{N}_0^{(0)}\left(r,\frac{1}{W''}\right)+S_0(r,W),\tag{4.47}$$

We have

$$N_0(r,W) \leq \overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0\left(r,\frac{1}{W'-1}\right)+\overline{N}_0^{(0)}\left(r,\frac{1}{W''}\right)+S_0(r,W).\tag{4.48}$$

Also we know from (4.47) that

$$N_0\left(r,\frac{1}{W'-1}\right) \leq \overline{N}_0\left(r,\frac{1}{W'}\right)+\overline{N}_0\left(r,\frac{1}{W'-1}\right)+\overline{N}_0^{(0)}\left(r,\frac{1}{W''}\right)+S_0(r,W).\tag{4.49}$$

From (4.49) and (4.48) we obtain

$$N_0(r,W)-\overline{N}_0(r,W)+N_0\left(r,\frac{1}{W'-1}\right)-2\nu\overline{N}_0\left(r,\frac{1}{W'-1}\right)+2\nu\overline{N}_0^{(0)}\left(r,\frac{1}{W''}\right) \leq 2\nu\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W).\tag{4.50}$$

Obviously,

$$N_0(r,W)-\overline{N}_0(r,W)\geq \overline{N}_0^{(2}(r,W). \tag{4.51}$$

and

$$N_0\left(r,\frac{1}{W'-1}\right)-2\nu\overline{N}_0\left(r,\frac{1}{W'-1}\right)\geq \overline{N}_0^{(3}\left(r,\frac{1}{W'-1}\right). \tag{4.52}$$

By (4.30), (4.50) and (4.52) we obtain

$$N_0^{(2}(r,W)+\overline{N}_0^{(3}\left(r,\frac{1}{W'-1}\right)-2\nu\overline{N}_0\left(r,\frac{1}{W''}\right)\leq 2\nu\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W).$$

From this and (4.46) we deduce that

$$\overline{N}_0^{(2}\left(r,\frac{1}{W'-1}\right)\leq (4\nu+1)\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W).$$

Together with (4.30) we have

$$\overline{N}_0\left(r,\frac{1}{W'-1}\right)\leq (4\nu+1)\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W). \tag{4.53}$$

From (4.53) and (4.48), it follows that

$$\overline{N}_0\left(r,W\right)\leq 6\nu\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W). \tag{4.54}$$

Finally, from (4.47), (4.53) and (4.54) we find that

$$T_0(r,W')\leq 12\nu\overline{N}_0\left(r,\frac{1}{W'}\right)+S_0(r,W).$$

This is the conclusion (4.25).

We set

$$G=\frac{1}{W}\left(\frac{W''}{W'-1}-2\nu\frac{W'}{W-1}\right). \tag{4.55}$$

Then

$$m_0(r,G)\ \leq\ m_0\left(r,\frac{W'}{W}\left(\frac{W''}{W'(W'-1)}\right)\right)+m_0\left(r,\frac{W'}{W(W-1)}\right)+O(1)$$

$$\begin{aligned}
&\leq 2\nu m_0\left(r, \frac{W'}{W}\right) + m_0\left(r, \frac{W''}{W'}\right) + m_0\left(r, \frac{W''}{W'-1}\right) \\
&\quad + m_0\left(r, \frac{W'}{W-1}\right) + O(1) \\
&= S_0(r, W). \qquad (4.56)
\end{aligned}$$

Suppose z_2 be a zero of $W'-1$ with multiplicity 2 on annuli. Since W and W' share 1 DM on annuli, we see from (4.55), (4.29) and (4.30) that

$$G(z_2) = O(1). \qquad (4.57)$$

If z_∞ is a pole of W with multiplicity $p(\geq 1)$ on annuli, then an elementary calculation gives that

$$G(z) = O((z - z_\infty)), \quad \text{if} \;\; p = 1 \qquad (4.58)$$

and

$$G(z) = O((z - z_\infty)^{p-1}), \quad \text{if} \;\; p \geq 2\nu. \qquad (4.59)$$

It follows from (4.29), (4.30), (4.57), (4.58) and (4.59) that the pole of G can only occur at zeros of $W'-1$ with multiplicity not less than three and zeros of W on annuli. Thus

$$N_0(r, G) \leq N_0^2\left(r, \frac{1}{W}\right) + N_0^{(3}\left(r, \frac{1}{W'-1}\right). \qquad (4.60)$$

From (4.56) and (4.60), we obtain

$$T_0(r, G) \leq N_0^2\left(r, \frac{1}{W}\right) + N_0^{(3}\left(r, \frac{1}{W'-1}\right) + S_0(r, W). \qquad (4.61)$$

We consider two cases:

Case I. $G \equiv 0$. Then (4.55) becomes

$$\frac{W''}{W'-1} - 2\nu \frac{W;}{W-1}$$

By integration, we get $W'-1 = l(W-1)^2$. We rewrite this in the form

$$\frac{W'}{W-1-b} - \frac{W'}{W-1+b} = 2\nu b l, \qquad (4.62)$$

where $b^2 l = -1$.Integrating once we arrive at the conclusion (4.24).

Case II. $G \not\equiv 0$. Then from (4.57), (4.58) and (4.59), we get

$$\begin{aligned} N_0(r,W) - \overline{N}_0^{(2} &\le N_0\left(r, \frac{1}{G}\right) \le -m_0\left(r, \frac{1}{G}\right) + T_0(r,G) + O(1) \\ &\le -m_0\left(r, \frac{1}{G}\right) + N_0^2\left(r, \frac{1}{W}\right) + \overline{N}_0^{(3}\left(r, \frac{1}{W'-1}\right) + S_0(r,W). \end{aligned} \tag{4.63}$$

By (4.55), we have

$$W = \frac{1}{G}\left(\frac{W''}{W'-1} - 2\nu\frac{W'}{W-1}\right)$$

Therefore

$$\begin{aligned} m_0(r,W) &\le m_0\left(r, \frac{1}{G}\right) + m_0\left(r, \frac{W''}{W'-1}\right) + m_0\left(r, \frac{W'}{W-1}\right) + O(1) \\ &\le m_0\left(r, \frac{1}{G}\right) + S_0(r,W). \end{aligned} \tag{4.64}$$

From (4.63) and (4.64), we get

$$T_0(r,W) \le N_0^2\left(r, \frac{1}{W}\right) + \overline{N}_0^{(3}\left(r, \frac{1}{W'-1}\right) + \overline{N}_0^{(2}(r,W) + S_0(r,W). \tag{4.65}$$

From (4.64) and (4.65), we get

$$\begin{aligned} N_0\left(r, \frac{1}{W'-1}\right) &\le T_0(r,W') + O(1) = m_0(r,W') + N_0(r,W') + O(1) \\ &\le m_0\left(r, \frac{W'}{W}\right) + m_0(r,W) + N_0(r,W) + \overline{N}_0(r,W) + O(1) \\ &\le T_0(r,W) + \overline{N}_0(r,W) + S_0(r,W) \\ &\le 2\nu N_0^2\left(r, \frac{1}{W}\right) + 2\nu\overline{N}_0^{(3}\left(r, \frac{1}{W'-1}\right) + \overline{N}_0^{(2}(r,W) \\ &\quad + S_0(r,W). \end{aligned} \tag{4.66}$$

Set

$$H = \frac{1}{W}\left(\frac{W''}{W'-1} - (2\nu+1)\frac{W'}{W-1}\right). \tag{4.67}$$

Proceeding as above, we have

$$m_0(r,H) = S_0(r,W), \tag{4.68}$$

$$W(z_3) = O(1), \tag{4.69}$$
$$W(z) = O((z - z_\infty)^{p-1}), \tag{4.70}$$

where z_3 is a zero of $W' - 1$ with multiplicity three on annuli and also z_∞ is a pole of W with multiplicity $p(\geq 1)$ on annuli. Thus

$$N_0(r, H) \leq N_0^2\left(r, \frac{1}{W}\right) + \overline{N}_0^{2)}\left(r, \frac{1}{W' - 1}\right) + \overline{N}_0^{(4}\left(r, \frac{1}{W' - 1}\right). \tag{4.71}$$

From (4.68) and (4.71), we get

$$T_0(r, H) \leq N_0^2\left(r, \frac{1}{W}\right) + \overline{N}_0^{2)}\left(r, \frac{1}{W' - 1}\right) + \overline{N}_0^{(4}\left(r, \frac{1}{W' - 1}\right) + S_0(r, W). \tag{4.72}$$

If $H \equiv 0$, then

$$\frac{W''}{W' - 1} - (2\nu + 1)\frac{W'}{W - 1} = 0$$

Therefore, we get $W' - 1 = c(W - 1)^3$. this implies that

$$N_0(r, H) = 0, \tag{4.73}$$

and $m_0(r, W') = (2\nu + 1)m_0(r, W) + O(1)$. Hence $m_0(r, W) = S_0(r, W)$. This together with (4.73) gives the contradiction $T_0(r, W) = S_0(r, W)$. Therefore $H \not\equiv 0$. From this, (4.70) and (4.72), we get

$$\begin{aligned} N_0^{(2}(r, W) &\leq N_0\left(r, \frac{1}{H}\right) \leq T_0(r, H) + O(1) \\ &\leq N_0^2\left(r, \frac{1}{W}\right) + \overline{N}_0^{2)}\left(r, \frac{1}{W' - 1}\right) + \overline{N}_0^{(4}\left(r, \frac{1}{W' - 1}\right) + S_0(r, W). \end{aligned} \tag{4.74}$$

It follows from (4.30), (4.66) and (4.74) that

$$N_0\left(r, \frac{1}{W - 1}\right) = \overline{N}_0\left(r, \frac{1}{W' - 1}\right) \leq (2\nu + 1)\overline{N}_0^2\left(r, \frac{1}{W}\right) + S_0(r, W). \tag{4.75}$$

Also from (4.30), (4.65) and (4.74) that

$$m_0\left(r, \frac{1}{W - 1}\right) = 2\nu N_0^{2)}\left(r, \frac{1}{W}\right) + \overline{N}_0^{(4}\left(r, \frac{1}{W' - 1}\right) + S_0(r, W). \tag{4.76}$$

Set

$$I = \frac{W''}{W(W - 1)}. \tag{4.77}$$

It is clear that

$$m_0(r, I) \leq m_0\left(r, \frac{W''}{W'}\left(\frac{W'}{W(W(W-1))}\right)\right) = S_0(r, W). \tag{4.78}$$

If z_∞ is a pole of W with multiplicity $p(\geq 1)$ on annuli, then from (4.77) we see that

$$I(z) = O((z - z_\infty)^{p-2}). \tag{4.79}$$

Also, if z_∞ is a zero of $W' - 1$ with multiplicity $q(\geq 2)$ on annuli, then from (4.77) we see that

$$I(z) = O((z - z_q)^{q-2}). \tag{4.80}$$

Therefore from (4.77), (4.79) and (4.80), we conclude that

$$N_0(r, I) \leq N_0^2\left(r, \frac{1}{W}\right) + N_0^1(r, W). \tag{4.81}$$

From (4.78) and (4.81), we have

$$T_0(r, I) \leq N_0^2\left(r, \frac{1}{W}\right) + N_0^1(r, W) + S_0(r, W). \tag{4.82}$$

If $I \equiv 0$, then W then is a linear function on annuli. So W and W' can not share 1 DM on annuli which contradicts the condition of Theorem 4.2. Next we assume that $I \not\equiv 0$. From this, (4.81) and (4.82) we obtain

$$\begin{aligned} N_0^{(3}\left(r, \frac{1}{W'-1}\right) - 2\nu\overline{N}_0^{(3}\left(r, \frac{1}{W'-1}\right) \;&\leq\; N_0\left(r, \frac{1}{I}\right) \leq T_0(r, I) + O(1) \\ &\leq\; N_0^2\left(r, \frac{1}{W}\right) + N_0^{1)}(r, W) + S_0(r, W). \end{aligned}$$

That is,

$$\begin{aligned} N_0^{(3}\left(r, \frac{1}{W'-1}\right) &+ \overline{N}_0^{(2}(r, W) \\ &\leq N_0^2\left(r, \frac{1}{W}\right) + 2\nu\overline{N}_0^{(3}\left(r, \frac{1}{W'-1}\right) + \overline{N}_0(r, W) + S_0(r, W). \end{aligned} \tag{4.83}$$

Hence from (4.83) and (4.63), we obtain

$$N_0^{(4}\left(r,\frac{1}{W'-1}\right)+\overline{N}_0^{(2}(r,W)\ \leq\ N_0^2\left(r,\frac{1}{W}\right)+S_0(r,W). \tag{4.84}$$

and eliminating $\overline{N}_0^{(2}(r,W)$ between (4.84) and (4.64) gives

$$m_0\left(r,\frac{1}{W-1}\right)+\overline{N}_0^{(4}\left(r,\frac{1}{W'-1}\right)\leq(2\nu+1)N_0^2\left(r,\frac{1}{W}\right)+S_0(r,W). \tag{4.85}$$

and eliminating $\overline{N}_0^{(4}\left(r,\frac{1}{W'-1}\right)$ between (4.85) and (4.76) lead to

$$m_0\left(r,\frac{1}{W-1}\right)\ \leq\ \frac{4\nu+1}{2\nu}N_0^2\left(r,\frac{1}{W}\right)+S_0(r,W). \tag{4.86}$$

Combining (4.86) with (4.75) we will arrive at the conclusion (4.26).

This proves the Theorem 4.2. □

Remark 4.1. From (4.24) we find that

(1) If $l=-1$, then $b=\pm1$. Hence (4.24) becomes $W(z)=\dfrac{2\nu a}{1-ce^{-2z}}$.

(2) If $b\neq\pm$, then $T_0(r,W)=N_0\left(r,\frac{1}{W}\right)+S_0(r,W)$.

(3) $N_0\left(r,\frac{1}{W'}\right)=0$.

(4) If $c=-1$, then $W(z)=a[1-b\tanh(blz)]$.

From Theorem 4.1 and Remark 4.1(4), we deduce the following corollaries

Corollary 4.1. *Let $W(z)$ be ν-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$. If W and W' share the value $a(\neq 0,\infty)$ DM on annuli and if $\overline{N}_0\left(r,\frac{1}{W'}\right)=S_0(r,W)$, then W is given as* (4.24).

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REFERENCES

[1] S. S. Bhoosnurmath, R. S. Dyavanal, M. Barki, A. Rathod. Value distribution for nth difference operator of meromorphic functions with maximal deficiency sum. *J. Anal.* **27**, *3* (2019), 797–811.

[2] T. Cao, H. Yi. On the uniqueness theory of algebroid functions. *Southeast Asian Bull. Math.* **33**, *1* (2009), 25–39.

[3] T. B. Cao, H.-X. Yi, H.-Y. Xu. On the multiple values and uniqueness of meromorphic functions on annuli. *Comput. Math. Appl.* **58**, *7* (2009), 1457–1465.

[4] R. S. Dyavanal, A. Rathod. Uniqueness theorems for meromorphic functions on annuli. Indian Journal of Mathematics and Mathematical Sciences **12**, *1* (2016), 1–10.

[5] R. S. Dyavanal, A. Rathod. Multiple values and uniqueness of meromorphic functions on annuli. *International Journal Pure and Applied Mathematics* **107**, *4* (2016), 983–995.

[6] R. S. Dyavanal, A. Rathod. On the value distribution of meromorphic functions on annuli. *Indian Journal of Mathematics and Mathematical Sciences* **12**, *2* (2016), 203–217.

[7] R. S. Dyavanal, A. Rathod. Some generalisation of Nevanlinna's five-value theorem for algebroid functions on annuli. *Asian Journal of Mathematics and Computer Research* **20**, *2* (2017), 85–95.

[8] R. S. Dyavanal, A. Rathod. Nevanlinna's five-value theorem for derivatives of meromorphic functions sharing values on annuli. *Asian Journal of Mathematics and Computer Research* **20**, *1* (2017), 13–21.

[9] R. S. Dyavanal, A. Rathod. Unicity theorem for algebroid functions related to multiple values and derivatives on annuli. *Intern. J. Fuzzy Mathematical Archive* **13**, *1* (2017), 25–39, `http://www.researchmathsci.org/IJFMAart/IJFMA-v13n1-4.pdf`.

[10] R. S. Dyavanal, A. Rathod. General Milloux inequality for algebroid functions on annuli. *Int. J. Math. Appl.* **5**, *3-C* (2017), 319-326, `http://ijmaa.in/v5n3-c/319-326.pdf`.

[11] Fang M. L. Unicity theorems for algebroid functions. *Acta Math. Sinica* **36**, *2* (1993), 217–222 (in Chinese).

[12] A. Fernandez. On the value distribution of meromorphic function in the punctured plane. Mat. Stud. **34**, *2* (2010) 136–144.

[13] H. S. Gopalakrishna, S. S. Bhoosnurmath. Exceptional values of meromorphic functions. *Ann. Polon. Math.* **32**, *1* (1976), 83–93.

[14] W. K. Hayman. Meromorphic functions. Oxford, Oxford Mathematical Monographs Clarendon Press, 1964.

[15] Y. Z. He, X. Z. Xiao. Algebroid Functions and Ordinary Differential Equations. Beijing, China, Science Press, 1988.

[16] Y. He, Y. Li. Some results on algebroid functions. *Complex Variables Theory Appl.* **43**, *3–4* (2001), 299–313.

[17] Y. Hongxun. On the multiple values and uniqueness of algebroid functions. *Chinese Journal of Engineering Mathematics* **8**, *4* (1991), 1–8.

[18] A. Ya. Khrystiyanyn, A. A. Kondratyuk. On the Nevanlinna theory for meromorphic functions on annuli. I. *Mat. Stud.* **24**, *1* (2005), 19–30.

[19] A. Ya. Khrystiyanyn, A. A. Kondratyuk. On the Nevanlinna theory for meromorphic functions on annuli. II. *Mat. Stud.* **24**, *1* (2005), 57–68.

[20] G. S. Prokopovich. Fix-points of meromorphic functions. *Ukrain. Mat. Zh.* **25** (1973) 248–260 (in Russian); English translation in *Ukrainian Math J.* **25**, *2* (1973), 198–208.

[21] A. Rathod. The multiple values of algebroid functions and uniqueness. *Asian Journal of Mathematics and Computer Research* **14**, *2* (2016), 150–157.

[22] A. Rathod. The uniqueness of meromorphic functions concerning five or more values and small functions on annuli. *Asian Journal of Current Research* **1**, *3* (2016), 101–107.

[23] A. Rathod. Uniqueness of algebroid functions dealing with multiple values on annuli. *Journal of Basic and Applied Research International* **19**, *3* (2016), 157–167.

[24] A. Rathod. On the deficiencies of algebroid functions and their differential polynomials. *Journal of Basic and Applied Research International* **20**, *1* (2017), 1–11.

[25] A. Rathod. The multiple values of algebroid functions and uniqueness on annuli. *Konuralp J. Math.* **5**, *2* (2017), 216–227.

[26] A. Rathod. Several uniqueness theorems for algebroid functions. J. Anal. **25**, *2* (2017), 203–213.

[27] A. Rathod. Nevanlinna's five-value theorem for algebroid functions. *Ufa Math. J.* **10**, *2* (2018), 127–132.

[28] A. Rathod. Nevanlinna's five-value theorem for derivatives of algebroid functions on annuli. *Tamkang J. Math.* **49**, *2* (2018), 129–142.

[29] A. Rathod. Characteristic function and deficiency of algebroid functions on annuli. *Ufa Math. J.* **11**, *1* (2019), 121–132.

[30] A. Rathod. Value distribution of a algebroid function and its linear combinations of derivatives on annuli. *Electron. J. Math. Anal. Appl.* **8**, *1* (2020), 182–191.

[31] A. Rathod. Uniqueness theorems for meromorphic functions on annuli. *Ufa Math. J.* **12**, *1* (2020), 114–120.

[32] A. Rathod, S. H. Naveenkumar. Uniqueness of algebroid functions in connection to Nevanlinna's five value theorem. *Jñānābha* **50**, *2* (2020), 160–166.

[33] A. Rathod. Exceptional values of algebroid functions on annuli. *J. Anal.* **29**, *1* (2021), 131–145.

[34] A. Rathod. The shared set and uniqueness of meromorphic functions in an angular domain. *Tbilisi Math. J.* **14**, *3* (2021), 95–109.

[35] A. Rathod, P. B. Burji. The shared set and uniqueness of algebroid functions on annuli. *Malaya J. Mat.* **9**, *1* (2021), 1047–1056.

[36] A. Rathod, P. B. Burji. Uniqueness and value sharing of meromorphic functions on annuli. *Malaya J. Mat.* **9**, *1* (2021), 1071–1079.

[37] A. Rathod, S. Patil. Nevanlinna's five-value theorem for derivatives of meromorphic functions in an angular domain. *J. Class. Anal.* **20**, *1* (2022), 1–11.

[38] A. Rathod. Some results on uniqueness of meromorphic functions for finite order in an angular domain. *J. Class. Anal.* **20**, *1* (2022), 13–22.

[39] A. Rathod. Some results on value distribution theory for E-valued meromorphic function. *Jñānābha* **52**, *1* (2022), 246–261.

[40] A. Rathod. Value distribution of meromorphic functions with relative (k, n) Valiron defect on annuli. Mat. Stud. **57**, *2* (2022), 172–185.

[41] S. K. Singh, H. S. Gopalakrishna. Exceptional values of entire and meromorphic functions. *Math. Ann.* **191** (1971), 121–142.

[42] D. C. Sun, Z. S. Gao. Theorems for algebroid functions. *Acta Math. Sinica (Chinese Ser.)* **49**, *5* (2006), 1027–1032 (in Chinese).

[43] D. Sun, Z. Gao. On the operation of algebroid functions. *Acta Math. Sci. Ser. B (Engl. Ed.)* **30**, *1* (2010), 247–256.

[44] D. C. Sun, Z. S. Gao. Value distribution theory of algebroid functions. Beijing, China, Science Press, 2014 (in Chinese).

[45] Y. Tan, Y. Wang. On the multiple values and uniqueness of algebroid functions on annuli. *Complex Var. Elliptic Equ.* **60**, *9* (2015), 1254–1269.

[46] Y. Tan. 2016. Several uniqueness theorems of algebroid functions on annuli. *Acta Math. Sci. Ser. B (Engl. Ed.)* **36**, *1* (2016), 295–316.

[47] E. Ullrich. Uber den Einfluss der Verzweigtheit einer Algebloide auf ihre Wertverteilung. *J. Reine Angew. Math.* **167** (1932) 198–220.

[48] G. Valiron. Sur quelques propriétés des fonctions algebroldes. *C. R. Acad. Sci., Paris* **189** (1929), 824–826.

[49] C. C. Yang, H. X. Yi. Uniqueness theory of meromorphic functions. Mathematics and its Applications, vol. **557**. Dordrecht, Kluwer Academic Publishers Group, 2003.

[50] H. X. Yi. 1989. Multiple values and uniqueness of meromorphic functions. *Chinese Ann. Math. Ser. A* **10**, *4* (1989), 421–427 (in Chinese).

[51] Q. Zhang. Uniqueness of algebroid functions. *Mathematics in Practice and Theory* **43**, *1* (2013), 183–187.

[52] P. Zhang, P. Hu. On uniqueness for algebroid functions of finite order. *Acta Math. Sci. Ser. B (Engl. Ed.)* **35**, *3* (2015), 630–638.

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