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A NOTE ON SUBCLASSES OF MULTIVALENT FUNCTIONS

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Abstract. In this paper, certain subclasses of strongly multivalent functions are introduced with generalized Sãlãgean operator. We establish various properties for these classes and the results obtained here will generalize some other known results.

1. Introduction. Let $\mathcal{A}_p$ $(p \geq 1)$, denotes the class of functions f, analytic in the open unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$($\mathbb{C}$ stands for complex plane) and having the Taylor-Maclaurin series of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.$$

For $p = 1$, the class $\mathcal{A}_p$ reduces to $\mathcal{A}_1$, which is the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

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By $\mathcal{U}$, we denote the class of Schwarzian functions of the form

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

which are analytic and satisfying the conditions $w(0) = 0, |w(z)| < 1$, in the unit disc E. Nehari [10] proved that, for $w \in \mathcal{U}$, $|c_1| \le 1$ and $|c_2| \le 1 - |c_1|^2$.

For two analytic functions f and g in E, f is said to be subordinate to g if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ and it is denoted by $f \prec g$. Further, if the function g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

For $f \in \mathcal{A}_p$ and $\delta \ge 0$, Goyal [6] introduced the following differential operator:

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z) = D_\delta f(z),$$

and in general,

$$D_\delta^k f(z) = D(D_\delta^{k-1} f(z)) = z^p + \sum_{n=p+1}^{\infty} \left[1 + \left(\frac{n}{p} - 1\right)\delta\right]^k a_n z^n, p \in N_0 = N \cup \{0\},$$

with $D_\delta^0 f(0) = 0$. For $p = 1$, this operator reduces to that introduced by Al-Oboudi [1] and for $p = 1, \delta = 1$, it agrees with the well known Sãlãgean operator introduced in [16].

For $0 \le \alpha < p$, the classes of p-valently starlike functions and p-valently convex functions of order α are denoted by $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ $(0 \le \alpha < p)$ respectively and were introduced by Goluzina [5]. For specific values of α and p, we have the following observations:
(i) $\mathcal{S}_1^*(\alpha) \equiv \mathcal{S}^*(\alpha)$, the class of starlike functions of order α introduced by Robertson [14].
(ii) $\mathcal{K}_1(\alpha) \equiv \mathcal{K}(\alpha)$, the class of convex functions of order α introduced by Robertson [14].
(iii) $\mathcal{S}_p^*(0) \equiv \mathcal{S}_p^*$, the class of p-valent starlike functions.
(iv) $\mathcal{K}_p(0) \equiv \mathcal{K}_p$, the class of p-valent convex functions.
(v) $\mathcal{S}_1^*(0) \equiv \mathcal{S}^*$, the well known class of starlike functions.
(vi) $\mathcal{K}_1(0) \equiv \mathcal{K}$, the well known class of convex functions.

Further, Umezawa [20] established the class $\mathcal{C}_p(\alpha)$, which is the class of p-valent close-to-convex functions defined as

$$\mathcal{C}_p(\alpha) = \left\{ f : f \in \mathcal{A}_p, \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha, g \in \mathcal{S}_p^*, z \in E \right\}.$$

For $p = 1$, $\alpha = 0$, the class $\mathcal{C}_p(\alpha)$ reduces to $\mathcal{C}$, the class of close-to-convex functions introduced by Kaplan [9].

Following the concept of close-to-convex functions, Noor [11] introduced the class $\mathcal{C}^*$ of quasi-convex functions as follows:

$$\mathcal{C}^* = \left\{ f : f \in \mathcal{A}_1, \operatorname{Re}\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in \mathcal{K}, z \in E \right\}.$$

Every quasi-convex function is convex and close-to-convex and so is univalent. Also $f \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$. Different subclasses of quasi-convex functions were studied by various authors including Selvaraj and Stelin [17], Selvaraj et al. [18], Xiong and Liu [21] and Singh and Singh [19].

Recently, Raina [13] defined the class of strongly close-to-convex functions of order γ, as below:

$$\mathcal{C}'_\gamma = \left\{ f : f \in \mathcal{A}_1, \left| \arg\left\{\frac{zf'(z)}{g(z)}\right\} \right| < \frac{\gamma\pi}{2}, g \in \mathcal{K}, 0 < \gamma \leq 1, z \in E \right\},$$

or equivalently

$$\mathcal{C}'_\gamma = \left\{ f : f \in \mathcal{A}_1, \frac{zf'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma, g \in \mathcal{K}, 0 < \gamma \leq 1, z \in E \right\}.$$

In particular, $\mathcal{C}'_1 \equiv \mathcal{C}'$, the subclass of close-to-convex functions.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [2] introduced the class $\mathcal{P}(A, B; p; \alpha)$, which is a subclass of $\mathcal{A}_p$ consisting of the functions of the form $p(z) = p + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz}$. For $p = 1$, $\mathcal{P}(A, B; p; \alpha)$ reduces to $\mathcal{P}(A, B; \alpha)$, the class introduced by Polatoglu et al. [12] and for $p = 1, \alpha = 0$, the class $\mathcal{P}(A, B; p; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}_1$ introduced by Janowski [8].

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [2, 3], introduced the following suclasses of $\mathcal{A}_p$:

$$\mathcal{S}^*(A, B; p; \alpha) = \left\{ f : f \in \mathcal{A}_p, \frac{zf'(z)}{f(z)} \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz}, z \in E \right\}$$

and

$$\mathcal{K}(A,B;p;\alpha)=\left\{f: f\in\mathcal{A}_p, \frac{(zf'(z))'}{f'(z)}\prec\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}, z\in E\right\}.$$

The following points are to be noted:
(i) $\mathcal{S}^*(1,-1;p;\alpha)\equiv\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}(1,-1;p;\alpha)\equiv\mathcal{K}_p(\alpha)$.

(ii) $\mathcal{S}^*(A,B;p;0)\equiv\mathcal{S}_p^*(A,B)$ and $\mathcal{K}(A,B;p;0)\equiv\mathcal{K}_p(A,B)$, the classes studied by Hayami and Owa [7].

(iii) $\mathcal{S}^*(A,B;1;\alpha)\equiv\mathcal{S}^*(A,B;\alpha)$, the class studied by Polatoglu et al. [12].

(iv) $\mathcal{S}^*(A,B;1;0)\equiv\mathcal{S}^*(A,B)$ and $\mathcal{K}(A,B;1;0)\equiv\mathcal{K}(A,B)$, the subclasses of starlike and convex functions respectively, introduced by Janowski [8] and studied further by Goel and Mehrok [4].

(v) $\mathcal{S}^*(1,-1;1;\alpha)\equiv\mathcal{S}^*(\alpha)$ and $\mathcal{K}(1,-1;1;\alpha)\equiv\mathcal{K}(\alpha)$.

(vi) $\mathcal{S}^*(1,-1;1;0)\equiv\mathcal{S}^*$ and $\mathcal{K}(1,-1;1;0)\equiv\mathcal{K}$.

To avoid repetition, throughout this paper, we assume that $-1\le D<C\le 1$, $-1\le B<A\le 1$, $0\le\alpha<p$, $0\le\beta<p$, $\delta\ge 0$, $k\ge 0$, $0<\gamma\le 1$ and $z\in E$.

Getting motivated by the above mentioned work, now we introduce the following subclasses of strongly multivalent functions with generalized Sãlãgean operator:

Definition 1. *Let $\mathcal{C}_\gamma(A,B;C,D;p;\delta;k;\beta;\alpha)$ denote the class of functions $f\in\mathcal{A}_p$ and satisfying the condition*

$$\frac{(D_\delta^k f(z))'}{h'(z)}\prec\left(\frac{p+[pD+(C-D)(p-\beta)]z}{1+Dz}\right)^\gamma,$$

where

$$h(z)=z^p+\sum_{k=p+1}^{\infty}b_kz^k\in\mathcal{K}(A,B;p;\alpha).$$

Definition 2. *$\mathcal{C}_\gamma^s(A,B;C,D;p;\delta;k;\beta;\alpha)$ is the class of functions $f\in\mathcal{A}_p$ which satisfy the condition*

$$\frac{(D_\delta^k f(z))'}{g'(z)}\prec\left(\frac{p+[pD+(C-D)(p-\beta)]z}{1+Dz}\right)^\gamma,$$

where

$$g(z)=z^p+\sum_{k=p+1}^{\infty}d_kz^k\in\mathcal{S}^*(A,B;p;\alpha).$$

The following observations are obvious:

(i) $\mathcal{C}_1(A, B; C, D; p; 1; 1; 0; 0) \equiv \mathcal{C}^*(A, B; C, D; p)$.

(ii) $\mathcal{C}_1(A, B; C, D; 1; 1; 1; 0; 0) \equiv \mathcal{C}^*(A, B; C, D)$, the subclass of quasi-convex functions investigated by Singh and Singh [19].

(iii) $\mathcal{C}_1(1, -1; C, D; 1; 1; 1; 0; 0) \equiv \mathcal{C}^*(C, D)$, the class studied by Xiong and Liu [21].

(iv) $\mathcal{C}_1(1, -1; 1, (1-2\alpha)\beta; \beta; 1; 1; 1; 0; 0) \equiv \mathcal{C}^*(\alpha, \beta)$, the subclass of quasi-convex functions introduced by Selvaraj and Stelin [17].

(v) $\mathcal{C}_1(1, -1; 1, -1; 1; 1; 1; 0; 0) \equiv \mathcal{C}^*$.

(vi) $\mathcal{C}_1^s(A, B; C, D; p; 1; 1; 0; 0) \equiv \mathcal{C}_s^*(A, B; C, D; p)$.

(vii) $\mathcal{C}_1^s(A, B; C, D; 1; 1; 1; 0; 0) \equiv \mathcal{C}_s^*(A, B; C, D)$, the subclass of quasi-convex functions investigated by Singh and Singh [19].

(viii) $\mathcal{C}_1^s(1, -1; C, D; 1; 1; 1; 0; 0) \equiv \mathcal{C}_s^*(C, D)$, the class discussed by Singh and Singh [19].

(ix) $\mathcal{C}_1^s(1, -1; 1, (1-2\alpha)\beta; \beta; 1; 1; 1; 0; 0) \equiv \mathcal{C}_s^*(\alpha, \beta)$, the subclass of quasi-convex functions studied by Selvaraj et al. [18].

(x) $\mathcal{C}_1^s(1, -1; 1, -1; 1; 1; 1; 0; 0) \equiv \mathcal{C}_s^*$, the subclass of quasi-convex functions discussed by Singh and Singh [19].

Lemma 1 ([15]). *If*

$$\left(\frac{p + [pD + (C-D)(p-\beta)]w(z)}{1 + Dw(z)}\right)^{\gamma} = p + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}(C, D; p; \beta),$$

then

$$|p_n| \leq \gamma(C-D)(p-\beta), n \geq p.$$

Lemma 2 ([13]). *Let* $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$, *then*

$$\left(\frac{1 + C_1 z}{1 + D_1 z}\right)^{\gamma} \prec \left(\frac{1 + C_2 z}{1 + D_2 z}\right)^{\gamma}.$$

This paper is concerned with the various properties such as the coefficient estimates, distortion theorems, growth theorems, argument theorems and inclusion relations for the classes $\mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$ and $\mathcal{C}_\gamma^s(A, B; C, D; p; \delta; k; \beta; \alpha)$. The results already proved follow as special cases.

2. Results for the class $\mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$.

Theorem 2.1. *Let $f \in \mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$, then for $n \geq p+1$,*

$$|a_n| \leq \frac{p^2}{\left[1+\left(\frac{n}{p}-1\right)\delta\right]^k [(n-p)!]} \Pi_{j=0}^{n-(p+1)} |(B-A)(p-\alpha)+Bj|$$

(1)

$$+\frac{\gamma(C-D)(p-\beta)}{n\left[1+\left(\frac{n}{p}-1\right)\delta\right]^k}\left[p+\sum_{m=p+1}^{n-1}\frac{p}{(m-p)!}\Pi_{j=0}^{m-(p+1)}|(B-A)(p-\alpha)+Bj|\right].$$

The bound is sharp.

Proof. For $f \in \mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$, we have

$$(D_\delta^k f(z))' = h'(z)(P(z))^\gamma, \tag{2}$$

where

$$h(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{K}(A, B; p; \alpha)$$

and

$$P(z) = p + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}(C, D; p; \beta).$$

On expanding and equating the coefficients of z^{n-p} in (2), it yields

$$\begin{aligned} (3)\quad & n\left[1+\left(\frac{n}{p}-1\right)\delta\right]^k a_n \\ &= pnb_n + (n-1)p_1 b_{n-1} + (n-2)p_2 b_{n-2} + \cdots + 2p_{n-2}b_2 + pp_{n-p}. \end{aligned}$$

Application of triangle inequality and using Lemma 1 in (3), it gives

$$\begin{aligned} (4)\quad & n\left[1+\left(\frac{n}{p}-1\right)\delta\right]^k |a_n| \\ &\leq pn|b_n| + \gamma(C-D)(p-\beta)\left[(n-1)|b_{n-1}| + (n-2)|b_{n-2}| + \cdots\right. \\ &\qquad \left. + (p+1)|b_{p+1}| + p\right]. \end{aligned}$$

It was proved in [3] that, for $h(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \in \mathcal{K}(A, B; p; \alpha)$,

$$|b_n| \leq \frac{p}{n[(n-p)!]} \Pi_{j=0}^{n-(p+1)} |(B-A)(p-\alpha) + Bj|, n \geq p+1. \tag{5}$$

Using (5) in (4), the result (1) is obvious.
Equality sign in (1) is attained for the functions $f_n(z)$ defined by

$$(D_\delta^k f_n(z))' = pz^{p-1}(1-B\delta_1 z)^{\frac{(A-B)(p-\alpha)}{B}} \left[\frac{p + \{pD + (C-D)(p-\beta)\}\delta_2 z}{1 + D\delta_2 z}\right]^\gamma, \tag{6}$$

$$|\delta_1| = |\delta_2| = 1. \quad \square$$

Remark 1. (i) For $\gamma = 1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, Theorem 1 gives the result established by Singh and Singh [19].
(ii) By giving the values $\gamma = 1, A = 1, B = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result due to Xiong and Liu [21], can be easily obtained from Theorem 1.
(iii) Substituting for $\gamma = 1, A = 1, B = -1, C = (1-2\alpha)\beta, D = \beta, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$ in Theorem 1, we can easily get the result due to Selvaraj and Stelin [17].
(iv) For $\gamma = 1, A = 1, B = -1, C = 1, D = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result established by Noor [11], can be easily obtained from Theorem 1.

Theorem 2.2. *If $f \in \mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$, then for $|z| = r, 0 < r < 1$, we have:*
for $B \neq 0$,

$$\int_0^r pt^{p-1}(1-Bt)^{\frac{(A-B)(p-\alpha)}{B}} \left[\frac{p - \{pD + (C-D)(p-\beta)\}t}{1 - Dt}\right]^\gamma dt \leq |D_\delta^k f(z)|$$

$$\leq \int_0^r pt^{p-1}(1+Bt)^{\frac{(A-B)(p-\alpha)}{B}} \left[\frac{p + \{pD + (C-D)(p-\beta)\}t}{1 + Dt}\right]^\gamma dt; \tag{7}$$

and for $B = 0$,

$$\int_0^r pt^{p-1}e^{-A(p-\alpha)t} \left[\frac{p - \{pD + (C-D)(p-\beta)\}t}{1 - Dt}\right]^\gamma dt \leq |D_\delta^k f(z)|$$

$$\leq \int_0^r pt^{p-1}e^{A(p-\alpha)t} \left[\frac{p + \{pD + (C-D)(p-\beta)\}t}{1 + Dt}\right]^\gamma dt. \tag{8}$$

The results are sharp.

P r o o f. Equation (2) can be expressed as

$$|(D_\delta^k f(z))'| = |h'(z)||P(z)|^\gamma. \tag{9}$$

It was proved in [3] that, for $P(z) \in \mathcal{P}(A, B; p; \alpha)$,

$$\left(\frac{p - [pD + (C - D)(p - \beta)]r}{1 - Dr}\right)^\gamma \le |P(z)|^\gamma \le \frac{p + [pD + (C - D)(p - \beta)]r}{1 + Dr}. \tag{10}$$

Aouf [3] proved that, for $h(z) \in \mathcal{K}(A, B; p; \alpha)$,

$$\begin{cases} pr^{p-1}(1 - Br)^{\frac{A-B}{B}(p-\alpha)} \le |h'(z)| \le pr^{p-1}(1 + Br)^{\frac{A-B}{B}(p-\alpha)} & \text{if } B \ne 0, \\ pr^{p-1}e^{-A(p-\alpha)r} \le |h'(z)| \le pr^{p-1}e^{A(p-\alpha)r} & \text{if } B = 0. \end{cases} \tag{11}$$

Using (10) and (11) in (9), the results (7) and (8) can be easily obtained. Sharpness follows for the functions $f_n(z)$ defined as

$$(D_\delta^k f_n(z))' = \begin{cases} pz^{p-1}(1 + B\delta_3 z)^{\frac{(A-B)(p-\alpha)}{B}} \left[\frac{p + \{pD + (C - D)(p - \beta)\}\delta_4 z}{1 + D\delta_4 z}\right]^\gamma & \text{if } B \ne 0, \\ pz^{p-1}e^{A(p-\alpha)\delta_5 z} \left[\frac{p + \{pD + (C - D)(p - \beta)\}\delta_4 z}{1 + D\delta_4 z}\right]^\gamma & \text{if } B = 0, \end{cases} \tag{12}$$

where $|\delta_3| = |\delta_4| = |\delta_5| = 1$. □

Remark 2. (i) For $\gamma = 1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, Theorem 2 gives the result established by Singh and Singh [19].
(ii) By giving the values $\gamma = 1, A = 1, B = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result due to Xiong and Liu [21], can be easily obtained from Theorem 2.
(iii) Substituting for $\gamma = 1, A = 1, B = -1, C = (1 - 2\alpha)\beta, D = \beta, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$ in Theorem 2, we can easily get the result due to Selvaraj and Stelin [17].
(iv) For $\gamma = 1, A = 1, B = -1, C = 1, D = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result established by Noor [11], can be easily obtained from Theorem 2.

Theorem 2.3. *If $f \in \mathcal{C}_\gamma(A, B; C, D; p; \delta; k; \beta; \alpha)$, then*

(13)
$$\left|\arg \frac{(D_\delta^k f(z))'}{z^{p-1}}\right| \le \begin{cases} \dfrac{(A-B)(p-\alpha)}{B}\sin^{-1}(Br) \\ +\gamma \sin^{-1}\left(\dfrac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right) & \text{if } B \ne 0, \\ A(p-\alpha)r + \gamma \sin^{-1}\left(\dfrac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right) & \text{if } B = 0. \end{cases}$$

The bounds are sharp.

Proof. (2) can be expressed as

$$(D_\delta^k f(z))' = h'(z)(P(z))^\gamma.$$

Therefore, we have

(14)
$$\left|\arg \frac{(D_\delta^k f(z))'}{z^{p-1}}\right| \le \gamma |\arg P(z)| + \left|\arg \frac{pf_1(z)}{z^p}\right|,$$

where $f_1(z) = \dfrac{zh'(z)}{p}$.
Aouf [2], established that for $P(z) \in \mathcal{P}(A, B; p; \alpha)$,

(15)
$$|\arg P(z)| \le \sin^{-1}\left(\frac{(C-D)(p-\beta)r}{p-[pD+(C-D)(p-\beta)]Dr^2}\right).$$

It was proved by Aouf [3], that

(16)
$$\left|\arg \frac{pf_1(z)}{z^p}\right| \le \begin{cases} \dfrac{(A-B)(p-\alpha)}{B}\sin^{-1}(Br) & \text{if } B \ne 0, \\ A(p-\alpha)r & \text{if } B = 0. \end{cases}$$

Using (15) and (16) in (14), the results (13) are obvious.

Results are sharp for the function defined in (12). □

Remark 3. (i) By giving the values $\gamma = 1, A = 1, B = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result due to Xiong and Liu [21] can be easily obtained from Theorem 3.
(ii) Substituting for $\gamma = 1, A = 1, B = -1, C = (1-2\alpha)\beta, D = \beta, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$ in Theorem 3, we can easily get the result due to Selvaraj

and Stelin [17].
(iii) For $\gamma = 1, A = 1, B = -1, C = 1, D = -1, \alpha = 0, \beta = 0, p = 1, \delta = 1, k = 1$, the result established by Noor [11], can be easily obtained from Theorem 3.

Theorem 2.4. *Let $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \beta_2 \leq \beta_1 < p$, then*

$$\mathcal{C}_\gamma(A, B; C_1, D_1; p; \delta; k; \beta_1; \alpha) \subset \mathcal{C}_\gamma(A, B; C_2, D_2; p; \delta; k; \beta_2; \alpha).$$

Proof. As $f \in \mathcal{C}_\gamma(A, B; C_1, D_1; p; \delta; k; \beta_1; \alpha)$, so

$$\frac{(D_\delta^k f(z))'}{h'(z)} \prec \left(\frac{p + [pD_1 + (C_1 - D_1)(p - \beta_1)]z}{1 + D_1 z}\right)^\gamma.$$

As $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \beta_2 \leq \beta_1 < p$, we have

$$-1 \leq D_1 + \frac{(p - \beta_1)(C_1 - D_1)}{p} \leq D_2 + \frac{(p - \beta_2)(C_2 - D_2)}{p} \leq 1.$$

So by Lemma 2, we obtain

$$\frac{(D_\delta^k f(z))'}{h'(z)} \prec \left(\frac{p + [pD_2 + (C_2 - D_2)(p - \beta_2)]z}{1 + D_2 z}\right)^\gamma,$$

which implies $f \in \mathcal{C}_\gamma(A, B; C_2, D_2; p; \delta; k; \beta_2; \alpha)$. □

3. Results for the class $\mathcal{C}_\gamma^s(A, B; C, D; p; \delta; k; \beta; \alpha)$.

Theorem 3.1. *Let $f(z) \in \mathcal{C}_\gamma^s(A, B; C, D; p; \delta; k; \beta; \alpha)$, then for $n \geq p+1$,*

$$|a_n| \leq \frac{p}{n\left[1 + \left(\frac{n}{p} - 1\right)\delta\right]^k} \Pi_{j=0}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bj|}{j + 1}$$

$$+\frac{\gamma(C - D)(p - \beta)}{n^2\left[1 + \left(\frac{n}{p} - 1\right)\delta\right]^k}\left[p + \sum_{m=p+1}^{n-1} m\Pi_{j=0}^{m-(p+1)} \frac{|(B - A)(p - \alpha) + Bj|}{j + 1}\right]. \tag{17}$$

The result is sharp.

Proof. Using the result due to Aouf [2] that, for $g(z) = z^p + \sum_{k=p+1}^{\infty} d_k z^k \in \mathcal{S}^*(A, B; p; \alpha)$,

$$|d_n| \leq \Pi_{j=0}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bj|}{j + 1}, n \geq p + 1,$$

and following the procedure of Theorem 1, the proof is obvious.

Equality sign in (17) hold for the functions $f_n(z)$ defined by

$$(D_\delta^k f_n(z))' = z^{p-1}(1-B\delta_6 z)^{\frac{(A-B)(p-\alpha)}{B}}\left[p-\frac{\delta_7 z(A-B)(p-\alpha)}{1-B\delta_7 z}\right] \times\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_8 z}{1+D\delta_8 z}\right]^\gamma, \tag{18}$$

where $|\delta_6|=|\delta_7|=|\delta_8|=1$. □

Remark 4. (i) For $\gamma,\alpha=0,\beta=0,p=1,\delta=1,k=1$, Theorem 5 yields the result due to Singh and Singh [19].
(ii) Putting $\gamma, A=1, B=-1,\alpha=0,\beta=0,p=1,\delta=1,k=1$ in Theorem 5, it yields the result for the class $\mathcal{C}_s^*(C,D)$.
(iii) Substituting for $\gamma, A=1, B=-1, C=(1-2\alpha)\beta, D=\beta,\alpha=0,\beta=0,p=1,\delta=1,k=1$, the result due to Selvaraj et al. [18], can be easily obtained from Theorem 5.
(iv) For $\gamma, A=1, B=-1, C=1, D=-1,\alpha=0,\beta=0,p=1,\delta=1,k=1$, Theorem 5 gives the result for the class $\mathcal{C}_s^*$.

Theorem 3.2. *If $f\in\mathcal{C}_\gamma^s(A,B;C,D;p;\delta;k;\beta;\alpha)$, then for $|z|=r, 0<r<1$, we have*
for $B\neq 0$,

$$\int_0^r t^p(1-Bt)^{\frac{A-B}{B}(p-\alpha)}\left[\frac{p}{t}-\frac{(A-B)(p-\alpha)}{1-Bt}\right]\left[\frac{p-\{pD+(C-D)(p-\beta)\}t}{1-Dt}\right]^\gamma dt$$

$$\le |D_\delta^k f(z)| \tag{19}$$

$$\le \int_0^r t^p(1+Bt)^{\frac{A-B}{B}(p-\alpha)}\left[\frac{p}{t}+\frac{(A-B)(p-\alpha)}{1+Bt}\right]\left[\frac{p+\{pD+(C-D)(p-\beta)\}t}{1+Dt}\right]^\gamma dt;$$

for $B=0$,

$$\int_0^r t^{p-1}e^{-A(p-\alpha)t}\left[p-A(p-\alpha)t\right]\left[\frac{p-\{pD+(C-D)(p-\beta)\}t}{1-Dt}\right]^\gamma dt\le|D_\delta^k f(z)|$$

$$\le \int_0^r t^{p-1}e^{A(p-\alpha)t}\left[p+A(p-\alpha)t\right]\left[\frac{p+\{pD+(C-D)(p-\beta)\}t}{1+Dt}\right]^\gamma dt. \tag{20}$$

Estimates are sharp.

P r o o f. Following the procedure of Theorem 2 and using the result that, for $g \in \mathcal{S}^*(A,B;p;\alpha)$,

$$\begin{cases} r^p(1-Br)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p}{r}-\frac{(A-B)(p-\alpha)}{1-Br}\right] & \\ \leq |g'(z)| \leq r^p(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p}{r}+\frac{(A-B)(p-\alpha)}{1+Br}\right] & \text{if } B \neq 0, \\ r^{p-1}e^{-A(p-\alpha)r}\left[p-A(p-\alpha)r\right] \leq |g'(z)| \leq r^{p-1}e^{A(p-\alpha)r}\left[p+A(p-\alpha)r\right] & \text{if } B = 0, \end{cases}$$

the results (19) and (20) can be easily derived.

Sharpness follows for the functions $f_n(z)$ defined as

$$(D_\delta^k f_n(z))' = \begin{cases} z^p(1+B\delta_9 z)^{\frac{(A-B)(p-\alpha)}{B}}\left[\frac{p}{z}+\frac{(A-B)(p-\alpha)}{1+B\delta_{10}z}\right] & \\ \times\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_{11}z}{1+D\delta_{11}z}\right]^\gamma & \text{if } B \neq 0, \\ z^{p-1}e^{A(p-\alpha)\delta_{12}z}\left[\frac{p}{z}+\frac{(A-B)(p-\alpha)}{1+B\delta_{10}z}\right] & \\ \times\left[\frac{p+\{pD+(C-D)(p-\beta)\}\delta_{11}z}{1+D\delta_{11}z}\right]^\gamma & \text{if } B = 0, \end{cases}$$

where $|\delta_9| = |\delta_{10}| = |\delta_{11}| = |\delta_{12}| = 1$. □

Remark 5. (i) For $\gamma = 1, \alpha = 0, \beta = 0, p = 1$, Theorem 6 yields the result due to Singh and Singh [19].
(ii) Putting $\gamma = 1, A = 1, B = -1, \alpha = 0, \beta = 0, p = 1$ in Theorem 6, it yields the result for the class $\mathcal{C}_s^*(C,D)$.
(iii) Substituting for $\gamma = 1, A = 1, B = -1, C = (1-2\alpha)\beta, D = \beta, \alpha = 0, \beta = 0, p = 1$, the result due to Selvaraj et al. [18], can be easily obtained from Theorem 6.
(iv) For $\gamma = 1, A = 1, B = -1, C = 1, D = -1, \alpha = 0, \beta = 0, p = 1$, Theorem 6 gives the result for the class $\mathcal{C}_s^*$.

Theorem 3.3. *Let* $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ *and* $0 \leq \beta_2 \leq \beta_1 < p$, *then*

$$\mathcal{C}_\gamma^s(A,B;C_1,D_1;p;\delta;k;\beta_1;\alpha) \subset \mathcal{C}_\gamma^s(A,B;C_2,D_2;p;\delta;k;\beta_2;\alpha).$$

P r o o f. Following the procedure of Theorem 4 and using Lemma 2, the proof is obvious. □

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