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ON GENERALIZED HELICOIDAL MINIMAL SURFACES IN MINKOWSKI 5-SPACE

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Abstract. In this paper, we study two kinds of generalized helicoidal surfaces in Minkowski 5-space. We give the necessary and sufficient conditions for such surfaces to be a minimal surface, which are ordinary differential equations. We solve those equations explicitly and discuss the behavior of solutions.

1. Introduction. One of the well-known surfaces in differential geometry is the helicoid. Helicoidal surfaces appear as a generalization of rotational surfaces. These surfaces are invariant by a subgroup of the group of isometries of the ambient space, called helicoidal group whose elements can be seen as a composition of a translation with a rotation for a given axis. In [4], in Euclidean 3-space, the space of all helicoidal surfaces with constant mean curvatures or constant Gaussian curvatures were studied. This space behaves as a circular cylinder, where a given generator corresponds to the rotational surfaces and each

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parallel corresponds to a periodic family of helicoidal surfaces. In [2], the authors studied the cases with prescribed mean curvature or Gauss curvature.

Many researchers studied helicoidal surfaces in different spaces. In [6], under the cubic screw motion, the linear Weingarten helicoidal surfaces in Minkowski 3-space were constructed. In [5], the authors constructed a helicoidal surface with a light-like axis with prescribed mean curvature or Gauss curvature given by smooth function in Minkowski 3-space and solved an open problem left in [3]. Also, in [7], the authors classify all helicoidal non-degenerate surfaces in Minkowski 3-space with constant mean curvature whose generating curve is the graph of a polynomial or a Lorentzian circle.

Besides, in [1], the rotational surfaces in higher dimensional Euclidean spaces were studied. Some results related with the curvature properties of these surfaces were obtained. Also they give examples of rotational surfaces in Euclidean 5-space.

Lastly, in [8], we studied generalized helicoidal surfaces in Euclidean 5-space. We obtained the necessary and sufficient conditions for generalized helicoidal surfaces in Euclidean 5-space to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solved those equations and discussed the completeness of the surfaces.

Let $\mathbb{R}_1^5$ be the 5-dimensional Minkowski space with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2.$$

In the previous paper [9], we studied the generalized helicoidal surfaces in $\mathbb{R}_1^5$ parametrized by

$$M: \quad F(t,u) = (\alpha(t)\cos u, \alpha(t)\sin u, \beta(t)\cos u, \beta(t)\sin u, u) \tag{1.1}$$

where α and β are smooth functions satisfying

$$\alpha^2 + \beta^2 > 0, \quad (\alpha')^2 + (\beta')^2 > 0 \quad \text{and} \quad \alpha^2 + \beta^2 - 1 \neq 0.$$

In this paper, we consider other two kinds of the generalized helicoidal surfaces in $\mathbb{R}_1^5$. We call these surfaces the second kind of the generalized helicoidal surfaces and the third kind of the generalized helicoidal surfaces. The second kind of the generalized helicoidal surfaces in $\mathbb{R}_1^5$ is parametrized by

$$M_2: \quad F(t,u) = (u, \alpha(t)\cos u, \alpha(t)\sin u, \beta(t)\cosh u, \beta(t)\sinh u) \tag{1.2}$$

where α and β are smooth functions satisfying

$$\alpha^2 - \beta^2 \neq 0, \quad (\alpha')^2 + (\beta')^2 > 0 \quad \text{and} \quad 1 + \alpha^2 - \beta^2 \neq 0.$$

The third kind of the generalized helicoidal surfaces in $\mathbb{R}^5_1$ is parametrized by

$$(1.3) \qquad M_3: \quad F(t,u) = (u, \alpha(t)\cos u, \alpha(t)\sin u, \beta(t)\sinh u, \beta(t)\cosh u)$$

where α and β are smooth functions satisfying

$$\alpha^2 + \beta^2 > 0 \quad \text{and} \quad (\alpha')^2 - (\beta')^2 \neq 0.$$

We give the necessary and sufficient conditions for M_2 and M_3 to be a minimal surface, which are ordinary differential equations. We solve those equations explicitly and discuss the behavior of solutions.

2. Preliminaries. Let $\mathbb{R}^n_q$ be the n-dimensional semi-Euclidean space of index q with inner product $\langle\ ,\ \rangle$ and flat connection D. Let M be a semi-Riemannian submanifold in $\mathbb{R}^n_q$. According to the decomposition

$$\mathbb{R}^n_q|_M = TM \perp TM^\perp,$$

we have

$$D_X Y = \nabla_X Y + h(X,Y),$$

and

$$D_X \xi = -A_\xi X + {}^\perp\nabla_X\, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^\perp\right)$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^\perp\nabla$ is the normal connection. We note that

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

Let $\mathbb{R}^5_1$ be the 5-dimensional Minkowski space with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2.$$

In the following, we assume that M is a surface in $\mathbb{R}^5_1$. We use the following convention on the ranges of indices:

$$1 \leq A, B \ldots \leq 5, \quad 1 \leq i, j \ldots \leq 2, \quad 3 \leq \alpha, \beta \ldots \leq 5.$$

Let $\{e_i\}$ be a local orthonormal frame field on M and $\{e_\alpha\}$ be a normal orthonormal frame field to M. Let $\varepsilon_A = \langle e_A, e_A\rangle = \pm 1$. Set

$$h^\alpha_{ij} = \varepsilon_\alpha \langle h(e_i, e_j), e_\alpha\rangle$$

which are the components of the second fundamental form h.

The mean curvature vector H of M is given by

$$H = \frac{1}{2}\sum_{\alpha} \left(\varepsilon_1 h_{11}^{\alpha} + \varepsilon_2 h_{22}^{\alpha}\right) e_{\alpha}.$$

A surface is called minimal if $H = 0$ identically.

3. The second kind of generalized helicoidal surfaces in $\mathbb{R}_1^5$.

In this section, we consider the second kind of generalized helicoidal surface M_2 parametrized by (1.2). Then we have

$$\begin{aligned} F_t &= \left(0, \alpha'(t)\cos u, \alpha'(t)\sin u, \beta'(t)\cosh u, \beta'(t)\sinh u\right), \\ F_u &= \left(1, -\alpha(t)\sin u, \alpha(t)\cos u, \beta(t)\sinh u, \beta(t)\cosh u\right) \end{aligned}$$

and

$$\langle F_t, F_t\rangle = \left(\alpha'(t)\right)^2 + \left(\beta'(t)\right)^2, \quad \langle F_t, F_u\rangle = 0, \quad \langle F_u, F_u\rangle = 1 + \alpha^2(t) - \beta^2(t).$$

Then we can choose the followings:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} F_t \\ &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} \left(0, \alpha'\cos u, \alpha'\sin u, \beta'\cosh u, \beta'\sinh u\right), \\ e_2 &= \frac{1}{\sqrt{\varepsilon_1\left(1 + \alpha^2 - \beta^2\right)}} F_u \\ &= \frac{1}{\sqrt{\varepsilon_1\left(1 + \alpha^2 - \beta^2\right)}} \left(1, -\alpha\sin u, \alpha\cos u, \beta\sinh u, \beta\cosh u\right), \\ e_3 &= \frac{1}{\sqrt{\varepsilon_2\left(\alpha^2 - \beta^2\right)}} \left(0, \beta\sin u, -\beta\cos u, -\alpha\sinh u, -\alpha\cosh u\right), \\ e_4 &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} \left(0, -\beta'\cos u, -\beta'\sin u, \alpha'\cosh u, \alpha'\sinh u\right), \\ e_5 &= \frac{1}{\sqrt{\varepsilon_2\left(\alpha^2 - \beta^2\right)}\sqrt{\varepsilon_1\left(1 + \alpha^2 - \beta^2\right)}} (\alpha^2 - \beta^2, \alpha\sin u, -\alpha\cos u, \\ &\qquad -\beta\sinh u, -\beta\cosh u) \end{aligned}$$

where $\varepsilon_1 = \operatorname{sgn}\left(1+\alpha^2-\beta^2\right)$ and $\varepsilon_2 = \operatorname{sgn}\left(\alpha^2-\beta^2\right)$. Here $\{e_1, e_2\}$ is an orthonormal frame field on M_2 with sign $(+, \varepsilon_1)$ and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame field to M_2 with sign $(-\varepsilon_2, +, \varepsilon_1\varepsilon_2)$. We note that $\varepsilon_2 = -1$ when $\varepsilon_1 = -1$.

Also we can easily obtain that

$$
\begin{aligned}
D_{e_1}e_1 &= \frac{(\beta'\alpha''-\alpha'\beta'')}{\left((\alpha')^2+(\beta')^2\right)^2}\left(0, \beta'\cos u, \beta'\sin u, -\alpha'\cosh u, -\alpha'\sinh u\right),\\
D_{e_2}e_1 &= \frac{1}{\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\varepsilon_1\left(1+\alpha^2-\beta^2\right)}}(0, -\alpha'\sin u, \alpha'\cos u,\\
&\qquad \beta'\sinh u, \beta'\cosh u),\\
D_{e_2}e_2 &= \frac{1}{\varepsilon_1\left(1+\alpha^2-\beta^2\right)}\left(0, -\alpha\cos u, -\alpha\sin u, \beta\cosh u, \beta\sinh u\right).
\end{aligned}
$$

The components of the second fundamental form h of M_2 are given as follows

$$
\begin{aligned}
h^4_{11} &= \frac{-\beta'\alpha''+\alpha'\beta''}{\left((\alpha')^2+(\beta')^2\right)^{3/2}},\\
h^3_{12} &= \frac{\varepsilon_2\left(\beta\alpha'-\alpha\beta'\right)}{\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\varepsilon_2\left(\alpha^2-\beta^2\right)}\sqrt{\varepsilon_1\left(1+\alpha^2-\beta^2\right)}},\\
h^5_{12} &= \frac{-\varepsilon_2\left(\alpha\alpha'-\beta\beta'\right)}{\left(1+\alpha^2-\beta^2\right)\sqrt{(\alpha')^2+(\beta')^2}\sqrt{\varepsilon_2\left(\alpha^2-\beta^2\right)}},\\
h^4_{22} &= \frac{\beta\alpha'+\alpha\beta'}{\varepsilon_1\left(1+\alpha^2-\beta^2\right)\sqrt{(\alpha')^2+(\beta')^2}},\\
h^3_{11} &= h^5_{11} = h^4_{12} = h^3_{22} = h^5_{22} = 0.
\end{aligned}
$$

Then we get the following theorem and corollary.

Theorem 1. *Let M_2 be a generalized helicoidal surface of the second kind parametrized by* (1.2)*. Then the mean curvature vector H of M_2 is given by*

$$
H = \frac{\left((\alpha')^2+(\beta')^2\right)\left(\beta\alpha'+\alpha\beta'\right)+\left(\alpha'\beta''-\beta'\alpha''\right)\left(1+\alpha^2-\beta^2\right)}{2\left(1+\alpha^2-\beta^2\right)\left((\alpha')^2+(\beta')^2\right)^{3/2}}e_4.
$$

Corollary 1. *Let M_2 be a generalized helicoidal surface of the second kind parametrized by* (1.2)*. Then M_2 is minimal if and only if*

$$\left((\alpha')^2+(\beta')^2\right)(\beta\alpha'+\alpha\beta')+(\alpha'\beta''-\beta'\alpha'')\left(1+\alpha^2-\beta^2\right)=0. \tag{3.1}$$

Let $\beta(t)=t$ in the equation (3.1). Then the minimal surface equation is

$$\left(1+\alpha^2-t^2\right)\alpha''-(t\alpha'+\alpha)\left((\alpha')^2+1\right)=0. \tag{3.2}$$

Multiplying (3.2) by $-2\alpha'/\left((\alpha')^2+1\right)^2$, we can get

$$\left(t^2\frac{(\alpha')^2}{(\alpha')^2+1}\right)'+\left(\frac{\alpha^2+1}{(\alpha')^2+1}\right)'=0.$$

Thus we have

$$t^2\frac{(\alpha')^2}{(\alpha')^2+1}+\frac{\alpha^2+1}{(\alpha')^2+1}=c_1$$

for a positive constant c_1. Then we get

$$\left(c_1-t^2\right)(\alpha')^2=\alpha^2+1-c_1 \tag{3.3}$$

or

$$\left(\alpha^2+1-c_1\right)(t')^2=c_1-t^2,\quad t'=\frac{dt}{d\alpha}. \tag{3.4}$$

So we have two cases (a) and (b) as

$$\begin{aligned}(a)\ \alpha^2+1-c_1 &> 0 \quad\text{and}\quad c_1-t^2>0,\\ (b)\ \alpha^2+1-c_1 &< 0 \quad\text{and}\quad c_1-t^2<0.\end{aligned}$$

In case (a), We have $\alpha^2+1-t^2>0$, so that M_2 is spacelike. In case (b), we have $c_1>1$ and $\alpha^2+1-t^2<0$, so that M_2 is timelike.

(a) Firstly, we consider the case that M_2 is spacelike. Then we have

$$\frac{d\alpha}{\sqrt{\alpha^2+1-c_1}}=\pm\frac{dt}{\sqrt{c_1-t^2}}. \tag{3.5}$$

Changing t to $-t$, we may only consider the $(+)$ case if necessary.

(a-1) When $c_1 = 1$, we have

$$\frac{d\alpha}{\alpha} = \frac{dt}{\sqrt{1-t^2}}$$

Integrating it, we have

$$\log|\alpha| = \arcsin t + c_2$$

for a constant c_2. On the extendibility, we can see that

$$t(\alpha) = \sin(\log|\alpha| - c_2)$$

is defined for $\alpha \neq 0$ and satisfies (3.4) for $c_1 = 1$.

(a-2) When $c_1 > 1$, integrating the equation (3.5), we have

$$\operatorname{arccosh}\left(\frac{\alpha}{\sqrt{c_1-1}}\right) = \pm\arcsin\left(\frac{t}{\sqrt{c_1}}\right) + c_2$$

for a constant c_2. Thus we get

$$\alpha(t) = \sqrt{c_1-1}\cosh\left(\pm\arcsin\left(\frac{t}{\sqrt{c_1}}\right) + c_2\right).$$

We can see that

$$t(\alpha) = \pm\sqrt{c_1}\sin\left(\operatorname{arccosh}\left(\frac{\alpha}{\sqrt{c_1-1}}\right) - c_2\right)$$

is defined for $\alpha > \sqrt{c_1-1}$ and satisfies (3.4).

On the extendibility, we can choose $c_2 = 0$ and let

$$t_{\pm}(\alpha) = \pm\sqrt{c_1}\sin\left(\operatorname{arccosh}\left(\frac{\alpha}{\sqrt{c_1-1}}\right)\right).$$

We note that

$$\lim_{\alpha\to\sqrt{c_1-1}} t_{\pm}(\alpha) = 0, \qquad \lim_{\alpha\to\sqrt{c_1-1}} t'_{\pm}(\alpha) = \pm\infty.$$

The graphs of $t_+(\alpha)$ and $t_-(\alpha)$ can be connected at the point $(\alpha, t) = \left(\sqrt{c_1-1}, 0\right)$ as a C^1 regular curve. Let $\alpha_+(t)$ and $\alpha_-(t)$ denote the inverse functions of $t_+(\alpha)$ and $t_-(\alpha)$ near the point $(\alpha, t) = \left(\sqrt{c_1-1}, 0\right)$, respectively. Connecting the graphs of $\alpha_+(t)$ and $\alpha_-(t)$, we get a C^1 function $\widetilde{\alpha}(t)$ for $t \in (-\delta, \delta)$. By (3.2), it satisfies

$$\left(1 + \widetilde{\alpha}^2 - t^2\right)\widetilde{\alpha}'' - \left(t\widetilde{\alpha}' + \widetilde{\alpha}\right)\left(\left(\widetilde{\alpha}'\right)^2 + 1\right) = 0$$

for $t \in (-\delta, \delta)$, $t \neq 0$. From it, we have

$$\lim_{t \to 0} \widetilde{\alpha}''(t) = \frac{\sqrt{c_1 - 1}}{c_1}.$$

Hence the graphs of $t_+(\alpha)$ and $t_-(\alpha)$ can be connected as a C^2 regular curve.

(a-3) When $0 < c_1 < 1$, integrating the equation (3.5) for the (+) case, we have

$$\operatorname{arcsinh}\left(\frac{\alpha}{\sqrt{1 - c_1}}\right) = \arcsin\left(\frac{t}{\sqrt{c_1}}\right) + c_2$$

for a constant c_2. Thus we get

$$\alpha(t) = \sqrt{1 - c_1} \sinh\left(\arcsin\left(\frac{t}{\sqrt{c_1}}\right) + c_2\right).$$

We can see that

$$t(\alpha) = \sqrt{c_1} \sin\left(\operatorname{arcsinh}\left(\frac{\alpha}{\sqrt{1 - c_1}}\right) - c_2\right)$$

is defined for any $\alpha \in \mathbb{R}$ and satisfies (3.4).

(b) Now, we consider the case that M_2 is timelike. Then we have

$$\frac{d\alpha}{\sqrt{c_1 - 1 - \alpha^2}} = \pm \frac{dt}{\sqrt{t^2 - c_1}}.$$

Integrating it, we get

$$\arcsin\left(\frac{\alpha}{\sqrt{c_1 - 1}}\right) = \pm \operatorname{arccosh}\left(\frac{t}{\sqrt{c_1}}\right) + c_2$$

for a constant c_2. We can see that

$$\alpha(t) = \pm\sqrt{c_1 - 1} \sin\left(\operatorname{arccosh}\left(\frac{t}{\sqrt{c_1}}\right) + c_3\right)$$

for a constant c_3, is defined for $t > \sqrt{c_1}$ and satisfies (3.3).

We can choose $c_3 = 0$ and let

$$\alpha_\pm(t) = \pm\sqrt{c_1 - 1} \sin\left(\operatorname{arccosh}\left(\frac{t}{\sqrt{c_1}}\right)\right).$$

We note that

$$\lim_{t \to \sqrt{c_1}} \alpha_\pm(t) = 0, \quad \lim_{t \to \sqrt{c_1}} \alpha'_\pm(t) = \pm\infty.$$

The graphs of $\alpha_+ (t)$ and $\alpha_- (t)$ can be connected at the point $(t, \alpha) = (\sqrt{c_1}, 0)$ as a C^1 regular curve. Let $t_+ (\alpha)$ and $t_- (\alpha)$ be the inverse functions of $\alpha_+ (t)$ and $\alpha_- (t)$ near the point $(t, \alpha) = (\sqrt{c_1}, 0)$, respectively. Connecting the graphs of $t_+ (\alpha)$ and $t_- (\alpha)$, we get a C^1 function $\tilde{t} (\alpha)$ for $\alpha \in (-\delta, \delta)$. It satisfies (3.1) where α is the parameter, $\beta = \tilde{t} (\alpha)$ and $\alpha \neq 0$. So

$$\left(1 + \alpha^2 - \tilde{t}^2\right) \tilde{t}'' + \left(1 + \left(\tilde{t}'\right)^2\right) \left(\tilde{t} + \alpha \tilde{t}'\right) = 0$$

for $\alpha \in (-\delta, \delta)$, $\alpha \neq 0$. From it, we have

$$\lim_{\alpha \to 0} \tilde{t}'' (\alpha) = \frac{\sqrt{c_1}}{c_1 - 1}.$$

Hence the graphs of $\alpha_+ (t)$ and $\alpha_- (t)$ can be connected as a C^2 regular curve.

Thus we can give the following corollary.

Corollary 2. *The solution of the minimal surface equation* (3.2) *is given by one of the followings*
(i) for a constant c_2,

$$\alpha (t) = \pm e^{\arcsin t + c_2}.$$

In this case, the surface M_2 *is a spacelike minimal surface.*
(ii) for constants $c_1 > 1$ *and* c_2,

$$\alpha (t) = \sqrt{c_1 - 1} \cosh \left(\pm \arcsin \left(\frac{t}{\sqrt{c_1}} \right) + c_2 \right).$$

In this case, the surface M_2 *is a spacelike minimal surface.*
(iii) for constants $0 < c_1 < 1$ *and* c_2,

$$\alpha (t) = \sqrt{1 - c_1} \sinh \left(\arcsin \left(\frac{t}{\sqrt{c_1}} \right) + c_2 \right).$$

In this case, the surface M_2 *is a spacelike minimal surface.*
(iv) for constants $c_1 > 0$ *and* c_2,

$$\alpha (t) = \sqrt{c_1 - 1} \sin \left(\operatorname{arccosh} \left(\frac{t}{\sqrt{c_1}} \right) + c_2 \right).$$

In this case, the surface M_2 *is a timelike minimal surface.*

4. The third kind of generalized helicoidal surfaces in $\mathbb{R}^5_1$. In this section, we consider the third kind of generalized helicoidal surface M_3 parametrized by (1.3). Then we have

$$\begin{aligned} F_t &= \left(0, \alpha'(t)\cos u, \alpha'(t)\sin u, \beta'(t)\sinh u, \beta'(t)\cosh u\right), \\ F_u &= (1, -\alpha(t)\sin u, \alpha(t)\cos u, \beta(t)\cosh u, \beta(t)\sinh u) \end{aligned}$$

and

$$\langle F_t, F_t\rangle = \left(\alpha'(t)\right)^2 - \left(\beta'(t)\right)^2, \quad \langle F_t, F_u\rangle = 0, \quad \langle F_u, F_u\rangle = 1 + \alpha^2(t) + \beta^2(t).$$

Then we can choose the followings:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{\varepsilon_3\left((\alpha')^2 - (\beta')^2\right)}} F_t \\ &= \frac{1}{\sqrt{\varepsilon_3\left((\alpha')^2 - (\beta')^2\right)}}\left(0, \alpha'\cos u, \alpha'\sin u, \beta'\sinh u, \beta'\cosh u\right), \\ e_2 &= \frac{1}{\sqrt{1+\alpha^2+\beta^2}} F_u \\ &= \frac{1}{\sqrt{1+\alpha^2+\beta^2}}(1, -\alpha\sin u, \alpha\cos u, \beta\cosh u, \beta\sinh u), \\ e_3 &= \frac{1}{\sqrt{\alpha^2+\beta^2}}(0, \beta\sin u, -\beta\cos u, \alpha\cosh u, \alpha\sinh u), \\ e_4 &= \frac{1}{\sqrt{\varepsilon_3\left((\alpha')^2 - (\beta')^2\right)}}\left(0, -\beta'\cos u, -\beta'\sin u, -\alpha'\sinh u, -\alpha'\cosh u\right), \\ e_5 &= \frac{1}{\sqrt{\alpha^2+\beta^2}\sqrt{1+\alpha^2+\beta^2}}\left(\alpha^2+\beta^2, \alpha\sin u, -\alpha\cos u, -\beta\cosh u, -\beta\sinh u\right) \end{aligned}$$

where $\varepsilon_3 = \operatorname{sgn}\left((\alpha')^2 - (\beta')^2\right)$. Here $\{e_1, e_2\}$ is an orthonormal frame field on M_3 with sign $(\varepsilon_3, +)$ and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame field to M_3 with sign $(+, -\varepsilon_3, +)$.

Also we can easily obtain that

$$D_{e_1}e_1 = \frac{\varepsilon_3(\alpha'\beta'' - \beta'\alpha'')}{\left((\alpha')^2 - (\beta')^2\right)^2}\left(0, \beta'\cos u, \beta'\sin u, \alpha'\sinh u, \alpha'\cosh u\right),$$

$$D_{e_2}e_1 = \frac{1}{\sqrt{\varepsilon_3\left((\alpha')^2-(\beta')^2\right)}\sqrt{1+\alpha^2+\beta^2}}(0,-\alpha'\sin u,\alpha'\cos u,$$
$$\beta'\cosh u,\beta'\sinh u),$$
$$D_{e_2}e_2 = \frac{1}{1+\alpha^2+\beta^2}\left(0,-\alpha\cos u,-\alpha\sin u,\beta\sinh u,\beta\cosh u\right).$$

The components of the second fundamental form h of M_3 are given as follows

$$h_{11}^4 = \frac{\varepsilon_3\left(\beta'\alpha''-\alpha'\beta''\right)}{\left[\varepsilon_3\left((\alpha')^2-(\beta')^2\right)\right]^{3/2}},$$
$$h_{12}^3 = \frac{\alpha\beta'-\beta\alpha'}{\sqrt{\varepsilon_3\left((\alpha')^2-(\beta')^2\right)}\sqrt{\alpha^2+\beta^2}\sqrt{1+\alpha^2+\beta^2}},$$
$$h_{12}^5 = \frac{-\left(\alpha\alpha'+\beta\beta'\right)}{\left(1+\alpha^2+\beta^2\right)\sqrt{\varepsilon_3\left((\alpha')^2-(\beta')^2\right)}\sqrt{\alpha^2+\beta^2}},$$
$$h_{22}^4 = \frac{-\varepsilon_3\left(\beta\alpha'+\alpha\beta'\right)}{\left(1+\alpha^2+\beta^2\right)\sqrt{\varepsilon_3\left((\alpha')^2-(\beta')^2\right)}},$$
$$h_{11}^3 = h_{11}^5=h_{12}^4=h_{22}^3=h_{22}^5=0.$$

Then we get the following theorem and corollary.

Theorem 2. *Let M_3 be a generalized helicoidal surface of the third kind parametrized by* (1.3)*. Then the mean curvature vector H of M_3 is given by*

$$H=-\frac{\left((\alpha')^2-(\beta')^2\right)\left(\beta\alpha'+\alpha\beta'\right)+\left(\alpha'\beta''-\beta'\alpha''\right)\left(1+\alpha^2+\beta^2\right)}{2\left(1+\alpha^2+\beta^2\right)\left[\varepsilon_3\left((\alpha')^2-(\beta')^2\right)\right]^{3/2}}e_4.$$

Corollary 3. *Let M_3 be a generalized helicoidal surface of the third kind parametrized by* (1.3)*. Then M_3 is minimal if and only if*

$$\left((\alpha')^2-(\beta')^2\right)\left(\beta\alpha'+\alpha\beta'\right)+\left(\alpha'\beta''-\beta'\alpha''\right)\left(1+\alpha^2+\beta^2\right)=0. \tag{4.1}$$

Let $\beta(t)=t$ in the equation (4.1). Then the minimal surface equation is

$$\left(1+\alpha^2+t^2\right)\alpha''+\left(t\alpha'+\alpha\right)\left(1-(\alpha')^2\right)=0. \tag{4.2}$$

Multiplying (4.2) by $2\alpha'/\left(1-(\alpha')^2\right)^2$, we can get

$$\left(t^2\frac{(\alpha')^2}{1-(\alpha')^2}\right)'+\left(\frac{\alpha^2+1}{1-(\alpha')^2}\right)'=0.$$

Thus we have

$$t^2\frac{(\alpha')^2}{1-(\alpha')^2}+\frac{\alpha^2+1}{1-(\alpha')^2}=c_1$$

for a constant c_1. Then

$$\left(c_1+t^2\right)\left(\alpha'\right)^2=c_1-1-\alpha^2 \tag{4.3}$$

or

$$\left(c_1-1-\alpha^2\right)\left(t'\right)^2=c_1+t^2,\quad t'=\frac{dt}{d\alpha}. \tag{4.4}$$

So we have two cases (a) and (b) as

$$\begin{array}{lcl} (a)\ c_1-1-\alpha^2 & > & 0 \quad \text{and} \quad c_1+t^2>0, \\ (b)\ c_1-1-\alpha^2 & < & 0 \quad \text{and} \quad c_1+t^2<0. \end{array}$$

In case (a), we have $c_1>1$ and $\left(\alpha'\right)^2<1$, so that M_3 is timelike. In case (b), we have $c_1<0$ and $\left(\alpha'\right)^2>1$, so that M_3 is spacelike.

(a) Firstly, we consider the case that $c_1-1-\alpha^2>0$ and $c_1+t^2>0$. Then we have $c_1>1$ and

$$\frac{d\alpha}{\sqrt{c_1-1-\alpha^2}}=\pm\frac{dt}{\sqrt{c_1+t^2}}. \tag{4.5}$$

Integrating the equation (4.5) by considering $(+)$ case, we have

$$\arcsin\left(\frac{\alpha}{\sqrt{c_1-1}}\right)=\operatorname{arcsinh}\left(\frac{t}{\sqrt{c_1}}\right)+c_2$$

for a constant c_2. We can see that

$$\alpha(t)=\sqrt{c_1-1}\sin\left(\operatorname{arcsinh}\left(\frac{t}{\sqrt{c_1}}\right)+c_2\right)$$

is defined for any $t\in\mathbb{R}$, and satisfies (4.3).

(b) Now, we consider the case that $c_1 - 1 - \alpha^2 < 0$ and $c_1 + t^2 < 0$. Then we have $c_1 < 0$. Let c_2 is a positive constant such that $c_2 = -c_1$. So we get

$$\frac{d\alpha}{\sqrt{\alpha^2 + 1 + c_2}} = \pm \frac{dt}{\sqrt{c_2 - t^2}}.$$

Integrating it by considering $(+)$ case, we get

$$\operatorname{arcsinh}\left(\frac{\alpha}{\sqrt{c_2 + 1}}\right) = \arcsin\left(\frac{t}{\sqrt{c_2}}\right) + c_3$$

for a constant c_3. So we have

$$\alpha(t) = \sqrt{c_2 + 1}\sinh\left(\arcsin\left(\frac{t}{\sqrt{c_2}}\right) + c_3\right).$$

On the extendibility, we can see that

$$t(\alpha) = \sqrt{c_2}\sin\left(\operatorname{arcsinh}\left(\frac{\alpha}{\sqrt{c_2 + 1}}\right) - c_3\right)$$

is defined for any $\alpha \in \mathbb{R}$ and satisfies (4.4).

Corollary 4. *The solution of the minimal surface equation* (4.2) *is given by one of the followings*
(i) *for constants* $c_1 > 1$ *and* c_2,

$$\alpha(t) = \sqrt{c_1 - 1}\sin\left(\operatorname{arcsinh}\left(\frac{t}{\sqrt{c_1}}\right) + c_2\right).$$

In this case, M_3 *is timelike.*
(ii) *for constants* $c_2 > 0$ *and* c_3,

$$\alpha(t) = \sqrt{c_2 + 1}\sinh\left(\arcsin\left(\frac{t}{\sqrt{c_2}}\right) + c_3\right).$$

In this case, M_3 *is spacelike.*

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