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S_4-DECOMPOSITION OF THE LINE GRAPH OF THE COMPLETE GRAPH

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Abstract. Let S_k denote a star with k edges. The line graph of the complete graph K_n is denoted by $L(K_n)$. In this paper, we prove that the graph $L(K_n)$ has an S_4-decomposition if and only if $n \geq 6$ and $n \equiv 0, 1, 2, 4, 6 \pmod 8$.

1. Introduction. All graphs considered here are finite and simple. Let S_k denote a *star* with k edges. If a graph G has no edges, then it is called a *null graph.* For an integer $p \geq 2$ and a graph G, pG denotes p vertex disjoint copies of G. The degree of a vertex v of G, denoted by $D_G v$ is the number of edges incident with v in G. A graph G is said to be *k-regular*, if $D_G v = k$, for all $v \in V(G)$. If $H_1, H_2, \ldots, H_l$ are edge disjoint subgraphs of a graph G such that $E(G) = \bigcup_{i=1}^{l} E(H_i)$, then we say that $H_1, H_2, \ldots, H_l$ *decompose* G and we denote

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it by $G = \oplus_{i=1}^{l} H_i$. If $H_i \cong S_k$ for $i = 1, 2, \dots, l$, then we say that G has an S_k-decomposition or a k-star decomposition and we denote it by $S_k|G$. The *line graph* $L(G)$ of a graph G is the graph with the vertex set $V(L(G)) = \{\{a,b\}|ab \in E(G)$, for all $a, b \in V(G)\}$ and the edge set $E(L(G)) = \{\{a,b\}\{c,d\}|\{a,b\},\{c,d\} \in V(L(G))$ and $|\{a,b\} \cap \{c,d\}| = 1\}$. The complete graph on n vertices is denoted by K_n. The line graph of the complete graph K_n is denoted by $L(K_n)$. Let G be a graph on n vertices with $V(G) = \{1, 2, 3, \dots, n\}$. The notation $(1; 2, 3, \dots, k, k+1)$ denotes a star with a center vertex 1 and k pendent edges $12, 13, \dots, 1k, 1(k+1)$. Let $S \subset V(G)$ and $T \subset E(G)$, then the subgraph of G induced by S is denoted by $\langle S \rangle$ and the subgraph of G obtained by deleting T is denoted by $G \setminus T$. Let X and Y be two disjoint subsets of $V(G)$. Then $E(X,Y)$ denotes the set of edges in G, whose one end vertex is in X and the other end vertex is in Y. The notation $\langle E(X,Y) \rangle$ denotes the graph induced by the edges of $E(X,Y)$.

In 2019, Ganesamurthy et al. [1] obtained the necessary and sufficient conditions for an S_3-decomposition of $L(K_n)$. In this paper, we show that the graph $L(K_n)$ has an S_4-decomposition if and only if $n \geq 6$ and $n \equiv 0, 1, 2, 4, 6 \pmod 8$. To prove our results we use the following theorem.

Theorem 1.1 (Yamamoto et al. [2]). *Let n and m be positive integers. The complete graph K_n has an S_m-decomposition if and only if $2m \leq n$ and $\binom{n}{2} \equiv 0 \pmod m$.*

2. S_4-decomposition of $L(K_n)$.

We know that, $|E(L(K_n))| = \binom{n}{2}(n-2)$ and is divisible by 4 only if $n \geq 4$ and $n \equiv 0, 1, 2, 4, 6 \pmod 8$. In the following Lemma, we prove that there doesn't exist an S_4-decomposition in $L(K_4)$.

Lemma 2.1. *There doesn't exist an S_4-decomposition in $L(K_4)$.*

Proof. $|V(L(K_4))| = 6$ and $|E(L(K_4))| = 12$. The graph $L(K_4)$ is 4-regular. Choose any S_4 in $L(K_4)$. In $L(K_4) \setminus S_4$, the degrees of the vertices are 4,3,3,3,3 and 0. Since there is only one vertex of degree 4, it is not possible to get two S_4 in $L(K_4) \setminus S_4$. $\square$

Therefore, we look for an S_4-decomposition of $L(K_n)$, when $n \geq 6$. Let $n_1 \geq 2$, $n_2 \geq 4$ and $n \geq 6$ be positive integers such that $n = n_1 + n_2$. We partition the vertex set of $L(K_n) = \{\{a,b\}|1 \leq a < b \leq n\}$ into three types of sets as follows:

Type I: $X_{n_1} = \{\{a,b\} | a < b \text{ and } a,b \in \{1,2,\ldots,n_1\}\}$

Type II: $X_{n_2} = \{\{a,b\} | a < b \text{ and } a,b \in \{n_1+1, n_1+2,\ldots,n\}\}$

Type III: $X_{n_1,n_2} = \{\{a,b\} | a \in \{1,2,\ldots,n_1\},\ b \in \{n_1+1, n_1+2,\ldots,n\}\}$

Therefore, $V(L(K_n)) = X_{n_1} \cup X_{n_2} \cup X_{n_1,n_2}$. Let $A = \{a_1, b_1\}$ and $B = \{a_2, b_2\}$. Now, we define the graphs G^i, $1 \le i \le 6$ as follows:

$$V(G^i) = \begin{cases} X_{n_1} & \text{if } i = 1 \\ X_{n_2} & \text{if } i = 2 \\ X_{n_1,n_2} & \text{if } i = 3 \\ X_{n_1,n_2} & \text{if } i = 4 \\ X_{n_1} \cup X_{n_1,n_2} & \text{if } i = 5 \\ X_{n_2} \cup X_{n_1,n_2} & \text{if } i = 6 \end{cases}$$

and

$$E(G^i) = \begin{cases} \{AB | A,B \in X_{n_1} \text{ and } |A \cap B| = 1\} & \text{if } i = 1 \\ \{AB | A,B \in X_{n_2} \text{ and } |A \cap B| = 1\} & \text{if } i = 2 \\ \{AB | A,B \in X_{n_1,n_2} \text{ and } a_1 = a_2\} & \text{if } i = 3 \\ \{AB | A,B \in X_{n_1,n_2} \text{ and } b_1 = b_2\} & \text{if } i = 4 \\ \{AB | A \in X_{n_1}, B \in X_{n_1,n_2} \text{ and } |A \cap B| = 1\} & \text{if } i = 5 \\ \{AB | A \in X_{n_2}, B \in X_{n_1,n_2} \text{ and } |A \cap B| = 1\} & \text{if } i = 6 \end{cases}$$

Note that, $G^1 \cong L(K_{n_1})$, $G^2 \cong L(K_{n_2})$, $G^3 \cong n_1 K_{n_2}$, $G^4 \cong n_2 K_{n_1}$, $G^5 \cong \langle E(X_{n_1}, X_{n_1,n_2}) \rangle$, $G^6 \cong \langle E(X_{n_2}, X_{n_1,n_2}) \rangle$ and $L(K_n) = \oplus_{i=1}^{6} G^i$.

Lemma 2.2. *There exists an S_4-decomposition in $L(K_6)$.*

Proof. Let n_1=2 and n_2=4. The vertex set of $L(K_6) = X_{n_1} \cup X_{n_2} \cup X_{n_1,n_2}$, where $X_{n_1} = \{1,2\}$, $X_{n_2} = \{\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$ and $X_{n_1,n_2} = \{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}$. The graph $G^1 (\cong L(K_2))$ is a null graph. In G^5, observe that $D_{G^5}\{1,2\}$=8. So, we can choose the vertex $\{1,2\}$ as a center vertex twice, to get an S_4-decomposition in

G^5. The graph $G^2 \cup G^3 \cup G^4 \cup G^6$ can be decomposed into 13 copies of S_4 as follows:

$(\{1,3\};\{2,3\},\{1,6\},\{1,5\},\{3,4\})$, $(\{1,4\};\{3,4\},\{1,5\},\{1,6\},\{1,3\})$,
$(\{1,5\};\{5,6\},\{2,5\},\{3,5\},\{1,6\})$, $(\{2,3\};\{2,5\},\{3,4\},\{3,5\},\{2,6\})$,
$(\{2,4\};\{2,3\},\{2,5\},\{3,4\},\{1,4\})$, $(\{2,6\};\{2,5\},\{2,4\},\{4,6\},\{1,6\})$,
$(\{3,5\};\{3,4\},\{1,3\},\{2,5\},\{5,6\})$, $(\{3,6\};\{1,6\},\{1,3\},\{2,3\},\{2,6\})$,
$(\{3,6\};\{3,5\},\{3,4\},\{4,6\},\{5,6\})$, $(\{4,5\};\{1,4\},\{1,5\},\{2,4\},\{2,5\})$,
$(\{4,5\};\{4,6\},\{5,6\},\{3,5\},\{3,4\})$, $(\{4,6\};\{1,4\},\{1,6\},\{2,4\},\{3,4\})$ and
$(\{5,6\};\{4,6\},\{1,6\},\{2,5\},\{2,6\})$.

□

Lemma 2.3. *If $n = 8m$, $m \geq 1$, then there exists an S_4-decomposition in $L(K_n)$.*

Proof. We split the proof into two cases.

Case (i): $m = 1$. Let n_1=2 and n_2=6. The graph G^1 is a null graph and G^2 has an S_4-decomposition, by Lemma 2.2. In G^6, $D_{G^6}\{a,b\}$=4, for all $\{a,b\} \in X_6$. Now, we choose each $\{a,b\} \in X_6$ as a center vertex, to get an S_4-decomposition in G^6. Now, the graph $G^3 \cup G^4 \cup G^5$ can be decomposed into 12 copies of S_4 as follows:

$(\{1,3\};\{1,2\},\{2,3\},\{1,8\},\{1,7\})$, $(\{1,4\};\{1,3\},\{1,2\},\{2,4\},\{1,8\})$,
$(\{1,5\};\{1,4\},\{1,3\},\{1,2\},\{2,5\})$, $(\{1,6\};\{1,5\},\{1,4\},\{1,3\},\{1,2\})$,
$(\{1,7\};\{1,6\},\{1,5\},\{1,4\},\{1,2\})$, $(\{1,8\};\{1,7\},\{1,6\},\{1,5\},\{1,2\})$,
$(\{2,3\};\{2,4\},\{2,5\},\{2,6\},\{1,2\})$, $(\{2,4\};\{2,5\},\{2,6\},\{2,7\},\{1,2\})$,
$(\{2,5\};\{2,6\},\{2,7\},\{2,8\},\{1,2\})$, $(\{2,6\};\{2,7\},\{2,8\},\{1,2\},\{1,6\})$,
$(\{2,7\};\{2,8\},\{1,2\},\{1,7\},\{2,3\})$ and $(\{2,8\};\{1,2\},\{1,8\},\{2,3\},\{2,4\})$.

Case (ii): $m \geq 2$. Let $n_1 = 8(m-1)$ and n_2=8. We apply mathematical induction on m, to prove this case. If $m = 2$, then $n_1 = n_2$=8. The graphs G^1 and G^2 has an S_4-decomposition, by Case (i). By Theorem 1.1, there exists an S_4-decomposition in G^3 and G^4. Note that $D_{G^5}\{a,b\}$=16, for all $\{a,b\} \in X_8$. We choose each $\{a,b\} \in X_8$ as a center vertex four times, to get an S_4-decomposition in G^5. Note that $D_{G^6}\{a,b\}$=16, for all $\{a,b\} \in X_8$. We choose each $\{a,b\} \in X_8$ as a center vertex four times, to get an S_4-decomposition in G^6. Hence, the result is true for $m = 2$. Assume that the result is true for all $2 < m < k$. Now, we prove that the result is true for $m = k$. The graph G^1 has an S_4-decomposition, by our assumption and the graph G^2 has an S_4-decomposition, by Case (i). By Theorem 1.1, there exists an S_4-decomposition in G^3 and G^4. In G^5, note that $D_{G^5}\{a,b\}$=16, for all $\{a,b\} \in X_{n_1}$. We choose each $\{a,b\} \in X_{n_1}$ as a center vertex four times, to get an S_4-decomposition in G^5. In G^6, note that $D_{G^6}\{a,b\} = 2n_1$, for all $\{a,b\} \in X_{n_2}$ and $n_1 \equiv 0 \pmod 8$. Therefore, we choose each $\{a,b\} \in X_{n_2}$ as a center vertex $\frac{2n_1}{4}$ times, to get an S_4-decomposition in G^6. □

Lemma 2.4. *If $n = 8m+1$, $m \geq 1$, then there exists an S_4-decomposition in $L(K_n)$.*

Proof. We split the proof into two cases.

Case (i): m=1. The graph $L(K_9)$ can be written as $[L(K_9) \setminus L(K_8)] \cup L(K_8)$, where $V(L(K_9) \setminus L(K_8)) = \{\{a,9\} | 1 \leq a \leq 8\}$. The graph $L(K_8)$ has an S_4-decomposition, by Lemma 2.3. Now, the graph $L(K_9) \setminus L(K_8)$ can be decomposed into 12 copies of S_4 as follows:

$$(\{i,9\}; \{i,6\}, \{i,7\}, \{i,8\}, \{i+1,9\}), 1 \leq i \leq 5,$$
$$(\{i,9\}; \{1,i\}, \{2,i\}, \{3,i\}, \{4,i\}), 5 \leq i \leq 8,$$
$$(\{i,9\}; \{i+2,9\}, \{i+3,9\}, \{i+4,9\}, \{i+5,9\}), 1 \leq i \leq 3,$$
$$(\{7,9\}; \{4,9\}, \{5,9\}, \{6,9\}, \{1,9\}),$$
$$(\{8,9\}; \{4,9\}, \{5,9\}, \{6,9\}, \{2,9\}),$$
$$(\{1,9\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}),$$
$$(\{2,9\}; \{1,2\}, \{2,3\}, \{2,4\}, \{2,5\}),$$
$$(\{3,9\}; \{1,3\}, \{2,3\}, \{3,4\}, \{3,5\}),$$
$$(\{4,9\}; \{1,4\}, \{2,4\}, \{3,4\}, \{4,5\}),$$
$$(\{6,9\}; \{5,6\}, \{6,7\}, \{6,8\}, \{4,9\}),$$
$$(\{7,9\}; \{5,7\}, \{6,7\}, \{7,8\}, \{8,9\}) \text{ and}$$
$$(\{8,9\}; \{5,8\}, \{6,8\}, \{7,8\}, \{1,9\}).$$

Case (ii): $m \geq 2$. Let $n_1 = 9 + 8(m-2)$ and n_2=8. We apply mathematical induction on m, to prove this case. If $m = 2$, then n_1=9 and n_2=8. The graph G^1 has an S_4-decomposition, by Case (i) and G^2 has an S_4-decomposition, by Lemma 2.3. By Theorem 1.1, there exists an S_4-decomposition in G^3 and G^4. Let $N_1 = \{\{i,i+1\} | 1 \leq i \leq 8\} \cup \{1,9\}$. Then $N_1 \subset X_9 \subset V(G^5)$. In G^5, consider the induced subgraph $F = \oplus_{i=1}^{8}\langle E(\{i,b\},\{i,i+1\})\rangle \cup \langle E(\{9,b\},\{1,9\})\rangle$, where $10 \le b \le 17$, see Fig. 2.1.

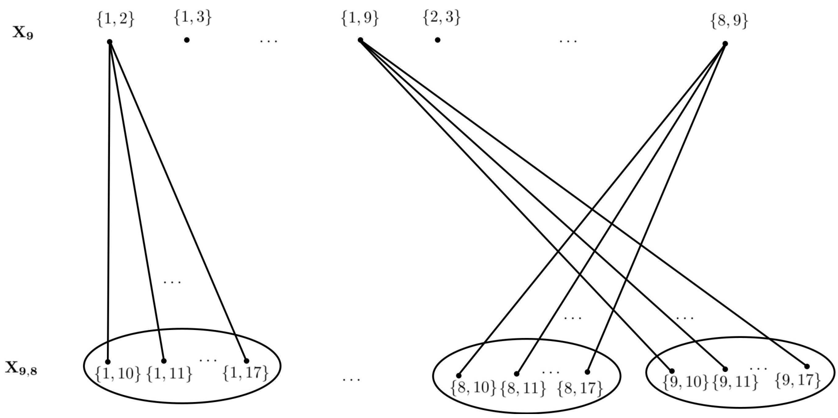

Fig. 2.1. The induced subgraph F of $\langle E(X_9, X_{9,8})\rangle$

In G^6, observe that $D_{G^6}\{a,b\}$=7, for all $\{a,b\} \in X_{9,8}$. In $G^6 \cup F$, $D_{G^6 \cup F}\{a,b\}$=8, for all $\{a,b\} \in X_{9,8}$, see Fig. 2.2. So, we choose each $\{a,b\} \in X_{9,8}$ as a center vertex twice, to get an S_4-decomposition in $G^6 \cup F$. Note that, $D_{G^5 \setminus E(F)}\{a,b\}$=8, for all $\{a,b\} \in N_1$ and $D_{G^5 \setminus E(F)}\{a,b\}$=16, for all $\{a,b\} \in X_9 \setminus N_1$. So, we can choose $\{a,b\} \in N_1$ as a center vertex twice and we choose $\{a,b\} \in X_9 \setminus N_1$ as a center vertex four times, to get an S_4-decomposition in $G^5 \setminus E(F)$. Hence, the result is true for $m = 2$. Assume that the result is true for all $2 < m < k$. We prove that the result is true for $m = k$. The graph G^1 has an S_4-decomposition, by our assumption and G^2 has an S_4-decomposition, by Lemma 2.3. By Theorem 1.1, there exists an S_4-decomposition in G^3 and G^4. Let $N_1 = \{\{i,i+1\} | 1 \le i \le n_1 - 1\} \cup \{1, n_1\}$, then $N_1 \subset X_{n_1} \subset V(G^5)$. In G^5, consider the induced subgraph $F = \oplus_{i=1}^{n_1-1}\langle E(\{i,b\},\{i,i+1\})\rangle \cup \langle E(\{n_1,b\},\{1,n_1\})\rangle$, where $n_1+1 \le b \le n$. In G^6, observe that $D_{G^6}\{a,b\}$=7, for all $\{a,b\} \in X_{n_1,n_2}$. In $G^6 \cup F$, $D_{G^6 \cup F}\{a,b\}$=8, for all $\{a,b\} \in X_{n_1,n_2}$. So, we choose each $\{a,b\} \in X_{n_1,n_2}$ as a center vertex twice, to get an S_4-decomposition in $G^6 \cup F$. Note that, $D_{G^5 \setminus E(F)}\{a,b\}$=8, for all $\{a,b\} \in N_1$ and $D_{G^5 \setminus E(F)}\{a,b\}$=16, for all $\{a,b\} \in X_{n_1} \setminus N_1$. So, we can choose $\{a,b\} \in N_1$ as a center vertex twice and we choose $\{a,b\} \in X_{n_1} \setminus N_1$ as a center vertex four times, to get an S_4-decomposition in $G^5 \setminus E(F)$. $\square$

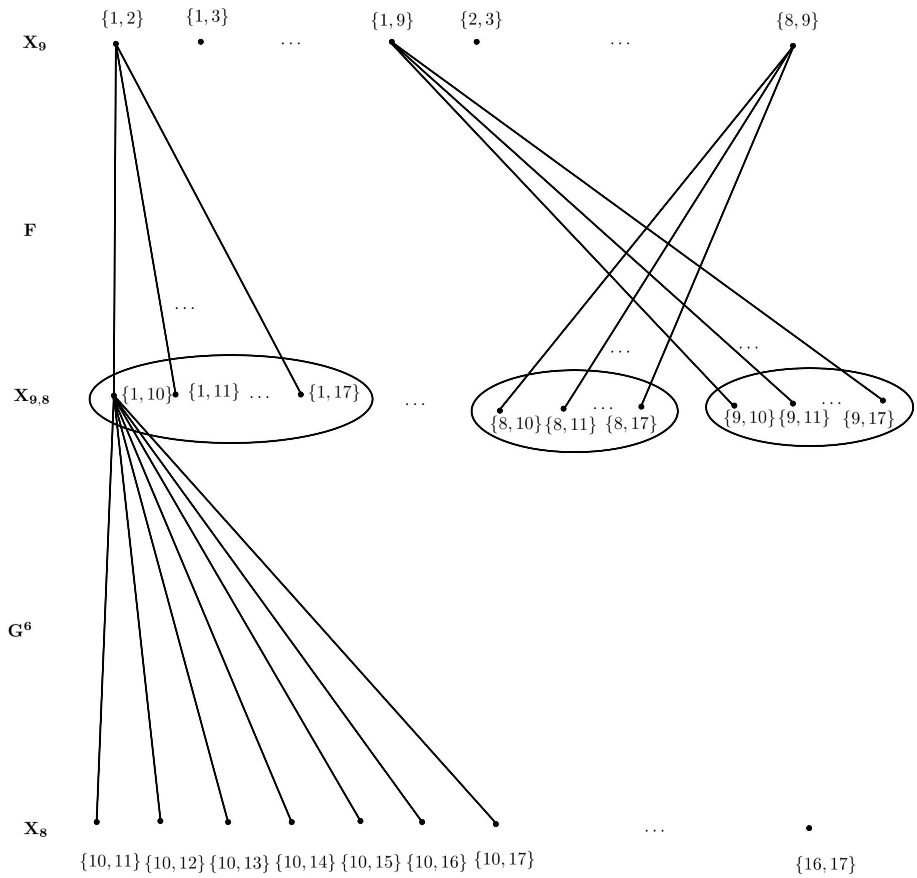

Fig. 2.2. The graph $G^6 \cup F$

Lemma 2.5. *If $n = 8m+2$, $m \geq 1$, then there exists an S_4-decomposition in $L(K_n)$.*

P r o o f. We split the proof into two cases.

Case (i): m=1. Let $A_1 = \{\{1, b\} | 2 \leq b \leq 10\}$ and $A_2 = \{\{a, b\} | 2 \leq a < b \leq 10\}$. Then $V(L(K_{10})) = A_1 \cup A_2$ and $E(L(K_{10})) = \langle E(A_1)\rangle \cup \langle E(A_2)\rangle \cup \langle E(A_1, A_2)\rangle$. The subgraph $\langle E(A_1)\rangle (\cong K_9)$ has an S_4-decomposition, by Theorem 1.1 and $\langle E(A_2)\rangle (\cong L(K_9))$ has an S_4-decomposition, by Lemma 2.4. In $\langle E(A_1, A_2)\rangle$, observe that $D_{\langle E(A_1,A_2)\rangle}\{a, b\}$=8, for all $\{a, b\} \in A_1$. We choose each $\{a, b\} \in A_1$ as a center vertex twice, to get an S_4-decomposition in $\langle E(A_1, A_2)\rangle$.

Case (ii): $m \geq 2$. Let $n_1 = 9 + 8(m-2)$ and n_2=9. The graphs G^1 and G^2 has an S_4-decomposition, by Lemma 2.4. By Theorem 1.1, there exists an S_4-decomposition in G^3 and G^4. In G^5, note that $D_{G^5}\{a,b\} = n_1 - 1$, for all $\{a,b\} \in X_{n_1,n_2}$ and note that $n_1 - 1 \equiv 0 \pmod 8$. We choose each $\{a,b\} \in X_{n_1,n_2}$ as a center vertex $\dfrac{n_1-1}{4}$ times, to get an S_4-decomposition in G^5. In G^6, note that $D_{G^6}\{a,b\}$=8, for all $\{a,b\} \in X_{n_1,n_2}$. We choose each $\{a,b\} \in X_{n_1,n_2}$ as a center vertex twice, to get an S_4-decomposition in G^6. □

Lemma 2.6. *If $n = 8m+4$, $m \geq 1$, then there exists an S_4-decomposition in $L(K_n)$.*

Proof. We split the proof into two cases.

Case (i): m=1. Let n_1=8 and n_2=4. The graph G^1 has an S_4-decomposition, by Lemma 2.3 and G^4 has an S_4-decomposition, by Theorem 1.1. In G^5, note that $D_{G^5}\{a,b\}$=8, for all $\{a,b\} \in X_8$. So, we can choose each $\{a,b\} \in X_8$ as a center vertex twice, to get an S_4-decomposition in G^5. Let $H' = G^2 \setminus \{(\{9,11\};\{9,10\},\{9,12\},\{10,11\},\{11,12\}) \cup (\{10,12\};\{10,11\},\{9,12\},\{9,10\},\{11,12\})\}$. Now, consider the stars from $H' \cup G^3 \cup G^6$ as follows:

$$S^i = \begin{cases} (\{i,9\};\{i,10\},\{i,11\},\{9,11\},\{9,12\}) & \text{if } 1 \leq i \leq 2 \\ (\{i,9\};\{i,10\},\{i,11\},\{9,10\},\{9,12\}) & \text{if } 3 \leq i \leq 6 \\ (\{i,9\};\{i,10\},\{i,11\},\{9,10\},\{9,11\}) & \text{if } 7 \leq i \leq 8 \\ (\{i-8,10\};\{i-8,11\},\{i-8,12\},\{10,11\},\{10,12\}) & \text{if } 9 \leq i \leq 10 \\ (\{i-8,10\};\{i-8,11\},\{i-8,12\},\{9,10\},\{10,12\}) & \text{if } 11 \leq i \leq 12 \\ (\{i-8,10\};\{i-8,11\},\{i-8,12\},\{9,10\},\{9,11\}) & \text{if } 13 \leq i \leq 16 \\ (\{i-16,11\};\{i-16,12\},\{9,11\},\{10,11\},\{11,12\}) & \text{if } 17 \leq i \leq 24 \\ (\{i-24,12\};\{i-24,9\},\{9,12\},\{10,12\},\{11,12\}) & \text{if } 25 \leq i \leq 32 \end{cases}$$

Let $H'' = H' \cup G^3 \cup G^6 \setminus E\left(\bigcup_{i=1}^{32} S^i\right)$. In H'', consider the two stars as follows: S' : $(\{9,12\};\{7,9\},\{8,9\},\{9,10\},\{11,12\})$ and S'' : $(\{10,11\};\{3,10\},\{4,10\},\{9,10\},\{11,12\})$. Let $H''' = H'' \setminus E(S' \cup S'')$. Note that, $D_{H'''}\{9,10\} = D_{H'''}\{10,11\} = D_{H'''}\{10,12\}$=4. So, we choose $\{\{9,10\},\{10,11\},\{10,12\}\} \in X_4$ as a center vertex, to get an S_4-decomposition in H'''. This completes the proof of Case (i).

Case (ii): $m \geq 2$. Let $n_1 = 8(m-1)$ and n_2=12. The graphs G^1 and G^2 has an S_4-decomposition, by Lemma 2.3 and Case (i). By Theorem 1.1, there exists an S_4-decomposition in G^4. In G^5, note that $D_{G^5}\{a,b\}$=24, for all $\{a,b\} \in X_{n_1}$. So, we can choose $\{a,b\} \in X_{n_1}$ as a center vertex six times to get an S_4-decomposition in G^5. By our observation, the graph $G^3 \cong n_1K_{12}$ with the vertex set $V(n_1K_{12}) = \{\{i,n_1+1\},\{i,n_1+2\},\dots,\{i,n\}|1 \le i \le n_1\}$ and $|V(n_1K_{12})| = 12n_1$. Now, we write $n_1K_{12} = n_1K_8 \cup n_1K_4 \cup n_1K_{8,4}$ with the vertex set as follows: $V(n_1K_8) = \{\{i,n_1+1\},\{i,n_1+2\},\dots,\{i,n_1+8\}|1 \le i \le n_1\}$, $V(n_1K_4) = \{\{i,n_1+9\},\{i,n_1+10\},\{i,n_1+11\},\{i,n\}|1 \le i \le n_1\}$ and $V(n_1K_{8,4}) = \{\{a,b\}|a \in V(n_1K_8), b \in V(n_1K_4)\}$. By Theorem 1.1, the graph K_8 has an S_4-decomposition. The graph $K_{8,4}$ is a simple, 4-regular bipartite graph, hence it has an S_4-decomposition. Now, let $H' = n_1K_{12} \setminus n_1[K_8 \cup K_{8,4}] = n_1K_4$. Now, the graph H' can be decomposed into 3 copies of stars with two edges as follows: For $1 \le i \le n_1$, $(\{i,n_1+9\};\{i,n_1+10\},\{i,n_1+11\})$, $(\{i,n_1+11\};\{i,n_1+10\},\{i,n_1+12\})$ and $(\{i,n_1+12\};\{i,n_1+9\},\{i,n_1+10\})$, see Fig. 2.3. Let $N \subset$

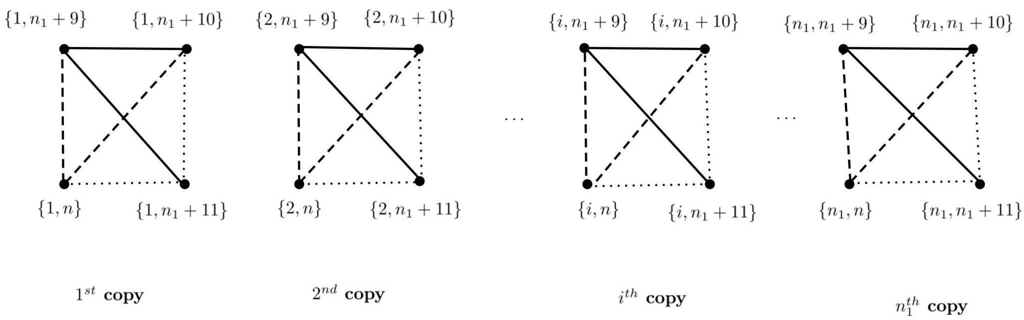

Fig. 2.3. The subgraph H' of n_1K_{12}

$X_{12} \subset V(G^6)$, where $N = \{\{n_1+9,n_1+11\},\{n_1+9,n_1+12\},\{n_1+11,n_1+12\}\}$. In $H' \cup G^6$, consider the 3 copies of stars with 4 edges as follows: For $1 \le i \le n_1$, S^1:$(\{i,n_1+9\};\{i,n_1+10\},\{i,n_1+11\}) \cup (\{i,n_1+9\};\{n_1+9,n_1+11\},\{n_1+9,n_1+12\})$, S^2:$(\{i,n_1+11\};\{i,n_1+10\},\{i,n_1+12\}) \cup (\{i,n_1+11\};\{n_1+9,n_1+11\},\{n_1+11,n_1+12\})$ and S^3:$(\{i,n_1+12\};\{i,n_1+9\},\{i,n_1+10\}) \cup (\{i,n_1+12\};\{n_1+9,n_1+12\},\{n_1+11,n_1+12\})$, see Fig. 2.4.

Remove the three set of stars S^1, S^2 and S^3 from $H' \cup G^6$, we denote it by H''. Note that $D_{H''}\{a,b\} = 2n_1$, for all $\{a,b\} \in X_{12} \setminus N$, $D_{H''}\{a,b\} = \emptyset$, for all $\{a,b\} \in N$ and note that $n_1 \equiv 0 \pmod 8$. Now, we choose $\{a,b\} \in X_{12} \setminus N$ as a center vertex $\dfrac{2n_1}{4}$ times to get an S_4-decomposition in H''. □

Lemma 2.7. *If $n = 8m+6$, $m \ge 1$, then there exists an S_4-decomposition in $L(K_n)$.*

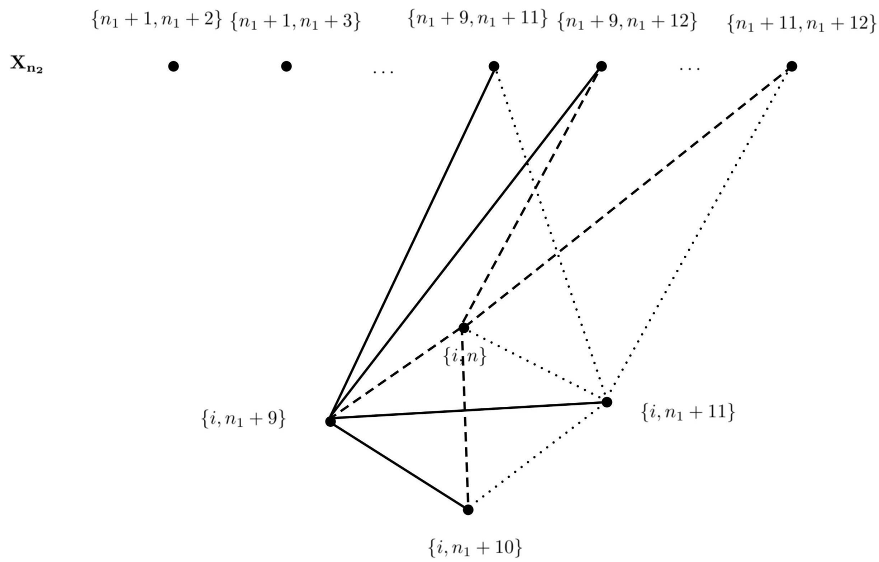

Fig. 2.4. The graph $H' \cup G^6$

Proof. Let $n_1 = 8m$ and n_2=6. The graphs G^1 and G^2 has an S_4-decomposition, by Lemma 2.2 and 2.3. By Theorem 1.1, there exists an S_4-decomposition in G^4. In G^5, note that $D_{G^5}\{a,b\}$=12, for all $\{a,b\} \in X_{n_1}$. We choose $\{a,b\} \in X_{n_1}$ as a center vertex three times to get an S_4-decomposition in G^5. Now, we show that the graph $G^3 \cup G^6$ has an S_4-decomposition. The graph G^3 can be decomposed into five copies of stars with three edges as follows: For $1 \le i \le n_1$, $(\{i, n_1+1\}; \{i, n_1+2\}, \{i, n_1+3\}, \{i, n_1+4\})$, $(\{i, n_1+3\}; \{i, n_1+2\}, \{i, n_1+4\}, \{i, n_1+5\})$, $(\{i, n_1+4\}; \{i, n_1+2\}, \{i, n_1+5\}, \{i, n\})$, $(\{i, n_1+$

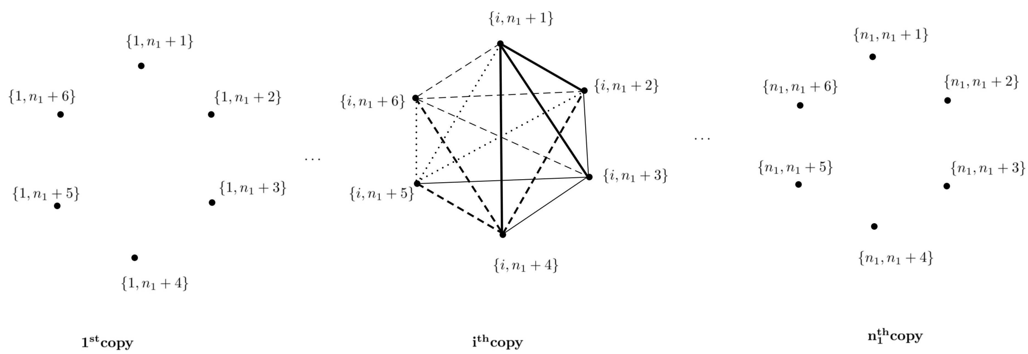

Fig. 2.5. The five copies of stars with 3 edges

$5\}; \{i, n_1+1\}, \{i, n_1+2\}, \{i, n\})$ and $(\{i, n\}; \{i, n_1+1\}, \{i, n_1+2\}, \{i, n_1+3\})$, see Fig. 2.5.

Let $B' \subset X_6 \subset V(G^6)$, where $B' = \{\{n_1+1, b\} | n_1+2 \le b \le n\}$. In $G^3 \cup G^6$, consider the five copies of stars with four edges as follows: For $1 \le i \le n_1$, S^1:$(\{i, n_1+1\}; \{i, n_1+2\}, \{i, n_1+3\}, \{i, n_1+4\}) \cup (\{i, n_1+1\}; \{n_1+1, n_1+2\})$, S^2:$(\{i, n_1+3\}; \{i, n_1+2\}, \{i, n_1+4\}, \{i, n_1+5\}) \cup (\{i, n_1+3\}; \{n_1+1, n_1+3\})$, S^3:$(\{i, n_1+4\}; \{i, n_1+2\}, \{i, n_1+5\}, \{i, n\}) \cup (\{i, n_1+4\}; \{n_1+1, n_1+4\})$, S^4:$(\{i, n_1+5\}; \{i, n_1+1\}, \{i, n_1+2\}, \{i, n\}) \cup (\{i, n_1+5\}; \{n_1+1, n_1+5\})$ and S^5:$(\{i, n\}; \{i, n_1+1\}, \{i, n_1+2\}, \{i, n_1+3\}) \cup (\{i, n\}; \{n_1+1, n\})$, see Fig. 2.6. Let

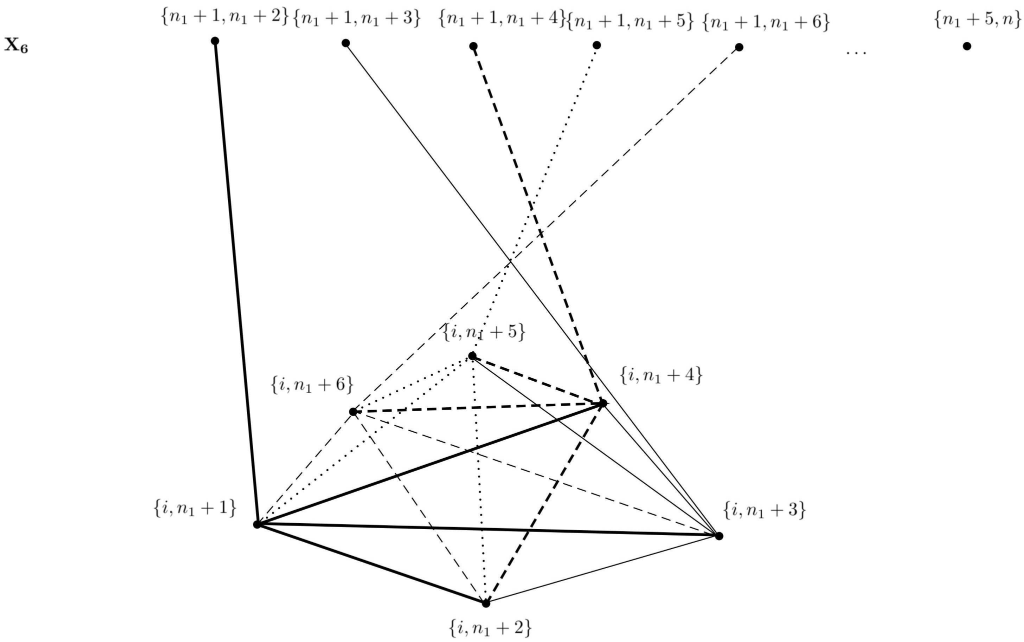

Fig. 2.6. The five copies of stars with 4 edges $E\left(\bigcup_{l=1}^{5} S^l\right)$ in $G^3 \cup G^6$

$H' = (G^3 \cup G^6) \setminus E\left(\bigcup_{l=1}^{5} S^l\right)$ be the subgraph obtained by removing $E\left(\bigcup_{l=1}^{5} S^l\right)$ from $G^3 \cup G^6$. In H', note that $D_{H'}\{a, b\} = n_1$, for all $\{a, b\} \in B'$, $D_{H'}\{a, b\} = 2n_1$, for all $\{a, b\} \in X_6 \setminus B'$ and note that $n_1 \equiv 0 \pmod 8$. So, we can choose $\{a, b\} \in B'$ as a center vertex $\frac{n_1}{4}$ times and we choose $\{a, b\} \in X_6 \setminus B'$ as a center vertex $\frac{2n_1}{4}$ times, to get an S_4-decomposition in H'. □

By combining the Lemmas 2.2 to 2.7, we get the following:

Theorem 2.1. *The graph* $L(K_n)$ *has an* S_4*-decomposition if and only if* $n \geq 6$ *and* $n \equiv 0, 1, 2, 4, 6 \pmod 8$.

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