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THE SOLUTION TO THE NON-SYMMETRIC COMPLETION PROBLEM OF QUASI-UNIFORM SPACES

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Dedicated to the memory of Professor Doitchin Doitchinov

Abstract. In this paper, we give a solution to the non-symmetric completion problem of quasi-uniform spaces. The proposed completion theory extends the uniform space completion theory and is also exceedingly well-behaved in the sense that it satisfies all of the requirements posed in the literature for a nice completion. The main contribution of this completion theory is the concept of the cut of nets, which is a common generalization of Doitchinov's D-Cauchy net and the Eudoxus-Dedekind-MacNeille cut.

1. Introduction. The completeness and completion problem of quasi-uniform spaces have primarily been considered in [1, 3, 5, 6, 9, 10, 11, 12, 14, 17, 19, 30, 31, 32]. A satisfactory extension of uniform space completion theory to arbitrary quasi-uniform spaces leads to Császár's double completeness, developed in the realm of syntopogenous spaces. In this direction, Fletcher and Lindgren [19]

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introduced the concept of bicompleteness and shown that any quasi-uniform space has a bicompletion, known as standard bicompletion. It turns out that this concept corresponds to the concept of double completeness for quasi-uniform spaces. Császár [9] and Deák [13] have introduced the notion of half-completeness in quasi-uniform spaces which generalizes the well known notion of bicompleteness. Andrikopoulos [4] constructed a half-completion, called standard half-completion. Because the idea underlying the bicompletion and half completion is symmetric, various authors have attempted to construct other non-symmetric completions as a result of the asymmetric nature of quasi-uniform spaces. This effort, however, did not bear fruit. And this because by definition the notion of completeness of a quasi-uniform space as well as the construction of the completion, depends on the choice of the definition of Cauchy net or Cauchy filter (often nets and filters lead to equivalent theories). Therefore, the failure to create a good asymmetric completion theory for the category of all quasi-uniform spaces results from the difficulty of extending the concept of the Cauchy net (Cauchy filter) from uniform to quasi-uniform spaces. More specifically, because uniform spaces are included in the class of quasi-uniform spaces, the notion of a Cauchy net in any quasi-uniform space must be defined in a way that satisfies the requirements that convergent nets are Cauchy and agree with the usual definition for uniform spaces. Furthermore, the proposed completion must be a monotone operator with respect to set inclusion and produce the usual uniform completion in the uniform case. Császár [9], Sieber and Pervin [31], and Stoltenberg [32] were the first to address the problem of defining Cauchy nets or filters in quasi-uniform spaces. Császár [9] introduced the notion of Cauchy filter in a quasi-uniform space and proved that every syntopogenous space can be embedded in a complete space. Stoltenberg [32] also gave a definition of Cauchy net in quasi-uniform spaces, which generalizes the definition of Kelly [23] for Cauchy sequences in quasi-metric spaces. The definition of Sieber and Pervin has been used by many authors and is becoming more widely accepted as the most appropriate way to generalize the concept of the Cauchy net in quasi-uniform spaces. On the other hand, there are several generalizations to the notion of Cauchy sequence (net) based on the definitions of Császár, Stoltenberg and Sieber and Pervin, but none of these generalizations can provide a satisfying completion theory for all quasi-uniform spaces. This problem has been studied in [29], where seven different notions of "Cauchy sequences (nets)"are presented. As a result of the existence of many definitions of Cauchyness, there exist many notions of quasi-uniform completeness in the literature. More precisely, by combining the seven definitions of "Cauchy sequence (net)"with the topologies $\tau_{\mathcal{U}}$, and $\tau_{\mathcal{U}^{-1}}$, we may reach a total of fourteen different definitions of "complete space"(considering the symmetry of using the $\mathcal{U}^{-1}$ instead of $\mathcal{U}$). Common place of all the completion theories for quasi-uniform spaces is the requirement to meet all the properties of uniform completeness.

More precisely, according to Doitchinov [17], the meaning of the Cauchy net in a quasi-uniform space $(X,\mathcal{U})$ must be defined in such a way that the following conditions are satisfied:

(i) Every convergent net is a Cauchy net;

(ii) In the uniform case (that is, when $(X,\mathcal{U})$ is a uniform space), the Cauchy nets are the usual ones.

On the other hand, a standard construction of a completion $(\widehat{X},\widehat{\mathcal{U}})$ of a quasi-uniform space $(X,\mathcal{U})$ should be able to satisfy the following:

(iii) If $(X,\mathcal{U}_X) \subset (Y,\mathcal{U}_Y)$, then $(\widehat{X},\widehat{\mathcal{U}_X}) \subset (\widehat{Y},\widehat{\mathcal{U}_Y})$, where the inclusions are understood as quasi-uniform embeddings and the second one is an extension of the former;

(iv) In the case when $(X,\mathcal{U})$ is a uniform space, $(\widehat{X},\widehat{\mathcal{U}})$ is nothing but the usual uniform completion of $(X,\mathcal{U})$.

We now add two more conditions for a nice quasi-uniform completion, in addition to those stated by Doitchinov, the necessity of which we will analyze in Section 5.

(v) In quasi-metric spaces, the sequential completeness and completeness via nets agree.

(vi) The completion of a quasi-uniform space $(X,\mathcal{U})$ as the quotient of equivalence classes of Cauchy nets can be replaced by working with Cauchy nets themselves.

In this paper, we present a completion theory for a quasi-uniform space $(X,\mathcal{U})$ based on Doitchinov and Stoltenberg's completion theories that meets all of the requirements presented in the introduction for a nice quasi-uniform completion. By using the definition of Cauchyness of Stoltenberg as well as Doitchinov's concept of Cauchy pair of nets, we define the concept of cut of nets as follows (see [27]): To each net, it corresponds a set of pairs of nets-conets (in the sense of Doitchinov) which lead to the notion of cut of nets, that is, a pair $(\mathcal{C},\mathcal{D})$ where $\mathcal{C}$ contains all equivalent nets of the given D-Cauchy net and $\mathcal{D}$ contains all the conets of the members of $\mathcal{C}$. The cuts of nets $(\mathcal{A},\mathcal{B})$ we use in our completion theory are called $\mathcal{U}$-cuts of nets, where the members of the $\mathcal{A}$ family contain Stoltenberg Cauchy nets for the space $(X,\mathcal{U})$ and the members of the $\mathcal{B}$ family contain Stoltenberg Cauchy nets for the space $(X,\mathcal{U}^{-1})$. It is shown here that a $\mathcal{U}$-cut is a point of a quasi-uniform completion of $(X,\mathcal{U})$. The new completion eliminates the weaknesses of Stoltenberg's completion and extends

Doitchinov's completion to arbitrary quasi-uniform spaces.

2. Notations and definitions. Let us recall some basic concepts, definitions, and results needed in the paper (see [20, 28, 34]). A quasi-pseudometric space (X, d) is a set X together with a non-negative real-valued function $d : X \times X \longrightarrow \mathbb{R}$ (called a quasi-pseudometric) such that, for every $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the additional condition (iii) $d(x, y) = 0$ implies $x = y$, then d is called a quasi-metric on X. A quasi-pseudometric is a pseudometric provided that $d(x, y) = d(y, x)$. Each quasi-pseudometric d on X induces a conjugate quasi-pseudometric d^{-1} on X defined by $d^{-1}(x, y) = d(y, x)$. By d^* we denote the pseudometric given by $d^* = \max\{d(x, y), d^{-1}(x, y)\}$. A quasi-pseudometric d on X induces a topology τ_d on X which has as a base the family of d-balls $\{B_d(x, r) : x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. A quasi-pseudometric space is T_0 if its associated topology τ_d is T_0. Axiom (i) and the T_0-condition can be replaced by (i′) $\forall x, y \in X$, $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$. In this case, we say that d is a T_0-*quasi-pseudometric.* A quasi-uniformity on a non-empty set X is a filter $\mathcal{U}$ on $X \times X$ that satisfies the following conditions: (i) for any $U \in \mathcal{U}$, $\Delta(X) = \{(x, x) | x \in X\} \subseteq U$, and (ii) given $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. The elements of the filter $\mathcal{U}$ are called *entourages.* If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$. A *uniformity* on X is a quasi-uniformity which also satisfies the additional axiom: (iii) For all $U \in \mathcal{U}$ we have $U^{-1} \in \mathcal{U}$ $(\mathcal{U} = \mathcal{U}^{-1})$. The pair $(X, \mathcal{U})$ is called a (*quasi-*)*uniform space.* Given a quasi-uniformity $\mathcal{U}$ on X, $\mathcal{U}^\star = \mathcal{U} \bigvee \mathcal{U}^{-1}$ will denote the coarsest uniformity on X which is finer than $\mathcal{U}$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^\star$ will be denoted by $U^\star$. A family $\mathcal{B}$ is a base for a quasi-uniformity $\mathcal{U}$ if and only if for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$. The family $\mathcal{B}$ is *subbase* for $\mathcal{U}$ if the family of finite intersections of members of $\mathcal{B}$ form a base for $\mathcal{U}$. Every quasi-uniformity $\mathcal{U}$ on X generates a topology $\tau(\mathcal{U})$. A neighborhood base for each point $x \in X$ is given by $\{U(x) | U \in \mathcal{U}\}$ where $U(x) = \{y \in X | (x, y) \in U\}$. If $(X, \mathcal{U})$ is a quasi-uniform space and Y any subset of X, then $\mathcal{U}_Y = \{U \cap (Y \times Y) | \ U \in \mathcal{U}\}$ is a quasi-uniformity on Y; it is the *subspace quasi-uniformity.* If (X, d) is a quasi-pseudometric space then $\mathcal{B} = \{U_{d,\epsilon} | \epsilon > 0\}$, where $U_{d,\epsilon} = \{(x, y) \in X \times X | d(x, y) < \epsilon\}$, is a base for a quasi-uniformity $\mathcal{U}_d$ on X such that $\tau_d = \tau(\mathcal{U}_d)$. For each quasi-uniformity $\mathcal{U}$ possessing a countable base there is a quasi-pseudometric $d_{\mathcal{U}}$ such that $\tau(\mathcal{U}) = \tau_{d_{\mathcal{U}}}$.

A function f from a quasi-uniform space $(X,\mathcal{U})$ to another quasi-uniform space $(X,\mathcal{V})$ is *quasi-uniformly continuous* if for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f(x), f(y)) \in \mathcal{V}$ whenever $(x, y) \in U$ i.e. the set $\{(x, y)|(f(x), f(y)) \in \mathcal{V}\} \in \mathcal{U}$. The function f is a quasi-uniform isomorphism if and only if f is one-to-one onto Y and f^{-1} is quasi-uniformly continuous. A quasi-uniform space $(X,\mathcal{U})$ can be embedded in a quasi-uniform space $(X,\mathcal{V})$ when there exists a quasi-uniform isomorphism from $(X,\mathcal{U})$ onto a subspace of $(Y,\mathcal{V})$. A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-pseudometric space (X, d) is called d_K-*Cauchy* (from Kelly [23, Definition 2.10]) if for each $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for each $n \geq m \geq k$. This is the notion of Cauchy sequence called in [29, Definition 1.iv] a *right* K-*Cauchy sequence*. Similarly, a sequence $(x_n)_{n\in\mathbb{N}}$ is *left* K-*Cauchy* if for each $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for each $n \geq m \geq k$ (see [29, Definition 1.v]). The space (X, d) is said to be *right* (resp. *left*) K-*sequentially complete* if every *right* (resp. *left*) K-*Cauchy* sequence converges on X. According to ([2, Definition 3]) a sequence $(x_n)_{n\in\mathbb{N}}$ on X is *right* (resp. *left*) *d-cofinal to* a sequence $(x_m)_{m\in\mathbb{N}}$ on X, if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ satisfying the following property: For each $n \geq n_\varepsilon$ there exists $m_n \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ (resp. $d(x_n, x_m) < \varepsilon$) whenever $m \geq m_n$. The sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_m)_{m\in\mathbb{N}}$ are *right* (resp. *left*) *d-cofinal* if $(x_n)_{n\in\mathbb{N}}$ is right (resp. left) d-cofinal to $(x_m)_{m\in\mathbb{N}}$ and vice versa. According to([17, Definition 1]) a sequence $(y_m)_{m\in\mathbb{N}}$ is called a *cosequence* to $(x_n)_{n\in\mathbb{N}}$, if for any $\varepsilon > 0$ there are $n_\varepsilon, m_\varepsilon \in \mathbb{N}$ such that $d(y_m, x_n) < \varepsilon$ when $n \geq n_\varepsilon$, $m \geq m_\varepsilon$. In this case, we write $d(y_m, x_n) \to 0$ or $\lim\limits_{m,n} d(y_m, x_n) = 0$. Generally speaking, when two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_m)_{m\in\mathbb{N}}$ are given in a quasi-pseudometric space (X, d), we will write $\lim\limits_{n,m} d(y_m, x_n) = r$ if for any $\varepsilon > 0$ there is an N_ε such that $|d(y_m, x_n) - r| < \varepsilon$ when $m, n > N_\epsilon$. We call κ-*cut* (of sequences) in X ([2, Definition 8]) an ordered pair $\xi = (\mathcal{A}, \mathcal{B})$ of families of right K-Cauchy sequences and left K-Cauchy cosequences, respectively, with the following properties: (i) For any $(x_n)_{n\in\mathbb{N}} \in \mathcal{A}$ and any $(x_m)_{m\in N} \in \mathcal{B}$, there holds $\lim\limits_{m,n} d(x_m, x_n) = 0$; (ii) Any two members of the family $\mathcal{A}$ (resp. $\mathcal{B}$) are right (resp. left) d-cofinal; (iii) The classes are maximal with respect to set inclusion. We call the member $\mathcal{A}$ (resp. $\mathcal{B}$) *first* (resp. *second*) class of ξ. A κ-*Cauchy sequence* is a right K-Cauchy sequence that belongs to the first class of a κ-cut (see [2, Definition 11]). A *directed set* is a nonempty set A plus a reflexive and transitive binary relation $\leq$ (i.e., a preorder), with the additional property that for each x and y in A, there must exist z in A with $x \leq z$ and $y \leq z$. Directed sets are a generalization of nonempty totally ordered sets. That is, all totally ordered sets are directed sets. A *net* in a topological space X is a function $\delta : A \to X$, where

A is a directed set. The point $\delta(a)$ is usually denoted x_a and the net is denoted by $(x_a)_{a\in A}$. A function $\varphi : D \to A$ is *cofinal in* A if for each $a \in A$, there exists some $\mu \in D$ such that $a \leq \varphi(\mu)$. A *subnet* of a net $\delta : A \to X$ is the composition $\delta \circ \varphi$, where $\varphi : D \to A$ is an increasing cofinal function from a directed set D to A. That is: (i) $\varphi(\mu_1) \leq \varphi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (φ is increasing); (ii) φ is cofinal in A. For each $\mu \in M$, the point $\delta \circ \varphi(\mu)$ is often written x_{a_μ}, and we usually speak of "the subnet $(x_{a_\mu})_{\mu\in M}$ of $(x_a)_{a\in A}$." A net $(x_a)_{a\in A}$ in quasi-uniform space $(X,\mathcal{U})$ is said to be *convergent* to $x \in X$ if for every $U \in \mathcal{U}$ there exists $a_U \in \mathbb{D}$ such that $(x, x_a) \in U$ for each $a \geq a_U$. The definition of net generalizes a key result about subsequences: A net $(x_a)_{a\in A}$ converges to x if and only if every subnet of $(x_a)_{a\in A}$ converges to x. According to Stoltenberg, a net $(x_a)_{a\in A}$ is a *Cauchy net* (also known as the *right-Cauchy net* in more current terminology), if for each $U \in \mathcal{U}$ there is a $a_U \in A$ such that $(x_\beta, x_\alpha) \in U$ whenever $a, \beta \in A$, $\alpha \geq a_U$, $\beta \geq a_U$ and $a \not\geq \beta$. Sieber and Pervin [31] defined the Cauchy filter in terms of a quasi-uniform space $(X,\mathcal{U})$, which permits the following equivalent description for nets: A net $(x_a)_{a\in A}$ is Cauchy in $(X,\mathcal{U})$ whenever given $U \in \mathcal{U}$ there is a point $x_U \in X$ and $a_0 \in A$ such that $(x_U, x_a) \in U$ for all $a \geq a_0$, $a \in A$.

Let X be a well-ordered set and let $R(x_\lambda)$, $\lambda \in \Lambda$ be a proposition with domain X. The *Principle of Transfinite Induction* asserts that if $\bigcup\limits_{\lambda<\mu} R(x_\lambda)$ implies $R(x_\mu)$, for all $\mu \in \Lambda$, then $R(x_\lambda)$ holds for all $\lambda \in \Lambda$.

Let X_a be a set, for each $a \in A$. The *Cartesian product* of the sets X_a is the set

$$\prod_{a\in A} X_a = \left\{ x : A \to \bigcup_{a\in A} X_a \,|\, x(a) \in X_a, \quad \text{for each} \quad a \in A \right\}.$$

The value of $x \in \prod\limits_{a\in A} X_a$ at a is usually denoted x_a, rather than $x(a)$, and x_a is referred to as the *ath coordinate* of x. The space X_a in the *ath factor space.* For each $\gamma \in A$ the map $\pi_\gamma : \prod\limits_{a\in A} X_a \to X_\gamma$, defined by $\pi_\gamma(x) = x_\gamma$,is called the *projection map* of $\prod\limits_{a\in A} X_a$ on X_γ, or the γth *projection map.* The Axiom of choice ensure that the Cartesian product of a nonempty collection of nonempty sets is nonempty.

Let $(X_i, \mathcal{U}_i)_{i\in I}$ be a family of quasi-uniform spaces. Assume $\prod\limits_{i\in A} X_i$ is the set-theoretic product of the family $(X_i)_{i\in I}$, and $\pi_j : \prod\limits_{i\in A} X_i \longrightarrow X_j$ $(j \in I)$ is its projection onto $(X_j, \mathcal{U}_j)$. Then the coarsest quasi-uniformity on $\prod_{i\in A} X_i$ that makes all projections quasi-uniformly continuous is called the product quasi-uniformity. It induces the product topology of the topological spaces $(X_i, \tau(\mathcal{U}_i))$. The open sets are the unions of subsets $\prod_{i\in I} U_i$, where U_i is an open subset of X_i with the additional condition that $U_i = X_i$ for all but finitely many indices i. It follows that sets of the form $\{(x,y) \mid (x^i, y^i) \in U_i\}$, for an i in I and U_i in $\mathcal{U}_i$, is a subbase for the product quasi-uniformity. It is easy to see that the topology generated from the product quasi-uniformity is the product topology of $(X, \tau(\mathcal{U}_i))$. The reason for this choice of open sets is that these are the least needed to make the projection onto the *ith* factor $p_i : X \to X_i$ quasi-uniformly continuous for all indices i. Also, a function f from a quasi-uniform space to a product of quasi-uniform spaces is *quasi- uniformly continuous* if and only if the composition $p_i \circ f$ is quasi- uniformly continuous for each $i \in I$.

3. The $\boldsymbol{\delta}$-completion. Throughout the paper $(X,\mathcal{U})$ will be an arbitrary quasi-uniform space, except the cases when it is explicitly stated that the space is T_0.

Stoltenberg [32] has given the following definition. The index S in our symbolisms is devoted to Stoltenberg.

Definition 3.1 ([32, Definition 2.1])**.** *A net $(x_a)_{a\in A}$ in a quasi-unifrom space $(X,\mathcal{U})$ is called right $\mathcal{U}_S$-Cauchy (left $\mathcal{U}_S$-Cauchy) if for each $U \in \mathcal{U}$ there is $a_U \in A$ such that $(x_\beta, x_\alpha) \in U$ (resp. $(x_\alpha, x_\beta) \in U$) whenever $\alpha \geq a_U$, $\beta \geq a_U$, $a \ngeq \beta$ and $a, \beta \in A$. Without loss of generality, we may suppose that for $U' \subseteq U$ we have $a_{U'} \geq a_U$ (resp. $\beta_{U'} \geq \beta_U$).*

An equivalent definition for quasi-pseudometric spaces is the following.

Definition 3.2. *A net $(x_a)_{a\in A}$ in a quasi-pseudometric space (X,d) is called right (resp.* left*) d_S*-Cauchy *if for each $\epsilon > 0$ there is $a_\epsilon \in A$ such that $d(x_\beta, x_\alpha) < \epsilon$ (resp. $d(x_\alpha, x_\beta) < \epsilon$) whenever $\alpha \geq a_\epsilon$, $\beta \geq a_\epsilon$, $a \ngeq \beta$ and $a, \beta \in A$. Without loss of generality, we may suppose that for $\varepsilon' \leq \varepsilon$, it is $a_{\varepsilon'} \geq a_\varepsilon$ (resp. $\beta_{\varepsilon'} \geq \beta_\varepsilon$).*

In the case of sequences, Definition 3.2 coincides with the definition of the notions of right K-Cauchy sequence and left K-Cauchy sequence, respectively.

An immediate consequence of Definitions 3.1 and 3.2 is the following proposition.

Proposition 3.1. *A net in a quasi-uniform space $(X,\mathcal{U})$ (resp. quasi-pseudometric space (X,d)) is left $\mathcal{U}_S$-Cauchy (resp. left d_S-Cauchy) if and only if it is right $(\mathcal{U}^{-1})_S$-Cauchy.*

Definition 3.3 ([32, Definition 2.2])**.** *A quasi-uniform space $(X,\mathcal{U})$ is $\mathcal{U}_S$-complete if and only if each right $\mathcal{U}_S$-Cauchy net converges in X with respect to $\tau_{\mathcal{U}}$. We say that a quasi-pseudometric space (X,d) is d_S-complete if and only if every right d_S-Cauchy net converges to a point in X with respect to τ_d.*

In the case of sequences, Definition 3.3 coincides with the definition of the notions of right K-sequentially complete quasi-uniform space and right K-sequentially complete quasi-pseudometric space, respectively.

In the following, we refer to the right $\mathcal{U}_S$-Cauchy nets in a quasi-uniform space $(X,\mathcal{U})$ as *right S-nets* in $(X,\mathcal{U})$ and the right d_S-Cauchy nets in a quasi-pseudometric space (X,d) as *right S-nets* in (X,d), respectively. Both the *left S-nets* in $(X,\mathcal{U})$ and the *left S-nets* in (X,d) have analogous symbolism.

In the case of sequences, we refer to the right (resp. left) K-Cauchy sequence in (X,d) as *right* (resp. *left*) *K-sequence* in (X,d).

Proposition 3.2 (see [32, Page 229])**.** *Let (X,d) be a quasi-pseudometric space. A net $(x_a)_{a\in A}$ is a right S-net (resp. left S-net) in (X,d) if and only if $(x_a)_{a\in A}$ is a right S-net (left S-net) in $(X,\mathcal{U}_d)$.*

Proof. By definition, we have that each quasi-pseudometric d on X generates a quasi-uniformity $\mathcal{U}_d$ which has as base the family $\{\{(x,y)\in X\times X|d(x,y)<\epsilon\}|\epsilon>0\}$. As a result, the Proposition's implication arises immediately from the Definition of $\mathcal{U}_d$. □

The following corollary is an immediate consequence of Proposition 3.2.

Corollary 3.3. *A quasi-pseudometric space (X,d) is d_S-complete if and only if $(X,\mathcal{U}_d)$ is $(\mathcal{U}_d)_S$-complete.*

Definition 3.4 ([17, Definition 1], [16])**.** *Let $(X,\mathcal{U})$ (resp. (X,d)) be a quasi-uniform space (quasi-pseudometric space) and let $(x_a)_{a\in A}$, $(y_\beta)_{\beta\in B}$ be two nets in $(X,\mathcal{U})$ (resp. (X,d)). The net $(y_\beta)_{\beta\in B}$ is called a conet of $(x_a)_{a\in A}$, if for any $U\in\mathcal{U}$ (resp. $\epsilon>0$) there are $a_U\in A$ and $\beta_U\in B$ (resp. $a_\epsilon\in A$ and $\beta_\epsilon\in B$) such that $(y_\beta,x_a)\in U$ (resp. $d(y_\beta,x_a)<\epsilon$) whenever $a\geq a_U$, $\beta\geq\beta_U$ (resp. $a\geq a_\epsilon$, $\beta\geq\beta_\epsilon$) and $a,\beta\in A$.*

Definition 3.5 ([2, Definition 3])**.** *A net $(x_a)_{a\in A}$ in a quasi-pseudometric space (X,d) is called right d-cofinal to a net $(x_\beta)_{\beta\in B}$ on X, if for each $\epsilon>0$ there exists $a_\epsilon\in A$ satisfying the following property: For each $a\geq a_\epsilon$ there exists*

$\beta_a \in B$ *such that* $d(x_\beta, x_a) < \epsilon$ *whenever* $\beta \geq \beta_a$. *The nets* $(x_a)_{a\in A}$ *and* $(x_\beta)_{\beta\in B}$ *are right d-cofinal if* $(x_a)_{a\in A}$ *is right d-cofinal to* $(x_\beta)_{\beta\in B}$ *and vice versa. The net* $(x_a)_{a\in A}$ *is called left d-cofinal to* $(x_\beta)_{\beta\in B}$ *if and only if it is right* d^{-1}*-cofinal to* $(x_\beta)_{\beta\in B}$. *The nets* $(x_a)_{a\in A}$ *and* $(x_\beta)_{\beta\in B}$ *are left d-cofinal if* $(x_a)_{a\in A}$ *is left d-cofinal to* $(x_\beta)_{\beta\in B}$ *and vice versa.*

Definition 3.6. *Let* (X, d) *be a quasi-pseudometric space. We call* δ*-cut in* X *an ordered pair* $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$ *of families of right* S*-nets and left* S*-nets respectively, with the following properties:*

(i) Any $(x_a)_{a\in A} \in \mathcal{A}_\xi$ *has as conet any* $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$*;*

(ii) Any two members of the family $\mathcal{A}_\xi$ *(resp.* $\mathcal{B}_\xi$*) are right (resp. left) d-cofinal.*

(iii) The classes $\mathcal{A}_\xi$ *and* $\mathcal{B}_\xi$ *are maximal with respect to set inclusion.*

Definition 3.7. *To every* $x \in X$ *we correspond the* δ*-cut* $\phi(x) = (\mathcal{A}_{\phi(x)}, \mathcal{B}_{\phi(x)})$ *where* $\mathcal{A}_{\varphi(x)}$ *contains all the nets which* τ_d*-converge to* x, $\mathcal{B}_{\varphi(x)}$ *contains all the nets which* $\tau_{d^{-1}}$*-converge to* x *and satisfy the requirements (i)–(iii) of Definition* 3.6. *The net* $(x) = x, x, x, \ldots$ *itself, belongs to both of the classes* $\mathcal{A}_{\varphi(x)}$ *and* $\mathcal{B}_{\varphi(x)}$.

Throughout the paper, if (X, d) is a quasi-pseudometric space, then $\widehat{X}$ denotes the set of all δ-cuts in (X, d) and for every $\xi \in \widehat{X}$, $\mathcal{A}_\xi$ and $\mathcal{B}_\xi$ denote the two classes of ξ. In this case, we write $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$.

Definition 3.8. *Let* (X, d) *be a quasi-pseudometric space. We call* δ-Cauchy net *any right* S*-net in* (X, d) *member of the first class of a* δ*-cut. The space* (X, d) *is* δ-complete *if and only if each* δ*-Cauchy net converges in* X.

Definition 3.9. *Two* δ*-Cauchy nets* $(x_a)_{a\in A}$ *and* $(x_\beta)_{\beta\in B}$ *in a quasi-pseudometric space* (X, d) *are called* δ-equivalent *if every conet to* $(x_a)_{a\in A}$ *is a conet to* $(x_\beta)_{\beta\in B}$ *and vice versa.*

It is straightforward to observe that δ-equivalence defines an equivalence relation on (X, d). As a result, $\mathcal{A}$ denotes the equivalence class of the δ-Cauchy nets considered equivalent by this equivalence relation.

Classes $\mathcal{A}$ and $\mathcal{B}$ in metric spaces are equal to the classical class of equivalent Cauchy nets and so coincide.

It is straightforward to observe that δ-equivalence defines an equivalence relation on (X, d). As a result, $\mathcal{A}$ denotes the equivalence class of the δ-Cauchy nets considered equivalent by this equivalence relation.

The notions of δ-cut of nets and δ-complete quasi-pseudometric space generalize the notions of κ-cut of sequences and κ-complete quasi-pseudometric space, respectively (see [2, Definition 11]).

Remark 3.4. If the space (X,d) is T_0, then the function ϕ defined above is an injective function (one-to-one) of X into $\widehat{X}$. Indeed, let $x, y \in X$ be such that $\phi(x) = \phi(y)$. Then, $(x), (y) \in \mathcal{A}_{\phi(x)} \cap \mathcal{B}_{\phi(x)} \cap \mathcal{A}_{\phi(y)} \cap \mathcal{A}_{\phi(y)}$. Thus, $d^*(x,y) = 0$ which implies that $x = y$. Generally, each quasi-pseudometric space (X,d) defines a T_0 quasi-pseudometric space $(\widetilde{X}, \widetilde{d})$. Indeed, let R be the equivalence relation in X defined by: xRy if and only if $d(x,y) = 0 = d(y,x)$. Let also $\widetilde{X} = X/R = \{[x] \mid x \in X\} = \{\{z \in X | zRx\} \mid x \in X\}$ be the quotient space (the set of equivalence classes). We define the function $\pi : X \to \widetilde{X}$ defined by $\pi(x) = \{[y] | \, x \in [y]\}$. Because each $x \in X$ belongs to only one equivalence class, the function π is well defined. In order to ensure that the projection map be continuous, we put an obligation in the topology we assign to $\widetilde{X}$: If a set G is open in $\widetilde{X}$ then $\pi^{-1}(G)$ is open in X. We define the *quotient topology* on $\widetilde{X}$ accepting all sets G that meet these criteria. On other words, a set G is open in $\widetilde{X}$ if and only if $\pi^{-1}(G)$ is open in X. The quotient topology on $\widetilde{X}$ is the finest topology on $\widetilde{X}$ for which the projection map π is continuous. Let $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ be a function defined by $\widetilde{d}([x],[y]) = d(x,y)$. Then, it is easy to check that $\widetilde{d}$ is a T_0 quasi-pseudometric in $\widetilde{X}$ that yields the quotient topology in $\widetilde{X}$.

Because of Remark 3.4, in what follows, (X,d) will be a T_0 quasi-pseudometric space, except the cases when it is explicitly stated that the space is a T_1 quasi-metric space.

Proposition 3.5. *Let (X,d) be a T_0 quasi-pseudometric space and let $(x_a)_{a\in A}$ be a right (resp. left) S-net in (X,d) with a subnet $(x_{a_\gamma})_{\gamma\in\Gamma}$. Then, $(x_a)_{a\in A}$ and $(x_{a_\gamma})_{\gamma\in\Gamma}$ are right (resp. left) d-cofinal.*

Proof. Let $(x_a)_{a\in A}$ be a right S-net in (X,d) and $(x_{a_\gamma})_{\gamma\in\Gamma}$ be a subnet of it. Let $\epsilon > 0$ be given. Then, there exists $a_\epsilon \in A$ such that for each $a, a' \in A$ with $a \geq a_\epsilon$, $a' \geq a_\epsilon$ and $a' \not\geq a$, we have $d(x_a, x_{a'}) < \epsilon$. Let $a_\gamma > a_\epsilon$ for some $\gamma \in \Gamma$. Then, for each $a > a_\gamma$ ($a_\gamma \not\geq a$) there exists $a_{a_\gamma} = a_\gamma \in A$ such that $d(x_a, x_{a_\gamma}) < \epsilon$. Hence, $(x_{a_\gamma})_{\gamma\in\Gamma}$ is right d-cofinal to $(x_a)_{a\in A}$. On the other hand, let $a > a_\epsilon$ for some $a \in A$. Let also $(a_\gamma)_a = a_\gamma$ for some $a_\gamma > a$, $\gamma \in \Gamma$. Then, for each $\gamma' > \gamma$ we have $d(x_{a_{\gamma'}}, x_a) < \epsilon$ which implies that $(x_a)_{a\in A}$ is right d-cofinal to $(x_{a_\gamma})_{\gamma\in\Gamma}$. The case of the left S-net is similar. □

Proposition 3.6. *In every T_0 quasi-pseudometric space (X,d) two right d-cofinal nets have the same conets.*

Proof. Let $(x_a)_{a\in A}$, $(x_\gamma)_{\gamma\in\Gamma}$ be two right d-cofinal nets. Suppose that $(y_\beta)_{\beta\in B}$ is a conet of $(x_a)_{a\in A}$. Fix $\epsilon > 0$. Then there exist $\beta_\epsilon \in B$ and $a_\epsilon \in A$ such that $d(y_\beta, x_a) < \frac{\epsilon}{2}$ for $\beta \geq \beta_\epsilon$, $a \geq a_\epsilon$. On the other hand, there is $\gamma_\epsilon \in \Gamma$ with the following property: For each $\gamma \geq \gamma_\epsilon$ there exists $a_\gamma \in A$ such that $d(x_a, x_\gamma) < \frac{\epsilon}{2}$ whenever $a \geq a_\gamma$. If $a^* = \max\{a_\epsilon, a_\gamma\}$, then from $d(y_\beta, x_{a^*}) < \frac{\epsilon}{2}$ and $d(x_{a^*}, x_\gamma) < \frac{\epsilon}{2}$ we can deduce that $d(y_\beta, x_\gamma) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $\beta \geq \beta_\epsilon$ and $\gamma \geq \gamma_\epsilon$. □

Similarly, we can demonstrate the following proposition.

Proposition 3.7. *In every quasi-pseudometric space (X, d), two left d-cofinal nets are conets of the same nets.*

The following two corollaries are an immediate consequence of Propositions 3.6 and 3.7.

Corollary 3.8. *In every T_0 quasi-pseudometric space (X, d), two right (left) d-cofinal nets have the same limit points for $\tau(d)$ (resp. $\tau(d^{-1})$).*

Corollary 3.9. *Let (X, d) be a T_0 quasi-pseudometric space and let $\xi, \xi' \in \widehat{X}$. Then, $\mathcal{A}_\xi \cap \mathcal{A}_{\xi'} \neq \emptyset$ implies $\mathcal{A}_\xi = \mathcal{A}_{\xi'}$.*

Definition 3.10. *Let (X, d) be a T_0 quasi-pseudometric space. Suppose that r is a nonnegative real number, $\xi', \xi'' \in \widehat{X}$, $(x_a)_{a\in A} \in \mathcal{A}_{\xi'}$ and $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{B}_{\xi''}$. We put $\widehat{d}(\xi', \xi'') \leq r$ if:*

(i) $\mathcal{A}_{\xi'} = \mathcal{A}_{\xi''}$ or

(ii) For each $\epsilon > 0$ there are $a_\epsilon \in A$, $\gamma_\varepsilon \in \Gamma$ such that

$$d(x_a, x_\gamma) < r + \epsilon \tag{1}$$

when $a \geq a_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$. If $\xi' = \phi(x)$ for some $x \in X$, then the net $(x_a)_{a\in A}$ always coincides with the fixed net, for which $x_a = x$ for all $a \in A$. Then, we let

$$\widehat{d}(\xi', \xi'') = inf\{r \mid \widehat{d}(\xi', \xi'') \leq r\}. \tag{2}$$

Proposition 3.10. *The truth of $\widehat{d}(\xi', \xi'') \leq r$ in Definition of 3.10(ii) is determined solely by ξ', ξ'' and r; it is unaffected by the choice of nets $(x_a)_{a\in A}$ and $(x_\gamma)_{\gamma\in\Gamma}$.*

Proof. Let $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ be two right S-nets in (X, d) of the class $\mathcal{A}_{\xi'}$ and $(x_\gamma)_{\gamma\in\Gamma}$ and $(x_\delta)_{\delta\in\Delta}$ be two right S-nets in (X, d) of the class $\mathcal{A}_{\xi''}$. Fix an $\varepsilon > 0$. Then, there are $a_\varepsilon \in A$ and $\gamma_\varepsilon \in \Gamma$ such that

$$d(x_a, x_\gamma) < r + \varepsilon$$

whenever $a \geq a_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$. Choose an arbitrary positive number ε' so that $0 < \varepsilon' < \frac{\varepsilon}{3}$. Then, there are $a_{\varepsilon'} \in A$ and $\gamma_{\varepsilon'} \in \Gamma$ such that

$$d(x_a, x_\gamma) < r + \varepsilon'$$

whenever $a \geq a_{\varepsilon'}$ and $\gamma \geq \gamma_{\varepsilon'}$.

Because $(x_a)_{a\in A}$ is right d-cofinal to $(x_\beta)_{\beta\in B}$, there exists $\widetilde{a}_\varepsilon \in A$, $\widetilde{a}_\varepsilon \geq a_{\varepsilon'}$ that fulfills the following property:

For each $a \geq \widetilde{a}_\varepsilon$ there is $\beta_a \in B$ such that

$$d(x_\beta, x_a) < \frac{\varepsilon}{3}$$

whenever $\beta \geq \beta_a$. Fix an $\widetilde{a}_\varepsilon \geq a_{\varepsilon'}$ and let $\widetilde{a}_\varepsilon = a^*$. Then, we have

$$d(x_\beta, x_\gamma) \leq d(x_\beta, x^*) + d(x^*, x_\gamma) < r + \varepsilon' + \frac{\varepsilon}{3}$$

whenever $\beta \geq \beta_{\widetilde{a}_\varepsilon}$ and $\gamma \geq \gamma_{\varepsilon'}$.

Similarly, since $(x_\delta)_{\delta\in\Delta}$ is right d-cofinal to $(x_\gamma)_{\gamma\in\Gamma}$, there is $\delta_\varepsilon \in \Delta$ which satisfies the following property: For each $\delta \geq \delta_\varepsilon$ there exists $\gamma_\delta \in \Gamma$ such that

$$d(x_\gamma, x_\delta) < \frac{\varepsilon}{3}$$

whenever $\gamma \geq \gamma_\delta$. Let $\gamma^* = \max\{\gamma_{\varepsilon'}, \gamma_\delta\}$. Then,

$$d(x_\beta, x_\delta) \leq d(x_\beta, x_{\gamma^*}) + d(x_{\gamma^*}, x_\delta) < r + \varepsilon' + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < r + \varepsilon$$

whenever $\beta \geq \beta_{\widetilde{a}_\varepsilon}$ and $\delta \geq \delta_\varepsilon$. □

Proposition 3.11. *Let (X, d) be a T_0 quasi-pseudometric space. Suppose that $\xi', \xi'' \in \widehat{X}$, $(x_a)_{a\in A} \in \mathcal{A}_{\xi'}$, $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_{\xi''}$ and $\mathcal{A}_{\xi'} \neq \mathcal{A}_{\xi''}$. Then,*

$$\widehat{d}(\xi', \xi'') = \lim_{a,\gamma} d(x_a, x_\gamma).$$

P r o o f. Let $\widehat{d}(\xi', \xi'') = r$. Then, for any $\varepsilon > 0$ there are $a_\varepsilon \in A$ and $\gamma_\varepsilon \in \Gamma$ such that

$$d(x_a, x_\gamma) < r + \varepsilon$$

whenever $a \geq a_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$. To prove that $r - \varepsilon < d(x_a, x_\gamma)$ for $a \geq a_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$, suppose to the contrary there exist a subnet $(x_{a_\lambda})_{\lambda\in\Lambda}$ of $(x_a)_{a\in A}$ and a subnet $(x_{\gamma_\mu})_{\mu\in M}$ of $(x_\gamma)_{\gamma\in\Gamma}$ such that for all x_{a_λ}, x_{γ_μ} there holds $d(x_{a_\lambda}, x_{\gamma_\mu}) \leq r - \varepsilon$. Then, by Propositions 3.5, 3.6, 3.10 and Definition 3.10 we have that $\widehat{d}(\xi', \xi'') \leq r - \varepsilon$, a contradiction. Therefore, we have $\widehat{d}(\xi', \xi'') = r = \lim_{a,\gamma} d(x_a, x_\gamma)$. □

Proposition 3.12. *Assume that (X, d) is a quasi-pseudometric space. Then, $(\widehat{X}, \widehat{d})$ is also a quasi-pseudometric space.*

Proof. It follows immediately from Definition 3.10 that $\widehat{d}(\xi, \xi) = 0$ and $\widehat{d}(\xi, \xi') \geq 0$ for all $\xi, \xi' \in \widehat{X}$. To demonstrate that $\widehat{d}$ meets the triangle inequality, consider $\xi, \xi', \xi'' \in \widehat{X}$. We will look at the following four cases: (i) $\mathcal{A}_\xi \neq \mathcal{A}_{\xi'}$ and $\mathcal{A}_{\xi'} \neq \mathcal{A}_{\xi''}$. Suppose that $\widehat{d}(\xi, \xi') = r_1$, $\widehat{d}(\xi', \xi'') = r_2$, $(x_a)_{a\in A} \in \mathcal{A}_\xi$, $(x_\beta)_{\beta\in B} \in \mathcal{A}_{\xi'}$ and $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_{\xi''}$. Then, for any $\varepsilon > 0$ there are $a_\varepsilon \in A$ and $\beta_\varepsilon \in B$ such that $d(x_a, x_\beta) < r_1 + \frac{\varepsilon}{2}$ whenever $a \geq a_\varepsilon$ and $\beta \geq \beta_\varepsilon$. Similarly, there are $\beta'_\varepsilon \in B$ and $\gamma_\varepsilon \in \Gamma$ such that $d(x_\beta, x_\gamma) < r_2 + \frac{\varepsilon}{2}$ whenever $\beta \geq \beta'_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$. Let $\beta^* = \max\{\beta_\varepsilon, \beta'_\varepsilon\}$. Then, $d(x_a, x_\gamma) \leq d(x_a, x_{\beta^*}) + d(x_{\beta^*}, x_\gamma) < r_1 + r_2 + \varepsilon$ whenever $a \geq a_\varepsilon$ and $\gamma \geq \gamma_\varepsilon$. Hence, according to Definition 3.10, we have

$$\widehat{d}(\xi, \xi'') \leq r_1 + r_2 = \widehat{d}(\xi, \xi') + \widehat{d}(\xi', \xi'').$$

(ii) $\mathcal{A}_\xi \neq \mathcal{A}_{\xi'}$ and $\mathcal{A}_{\xi'} = \mathcal{A}_{\xi''}$. Suppose that $\widehat{d}(\xi, \xi') = r$. Since $\mathcal{A}_{\xi'} = \mathcal{A}_{\xi''}$, Definition 3.10 implies that $\widehat{d}(\xi, \xi'') = r$ and $\widehat{d}(\xi', \xi'') = 0$. Therefore,

$$\widehat{d}(\xi, \xi'') = r = \widehat{d}(\xi, \xi') + \widehat{d}(\xi', \xi'').$$

(iii) $\mathcal{A}_\xi = \mathcal{A}_{\xi'}$ and $\mathcal{A}_{\xi'} \neq \mathcal{A}_{\xi''}$. This case's proof is similar to (ii).

(iv) $\mathcal{A}_\xi = \mathcal{A}_{\xi'}$ and $\mathcal{A}_{\xi'} = \mathcal{A}_{\xi''}$. This case is trivial. □

Proposition 3.13. *Let (X, d) be a quasi-pseudometric space. If (X, d) is T_0, so is $(\widehat{X}, \widehat{d})$.*

Proof. Let (X, d) be a T_0 quasi-pseudometric space. To prove that $(\widehat{X}, \widehat{d})$ is T_0, we will simply show that:

For each $\xi, \xi' \in \widehat{X}$, $\widehat{d}(\xi, \xi') = \widehat{d}(\xi', \xi) = 0$ if and only if $\xi = \xi'$.

If $\xi = \xi'$, then $\mathcal{A}_\xi = \mathcal{A}_{\xi'}$. Therefore, by Definition 3.10, we have that $\widehat{d}(\xi, \xi') = \widehat{d}(\xi', \xi) = 0$.

Conversely, let $\widehat{d}(\xi, \xi') = \widehat{d}(\xi', \xi) = 0$. Let $(x_a)_{a\in A} \in \mathcal{A}_\xi$, $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_{\xi'}$, $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$ and $(y_\delta)_{\delta\in\Delta} \in \mathcal{B}_{\xi'}$. Fix an $\varepsilon > 0$. Then, from $\widehat{d}(\xi', \xi) = 0 < \varepsilon$, there are $\gamma_\varepsilon \in \Gamma$ and $a_\varepsilon \in A$ such that

$$d(x_\gamma, x_a) < \frac{\varepsilon}{2} < \varepsilon \text{ whenever } \gamma \geq \gamma_\varepsilon \text{ and } a \geq a_\varepsilon. \tag{3}$$

On the other hand, since $(y_\delta)_{\delta\in\Delta}$ is a conet of $(x_\gamma)_{\gamma\in\Gamma}$, there are $\delta_\varepsilon \in \Delta$ and $\widetilde{\gamma}_\varepsilon \in \Gamma$ such that

$$d(y_\delta, x_\gamma) < \frac{\varepsilon}{2} < \varepsilon \text{ whenever } \delta \geq \delta_\varepsilon \text{ and } \gamma \geq \widetilde{\gamma}_\varepsilon. \tag{4}$$

Let $\gamma^* = \max\{\gamma_\varepsilon, \widetilde{\gamma}_\varepsilon\}$. Then,

$$d(y_\delta, x_a) < d(y_\delta, x_{\gamma^*}) + d(x_{\gamma^*}, x_a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{5}$$

Because ε is an arbitrary positive real, we can deduce from (5) that $(y_\delta)_{\delta\in\Delta}$ is a conet of $(x_a)_{a\in A}$. On the other hand, from $\widehat{d}(\xi, \xi') = 0 < \varepsilon$, we have that for each $\varepsilon > 0$ there exists $a_\varepsilon \in A$ with the property: For each $a \geq a_\varepsilon$ there exists $a_\gamma = \gamma_\varepsilon$ such that

$$d(x_\gamma, x_a) < \varepsilon \text{ whenever } \gamma \geq a_\gamma.$$

Therefore, $(x_a)_{a\in A}$ is right d-cofinal to $(x_\gamma)_{\gamma\in\Gamma}$.

Similarly, because $\widehat{d}(\xi, \xi') = 0$ is true, from proposition 3.10 we can use the same nets $(x_a)_{a\in A}$ and $(x_\gamma)_{\gamma\in\Gamma}$ reaching the conclusion that $(x_\gamma)_{\gamma\in\Gamma}$ is right d-cofinal to $(x_a)_{a\in A}$ and $(y_\beta)_{\beta\in B}$ is a conet of $(x_\gamma)_{\gamma\in\Gamma}$. Therefore, $(x_a)_{a\in A}$ and $(x_\gamma)_{\gamma\in\Gamma}$ are right d-cofinal and each conet of $(x_a)_{a\in A}$ is a conet of $(x_\gamma)_{\gamma\in\Gamma}$ and vice versa. Hence, $\mathcal{A}_\xi = \mathcal{A}_{\xi'}$ and $\mathcal{B}_\xi = \mathcal{B}_{\xi'}$ which implies that $\xi = \xi'$. □

Proposition 3.14. *Let (X, d) be a T_0 quasi-pseudometric space. Then, for any $x, y \in X$ we have*

$$\widehat{d}(\phi(x), \phi(y)) = d(x, y).$$

Proof. Since $(x) \in \mathcal{A}_{\phi(x)}$ and $(y) \in \mathcal{A}_{\phi(y)}$, the proposition's validity is an immediate consequence of Proposition 3.11. □

Proposition 3.15. *For any $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi) \in \widehat{X}$, $(x_a)_{a\in A} \in \mathcal{A}_\xi$ and $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$ we have that $\widehat{d}(\xi, \phi(x_a)) \to 0$ and $\widehat{d}(\phi(y_\beta), \xi) \to 0$.*

Proof. We now show that if $(x_a)_{a\in A} \in \mathcal{A}_\xi$, then $\phi(x_a)$ will converge to ξ. Fix an $\varepsilon > 0$ and an arbitrary right S-net $(x_\gamma)_{\gamma\in\Gamma}$ of $\mathcal{A}_\xi$. Since $(x_a)_{a\in A}$ is a right S-net, there exists $a_\varepsilon \in A$ such that

$$d(x_{a'}, x_a) < \frac{\varepsilon}{3}, \text{ whenever } a \geq a_\varepsilon,\ a' \geq a_\varepsilon \text{ and } a \ngeq a'. \tag{6}$$

Fix an $a \geq a_\varepsilon$ and let $(x_\delta)_{\delta\in\Delta} \in \mathcal{A}_{\phi(x_a)}$. Since $(x_a)_{a\in A}$ and $(x_\gamma)_{\gamma\in\Gamma}$ are right d-cofinal we have the property: There exists $\widetilde{a}_\varepsilon \in A$ such that for each $a'' \geq \widetilde{a}_\varepsilon$ there exists $\gamma_{a''} \in \Gamma$ satisfying

$$d(x_\gamma, x_{a''}) < \frac{\varepsilon}{3}, \text{ whenever } \gamma \geq \gamma_{a''}. \tag{7}$$

We have two cases to consider; the case where $a \ngeq a''$ and the case where $a \geq a''$.

Case 1: $a \not\geq a''$. Since $(x_\delta)_{\delta\in\Delta} \in \mathcal{A}_{\phi(x_a)}$, by definition, $(x_a, x_\delta) \to 0$ (x_a is constant). Therefore, there exists $\delta_\varepsilon \in \Delta$ such that

$$(8) \qquad d(x_a, x_\delta) < \frac{\varepsilon}{3}, \quad \text{whenever } \delta \geq \delta_\varepsilon.$$

By (6), (7) and (8), we have that

$$(9) \qquad d(x_\gamma, x_\delta) < d(x_\gamma, x_{a''}) + (x_{a''}, x_a) + (x_a, x_\delta) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Case 2: $a \geq a''$. In this case, since $(x_a)_{a\in A}$ and $(x_\gamma)_{\gamma\in\Gamma}$ are right d-cofinal and $a \geq a'' \geq \widetilde{a}_\varepsilon$, there exists $\gamma_a \in \Gamma$ such that

$$(10) \qquad d(x_\gamma, x_a) < \frac{\varepsilon}{3} \quad \text{whenever } \gamma \geq \gamma_a.$$

By (8) and (10) we have

$$(11) \qquad d(x_\gamma, x_\delta) < d(x_\gamma, x_a) + (x_a, x_\delta) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon.$$

Then, by (9) and (11), Definition 3.10 and Proposition 3.10, we conclude that

$$(12) \qquad d(\xi, \phi(x_a)) < \varepsilon \quad \text{for } \ a \geq a_\varepsilon \quad \text{which implies that } \widehat{d}(\xi, \phi(x_a)) \to 0.$$

To prove that $\widehat{d}(\phi(y_\beta), \xi) \to 0$, let $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_\xi$. Fix an $\varepsilon > 0$. Then, there are $\beta_\varepsilon \in B$, $\gamma_\varepsilon \in \Gamma$ such that

$$(13) \qquad d(y_\beta, x_\gamma) < \varepsilon \quad \text{whenever } \beta \geq \beta_\varepsilon, \ a \geq a_\varepsilon.$$

Since $(x_\gamma)_{\gamma\in\Gamma}$ is an arbitrary net of $\mathcal{A}_\xi$ and $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$ is a left S-net, similarly to the previous case by using Definition 3.10 we conclude that $\widehat{d}(\phi(y_\beta), \xi) < \varepsilon$ which completes the proof. □

Proposition 3.16. *Let (X, d) be a quasi-pseudometric space and let $(\xi_a)_{a\in A}$ be a non-constant right S-net in $(\widehat{X}, \widehat{d})$ without last element. Let also $a^* \in A$ be such that for each $a \geq a^*, a' \geq a^*$ and $a' \not\geq a$ there holds $\widehat{d}(\xi_a, \xi_{a'}) = 0$. Then, there exists a right S-net $(t_\sigma)_{\sigma\in\Sigma}$ in (X, d) such that $(\xi_a)_{a\in A}$ and $(\phi(t_\sigma))_{\sigma\in\Sigma}$ are right $\widehat{d}$-cofinal nets.*

Proof. We formulate the proof in two parts:

Part I: *The Construction of the right d_S-Cauchy net $(t_\sigma)_{\sigma\in\Sigma}$ in (X, d).*

Let $(\xi_a)_{a\in A}$ and a^* be as in the proposition. Since $\widehat{d}(\xi_a, \xi_{a'}) = 0$, for each $\epsilon > 0$ we have that

$$\widehat{d}(\xi_a, \xi_{a'}) < \epsilon \text{ whenever } a \geq a^*,\ a' \geq a^* \text{ and } a' \ngeq a. \tag{14}$$

Let $(\Gamma, \leq)$ be a cofinal well-ordered subset of $(A, \leq)$ (the existence of such Γ follows from the axiom of choice). Consider the subnet $(\xi_{a_\gamma})_{\gamma\in\Gamma} = (\xi_\gamma)_{\gamma\in\Gamma}$ of $(\xi_a)_{a\in A}$. Then, $(\xi_\gamma)_{\gamma\in\Gamma}$ is cofinal to $(\xi_a)_{a\in A}$ and by (14) we have that

$$\widehat{d}(\xi_\gamma, \xi_{\gamma'}) < \epsilon \text{ whenever } \gamma \geq \gamma' \geq a^*. \tag{15}$$

For any $\gamma \in \Gamma$, let $\xi_\gamma = (\mathcal{A}_{\xi_\gamma}, \mathcal{B}_{\xi_\gamma})$ and $(x_{\rho(k_\gamma)})_{\rho(k_\gamma)\in P_{k_\gamma}}$ be a member of $\mathcal{A}_{\xi_\gamma}$, $k_\gamma \in \mathcal{K}_\gamma$ where $\mathcal{K}_\gamma$ is the counter set of all the different right S-nets of $\mathcal{A}_{\xi_\gamma}$ and $\rho(k_\gamma)$ denote the indices sets of k_γ. Let also $\rho_\epsilon(k_\gamma)$ be the smallest index with the property

$$d(x_{\rho(k_\gamma)}, x_{\rho'(k_\gamma)}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k_\gamma) \geq \rho_\epsilon(k_\gamma),\ \rho'(k_\gamma) \geq \rho_\epsilon(k_\gamma) \text{ and } \rho'(k_\gamma) \ngeq \rho(k_\gamma). \tag{16}$$

For each $\gamma \in \Gamma$ fix an $k^*_\gamma \in \mathcal{K}_\gamma$. Without loss of generality (see Definition 3.2), we can assume that:

$$\text{For each } \epsilon' \leq \epsilon \text{ there holds } \rho_\epsilon(k^*_\gamma) \leq \rho_{\epsilon'}(k^*_\gamma). \tag{17}$$

Fix an $\varepsilon > 0$ and let $\gamma > \gamma' \geq a^*$. Then, $\widehat{d}(\xi_\gamma, \xi_{\gamma'}) = 0$ implies that there exist $\rho^\epsilon(k^*_\gamma) \in P_{k^*_\gamma}$, $\rho^\epsilon(k^*_{\gamma'}) \in P_{k^*_{\gamma'}}$ such that

$$d(x_{\rho(k^*_\gamma)}, x_{\rho(k^*_{\gamma'})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_\gamma) \geq \rho^\epsilon(k^*_\gamma) \text{ and } \rho(k^*_{\gamma'}) \geq \rho^\epsilon(k^*_{\gamma'}). \tag{18}$$

We advance to the construction of the demanded right d_S-Cauchy net $(t_\lambda)_{\lambda\in\Lambda}$ in (X, d) by using transfinite induction on the well-ordered set $(\Gamma, \leq)$. Let $\gamma_1 > \gamma_0 \geq a^*$ for some $\gamma_0, \gamma_1 \in \Gamma$ and let $(x_{\rho(k^*_{\gamma_1})})_{\rho(k^*_{\gamma_1})\in P_{k^*_{\gamma_1}}} \in \mathcal{A}_{\xi_{\gamma_1}}$, $(x_{\rho(k^*_{\gamma_0})})_{\rho(k^*_{\gamma_0})\in P_{k^*_{\gamma_0}}} \in \mathcal{A}_{\xi_{\gamma_0}}$. Then, by (16) we have

$$d(x_{\rho(k^*_{\gamma_1})}, x_{\rho'(k^*_{\gamma_1})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_1}) \geq \rho_\epsilon(k^*_{\gamma_1}),\ \rho'(k^*_{\gamma_1}) \geq \rho_\epsilon(k^*_{\gamma_1}) \text{ and } \rho'(k^*_{\gamma_1}) \ngeq \rho(k^*_{\gamma_1}). \tag{19}$$

and

$$(20)\quad d(x_{\rho(k^*_{\gamma_0})}, x_{\rho'(k^*_{\gamma_0})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_0}) \geq \rho_\epsilon(k^*_{\gamma_0}),\ \rho'(k^*_{\gamma_0}) \geq \rho_\epsilon(k^*_{\gamma_0}) \text{ and } \rho'(k^*_{\gamma_0}) \ngeq \rho(k^*_{\gamma_0}).$$

On the other hand, by (18) we have

$$(21)\quad d(x_{\rho(k^*_{\gamma_1})}, x_{\rho(k^*_{\gamma_0})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_1}) \geq \rho^\epsilon(k^*_{\gamma_1}) \text{ and } \rho(k^*_{\gamma_0}) \geq \rho^\epsilon(k^*_{\gamma_0}).$$

Let $\widetilde{\rho}_\epsilon(k^*_{\gamma_0}) = \rho_\epsilon(k^*_{\gamma_0})$ and $\widetilde{\rho}_\epsilon(k^*_{\gamma_1}) = min\{\rho_\epsilon(k^*_{\gamma_1}), \rho^\epsilon(k^*_{\gamma_1})\}$. Then, by (19), (20) and (21) we have

$$(22)\quad d(x_{\rho(k^*_{\gamma_1})}, x_{\rho(k^*_{\gamma_0})}) < \frac{2\varepsilon}{3} < \epsilon \text{ whenever } \rho(k^*_{\gamma_1}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_1}) \text{ and } \rho(k^*_{\gamma_0}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_0}).$$

We call equation (22), $T(1)$-Property.

Let now k_{γ_0} be an arbitrary member of the counter set $\mathcal{K}_{\gamma_0}$. Then, since $(x_{\rho(k_{\gamma_0})})_{\rho(k_{\gamma_0})\in P_{k_{\gamma_0}}}$ is right d-cofinal to $(x_{\rho(k^*_{\gamma_0})})_{\rho(k^*_{\gamma_0})\in P_{k^*_{\gamma_0}}}$ for each $\varepsilon > 0$ there exists $\overline{\rho}_\varepsilon(k_{\gamma_0})$ satisfying the following property: For each $\rho(k_{\gamma_0}) \geq \overline{\rho}_\varepsilon(k_{\gamma_0})$ there exists $\rho^*(k^*_{\gamma_0}) \in P_{k^*_{\gamma_0}}$ depending on $\rho(k_{\gamma_0})$ such that

$$(23)\qquad d(x_{\rho(k^*_{\gamma_0})}, x_{\rho(k_{\gamma_0})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_0}) \geq \rho^*(k^*_{\gamma_0})$$

Let $\widetilde{\widetilde{\rho}}_\epsilon(k^*_{\gamma_0}) = \min\{\widetilde{\rho}_\epsilon(k^*_{\gamma_0}), \rho^*(k^*_{\gamma_0})\}$. Then, by (22) and (23) we have

$$(24)\quad \begin{gathered} d(x_{\rho(k^*_{\gamma_1})}, x_{\rho(k_{\gamma_0})}) < d(x_{\rho(k^*_{\gamma_1})}, x_{\widetilde{\widetilde{\rho}}_\epsilon(k^*_{\gamma_0})}) + d(x_{\rho(k^*_{\gamma_0})}, x_{x_{\widetilde{\widetilde{\rho}}_\epsilon(k^*_{\gamma_0})}}) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \\ \text{whenever } \rho(k^*_{\gamma_1}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_1}) \text{ and } \rho(k_{\gamma_0}) \geq \overline{\rho}_\varepsilon(k_{\gamma_0}). \end{gathered}$$

We call equation (24), $\mathfrak{T}(1)$-Property.

Let $\gamma_2 \in \Gamma$ such that $\gamma_2 > \gamma_1 > \gamma_0 \geq a^*$ and let $(x_{\rho(k^*_{\gamma_2})})_{\rho(k^*_{\gamma_2})\in P_{k^*_{\gamma_1}}} \in \mathcal{A}_{\xi_{\gamma_2}}$ with $(x_{\rho(k^*_{\gamma_1})})_{\rho(k^*_{\gamma_1})\in P_{k^*_{\gamma_1}}} \in \mathcal{A}_{\xi_{\gamma_1}}$ and $(x_{\rho(k^*_{\gamma_0})})_{\rho(k^*_{\gamma_0})\in P_{k^*_{\gamma_1}}} \in \mathcal{A}_{\xi_{\gamma_0}}$ being as above. Then, by (16) we have

$$(25)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho'(k^*_{\gamma_2})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_2}) \geq \rho_\epsilon(k^*_{\gamma_2}),\ \rho'(k^*_{\gamma_2}) \geq \rho_\epsilon(k^*_{\gamma_2}) \text{ and } \rho'(k^*_{\gamma_2}) \ngeq \rho(k^*_{\gamma_2}).$$

On the other hand, since $\widehat{d}(\xi_{\gamma_2}, \xi_{\gamma_1}) = 0$ and $\widehat{d}(\xi_{\gamma_2}, \xi_{\gamma_0}) = 0$, by (18) we have

$$(26)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_1})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_2}) \geq \rho^\epsilon(k^*_{\gamma_2}) \text{ and } \rho(k^*_{\gamma_1}) \geq \rho^\epsilon(k^*_{\gamma_1})$$

and

$$(27)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_0})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_2}) \geq \rho^\epsilon(k^*_{\gamma_2}) \text{ and } \rho(k^*_{\gamma_0}) \geq \rho^\epsilon(k^*_{\gamma_0}).$$

Let $\widetilde{\rho}_\epsilon(k^*_{\gamma_2}) = \min\{\rho_\epsilon(k^*_{\gamma_2}), \rho^\epsilon(k^*_{\gamma_2})\}$. Then, by using $\widehat{d}(\xi_{\gamma_2}, \xi_{\gamma_1}) = 0$ we have that

$$(28)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_1})}) < \frac{\varepsilon}{3} \text{ whenever } \rho(k^*_{\gamma_2}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_2}) \text{ and } \rho(k^*_{\gamma_1}) \geq \rho'(k^*_{\gamma_1}) \text{ for some } \rho'(k^*_{\gamma_1}) \in P_{k^*_{\gamma_1}}.$$

By combining (19), (26) and (28) we have

$$(29)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_1})}) < \frac{2\varepsilon}{3} < \varepsilon \text{ whenever } \rho(k^*_{\gamma_2}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_2}) \text{ and } \rho(k^*_{\gamma_1}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_1}).$$

Moreover, by combining (22) and (28) we have

$$(30)\quad d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_0})}) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \epsilon \text{ whenever } \rho(k^*_{\gamma_2}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_2}) \text{ and } \rho(k^*_{\gamma_0}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_0}).$$

Finally, by (22), (29) and (30) we have that

$$(31)\quad \begin{gathered} d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_1})}) < \epsilon,\ d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k^*_{\gamma_0})}) < \epsilon \text{ and } d(x_{\rho(k^*_{\gamma_1})}, x_{\rho(k^*_{\gamma_0})}) < \epsilon, \\ \text{whenever } \rho(k^*_{\gamma_2}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_2}),\ \rho(k^*_{\gamma_1}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_1}) \text{ and } \rho(k^*_{\gamma_0}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_0}). \end{gathered}$$

We call equation (31), $T(2)$-Property.

Let now k_{γ_2} be an arbitrary member of the counter set $\mathcal{K}_{\gamma_2}$ and thus $(x_{\rho(k_{\gamma_2})})_{\rho(k_{\gamma_2}) \in P_{k_{\gamma_2}}} \in \mathcal{A}_{\xi_{\gamma_2}}$. Since $(x_{\rho(k_{\gamma_2})})_{\rho(k_{\gamma_2}) \in P_{k_{\gamma_2}}}$ is right d-cofinal to

$(x_{\rho(k^*_{\gamma_2})})_{\rho(k^*_{\gamma_2})\in P_{k^*_{\gamma_2}}}$ by using (29) and (30), as in the case of $\mathfrak{T}(1)$-property, we can prove that

$$\begin{aligned} &d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k_{\gamma_1})}) < \epsilon,\ d(x_{\rho(k^*_{\gamma_2})}, x_{\rho(k_{\gamma_0})}) < \epsilon \text{ and } d(x_{\rho(k^*_{\gamma_1})}, x_{\rho(k_{\gamma_0})}) < \epsilon, \\ &\text{whenever } \rho(k^*_{\gamma_2}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_2}),\ \rho(k^*_{\gamma_1}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_1}),\ \rho(k^*_{\gamma_0}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma_0}), \\ &\rho(k_{\gamma_1}) \geq \overline{\rho}_\varepsilon(k_{\gamma_1}) \text{and } \rho(k_{\gamma_0}) \geq \overline{\rho}_\varepsilon(k_{\gamma_0}) \end{aligned} \tag{32}$$

where the symbolism $\overline{\rho}_\varepsilon(k_\gamma)$ is as in the $\mathfrak{T}(1)$-property.

We call equation (32), $\mathfrak{T}(2)$-Property.

Let $\gamma \in \Gamma$ be an ordinal which has (i) the $T(\gamma)$-Property and (ii) the $\mathfrak{T}(\gamma)$-Property, that is:

(i) For each $\gamma' < \gamma$ we have

$$d(x_{\rho(k^*_\gamma)}, x_{\rho(k^*_{\gamma'})}) < \epsilon \text{ whenever } \rho(k^*_\gamma) \geq \widetilde{\rho}_\epsilon(k^*_\gamma) \text{ and } \rho(k^*_{\gamma'}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma'}) \tag{33}$$

and

$$d(x_{\rho(k^*_{\gamma'})}, x_{\rho(k^*_{\gamma''})}) < \epsilon \text{ whenever } \rho(k^*_{\gamma'}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma'}) \text{ and } \rho(k^*_{\gamma''}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma''}) \tag{34}$$

for all $\gamma'' < \gamma'$.

(ii) For each $\gamma' < \gamma$ we have

$$d(x_{\rho(k^*_\gamma)}, x_{\rho(k_{\gamma'})}) < \epsilon \text{ whenever } \rho(k^*_\gamma) \geq \widetilde{\rho}_\epsilon(k^*_\gamma) \text{ and } \rho(k_{\gamma'}) \geq \overline{\rho}_\varepsilon(k_{\gamma'}) \tag{35}$$

and

$$d(x_{\rho(k^*_{\gamma'})}, x_{\rho(k^*_{\gamma''})}) < \epsilon \text{ whenever } \rho(k^*_{\gamma'}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma'}) \text{ and } \rho(k_{\gamma''}) \geq \overline{\rho}_\varepsilon(k_{\gamma''}) \tag{36}$$

for all $\gamma'' < \gamma'$ (we recall that $(x_{\rho(k_{\gamma'})})_{\rho(k_{\gamma'})\in P_{k_{\gamma'}}}$ and $(x_{\rho(k_{\gamma''})})_{\rho(k_{\gamma''})\in P_{k_{\gamma''}}}$ are arbitrary right S-nets of $\mathcal{A}_{\xi_{\gamma'}}$ and $\mathcal{A}_{\xi_{\gamma''}}$, respectively).

According to transfinite induction, if $T(\gamma)$ is true whenever $T(\gamma')$ is true for all $\gamma' < \gamma$, then $T(\gamma)$ is true for all γ.

We follow two steps:

(i) *Successor case:* Prove that for any successor ordinal $\gamma + 1$, $T(\gamma + 1)$ follows from $T(\gamma)$.

(ii) *Limit case:* Prove that for any limit ordinal γ, $T(\gamma)$ follows from $[T(\gamma')$ for all $\gamma' < \gamma]$.

Step (i). Let γ be a successor ordinal. Let also δ be an ordinal such that $\gamma < \delta$. If δ is a limit ordinal, then there exists an ordinal ζ' such that $\gamma < \zeta' < \delta$, otherwise, there exists an ordinal ζ such that $\gamma < \delta < \zeta$. We only prove that $T(\gamma+1)$ follows from $T(\gamma)$ from the case $\gamma < \delta < \zeta$ with $\gamma+1=\zeta$ (the case $\gamma < \zeta' < \delta$ is similar for $\gamma+1=\delta$).

By repeating the steps (16)-(34) above for γ', δ, ζ, where $\gamma' \leq \gamma$, instead of $\gamma_0, \gamma_1, \gamma_2$, considering them with same layout, we conclude that

$$d(x_{\rho(k^*_\zeta)}, x_{\rho(k^*_\gamma)}) < \epsilon \text{ whenever } \rho(k^*_\zeta) \geq \widetilde{\rho}_\epsilon(k^*_\zeta) \text{ and } \rho(k^*_\gamma) \geq \widetilde{\rho}_\epsilon(k^*_\gamma) \tag{37}$$

and

$$d(x_{\rho(k^*_\zeta)}, x_{\rho(k^*_{\gamma'})}) < \epsilon \text{ whenever } \rho(k^*_\zeta) \geq \widetilde{\rho}_\epsilon(k^*_\zeta) \text{ and } \rho(k^*_{\gamma'}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma'}). \tag{38}$$

for all $\gamma' < \gamma$.

Therefore, $T(\gamma+1) = T(\zeta)$ holds.

Step (ii). Let γ be a limit ordinal. As it is well-known, an ordinal γ is a limit ordinal if and only if there is an ordinal less than γ, and whenever γ' is an ordinal less than γ, then there exists an ordinal γ'' such that $\gamma' < \gamma'' < \gamma$. Therefore, by repeating the steps (16)-(34) above for $\gamma', \gamma'', \gamma$ instead of $\gamma_0, \gamma_1, \gamma_2$, considering them with same layout, we conclude that

$$d(x_{\rho(k^*_\gamma)}, x_{\rho(k^*_{\gamma''})}) < \epsilon \text{ whenever } \rho(k^*_\gamma) \geq \widetilde{\rho}_\epsilon(k^*_\gamma) \text{ and } \rho(k^*_{\gamma''}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma''}). \tag{39}$$

and

$$d(x_{\rho(k^*_{\gamma''})}, x_{\rho(k^*_{\gamma'})}) < \epsilon \text{ whenever } \rho(k^*_{\gamma''}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma''}) \text{ and } \rho(k^*_{\gamma'}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma'}) \tag{40}$$

for all $\gamma, \gamma', \gamma''$ with $\gamma' < \gamma'' < \gamma$.

Therefore, $T(\gamma)$ holds. It follows that

$$T(\gamma) \text{ is true for all } \gamma \in \widehat{\Gamma}, \text{ where } \widehat{\Gamma} \text{ is cofinal to } \Gamma. \tag{41}$$

Similarly, we have that

$$\mathfrak{T}(\gamma) \text{ is true for all } \gamma \in \widehat{\Gamma}, \text{ where } \widehat{\Gamma} \text{ is cofinal to } \Gamma, \text{ that is,} \tag{42}$$

$$d(x_{\rho(k^*_\gamma)}, x_{\rho(k_{\gamma''})}) < \epsilon, \text{ whenever } \rho(k^*_\gamma) \geq \widetilde{\rho}_\epsilon(k^*_\gamma) \text{ and } \rho(k_{\gamma''}) \geq \overline{\rho}_\varepsilon(k_{\gamma''}) \tag{43}$$

and

(44) $\quad d(x_{\rho(k^*_{\gamma''})}, x_{\rho(k_{\gamma'})}) < \epsilon,$ whenever $\rho(k^*_{\gamma''}) \geq \widetilde{\rho}_\epsilon(k^*_{\gamma''})$ and $\rho(k_{\gamma'}) \geq \overline{\rho}_\varepsilon(k_{\gamma'})$

for all $\gamma, \gamma', \gamma''$ with $\gamma' < \gamma'' < \gamma$ (we recall that in contrast to the $T(\gamma)$-property where $k^*_{\gamma'}$ and $k^*_{\gamma''}$ are concrete members of the counter sets $\mathcal{K}_{\gamma'}$ and $\mathcal{K}_{\gamma''}$, respectively, in $\mathfrak{T}(\gamma)$-property $k_{\gamma'}$ and $k_{\gamma''}$ are arbitrary members of the counter sets $\mathcal{K}_{\gamma'}$ and $\mathcal{K}_{\gamma''}$ respectively).

By (17), for each $\epsilon' < \epsilon$, we have $\widetilde{\rho}_\epsilon(k^*_\gamma) \leq \widetilde{\rho}_{\epsilon'}(k^*_\gamma)$ which jointly to (39),(40) and (41) we conclude that

(45) $d(x_{\widetilde{\rho}_\epsilon(k^*_\gamma)}, x_{\widetilde{\rho}_{\epsilon'}(k^*_{\gamma'})}) < \epsilon$ whenever $\gamma \geq \gamma' \geq \gamma_0$, with $\gamma_0, \gamma', \gamma \in \widehat{\Gamma}$, and $\epsilon \leq \epsilon'$.

Let $\widetilde{\Gamma} = \{(\gamma, \epsilon) |\ \gamma \in \widehat{\Gamma},\ \epsilon > 0\}$. We define an order $\preceq$ on $\widetilde{\Gamma}$ as follows:

$$(46) \qquad \begin{array}{c} (\gamma', \epsilon') \prec (\gamma, \epsilon) \text{ if and only if } (i)\gamma' < \gamma \text{ or } (ii)\ \gamma' = \gamma \text{ and } \epsilon < \epsilon' \\ \text{and } (\gamma', \epsilon') = (\gamma, \epsilon) \text{ if and only if } \gamma' = \gamma \text{ and } \epsilon = \epsilon'. \end{array}$$

Clearly, $\widetilde{\Gamma}$ is directed with respect to $\preceq$. Let $(t_\sigma)_{\sigma\in\Sigma}$ be the net $(x_{(\gamma,\epsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ where $x_{(\gamma,\epsilon)} = x_{\widetilde{\rho}_\epsilon(k^*_\gamma)}$. Then, by (45), $(x_{(\gamma,\epsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ is a right S-net in (X, d). Therefore, the required net $(t_\sigma)_{\sigma\in\Sigma}$ of Part I is the net $(x_{(\gamma,\varepsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$.

Part II: *The right* $\widehat{d}_S$*-Cauchy nets* $(\xi_a)_{a\in A}$ *and* $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ *are right* $\widehat{d}$*-cofinal.*

Since $(\xi_a)_{a\in A}$ and $(\xi_\gamma)_{\gamma\in\Gamma}$ are right $\widehat{d}$-cofinal and $(\xi_\gamma)_{\gamma\in\Gamma}$ and $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$ are right $\widehat{d}$-cofinal, to prove that $(\xi_a)_{a\in A}$ and $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ are right $\widehat{d}$-cofinal, we have to prove that $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$ and $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ are right $\widehat{d}$-cofinal. We first prove that $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$ is right $\widehat{d}$-cofinal to $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$. Indeed, let $\epsilon^* > 0$. Choose a $\widehat{\gamma} \in \widehat{\Gamma}$ depending on $\epsilon^* > 0$ such that $\widehat{\gamma} = \gamma_{\epsilon^*} \geq a^*$. Then, if $\overline{\gamma} \geq \widehat{\gamma}$, by (35), (36) and property-$\mathfrak{T}(\gamma)$ we have that

(47)

$d(x_{\widetilde{\rho}_\epsilon(k^*_\gamma)}, x_{\rho(k_{\overline{\gamma}})}) < \epsilon$ whenever $\rho(k_{\overline{\gamma}}) \geq \overline{\rho}_0(k_{\overline{\gamma}})$ for some $\overline{\rho}_0(k_{\overline{\gamma}}) \in P_{k_{\overline{\gamma}}}$ and $\varepsilon > 0$,

where $(x_{\rho(k_{\overline{\gamma}})})_{\rho(k_{\overline{\gamma}})\in P_{k_{\overline{\gamma}}}}$ is an arbitrary right S-net of $\mathcal{A}_{\xi_{\overline{\gamma}}}$. Then, there exists $(\gamma, \epsilon)_{\overline{\gamma}} = \widehat{\gamma}$ with $\widehat{\gamma} > \overline{\gamma}$ such that

(48) $\qquad d((x_{(\gamma,\epsilon)}), x_{\rho(k_{\overline{\gamma}})}) < \epsilon$ whenever $\gamma \geq \widehat{\gamma}$ and $\epsilon > \epsilon^*$ for some $\epsilon^* > 0$.

Hence,

$$d(\phi(x_{(\gamma,\epsilon)})), \xi_{\overline{\gamma}}) < \epsilon \text{ whenever } (\gamma,\epsilon) \geq (\widehat{\gamma},\epsilon^*). \tag{49}$$

Hence, $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$ is right $\widehat{d}$-cofinal to $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$.

To prove the converse, that is, $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ is right $\widehat{d}$-cofinal to $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$, let $\varepsilon > 0$ and let $\gamma_\varepsilon \in \widehat{\Gamma}$ such that

$$\widehat{d}(\xi_\gamma, \xi_{\overline{\gamma}}) < \frac{\varepsilon}{3} \text{ whenever } \gamma \geq \overline{\gamma} \geq \gamma_\varepsilon. \tag{50}$$

Suppose that $(\gamma_\varepsilon, \widetilde{\varepsilon'}) \in \widetilde{\Gamma}$ with $\widetilde{\varepsilon'} \leq \frac{\varepsilon}{3}$. We prove that for each $(\overline{\gamma},\widetilde{\varepsilon}) \geq (\gamma_\varepsilon,\widetilde{\varepsilon'})$ there exists $\gamma^*_{(\overline{\gamma},\widetilde{\varepsilon})} \in \widehat{\Gamma}$ with the property:

For each $\gamma \geq \gamma^*_{(\overline{\gamma},\widetilde{\varepsilon})} \in \widehat{\Gamma}$ we have

$$\widehat{d}(\xi_\gamma, \phi(x_{(\overline{\gamma},\widetilde{\varepsilon})}) < \varepsilon. \tag{51}$$

Indeed, let $(x_\theta)_{\theta\in\Theta}$ be an arbitrary right S-net of $\mathcal{A}_{\xi_\gamma}$ and let $(x_{\rho(k^*_{\overline{\gamma}})})_{\rho(k^*_{\overline{\gamma}})\in P_{k^*_{\overline{\gamma}}}} \in \mathcal{A}_{\xi_{\overline{\gamma}}}$, where $k^*_{\overline{\gamma}}$ is the concrete member of the counter set $\mathcal{K}_{\overline{\gamma}}$ defined in (17). Then, by (50) we have that

$$\begin{aligned} &d(x_\theta, x_{\rho(k^*_{\overline{\gamma}})}) < \frac{\varepsilon}{3} \text{ whenever } \theta \geq \theta_0 \text{ for some } \theta_0 \in \Theta \\ &\text{and } \rho(k^*_{\overline{\gamma}}) \geq \rho_0(k^*_{\overline{\gamma}}) \text{ for some } \rho_0(k^*_{\overline{\gamma}}) \in P_{k^*_{\overline{\gamma}}}. \end{aligned} \tag{52}$$

If $\rho_0(k^*_{\overline{\gamma}}) \geq \widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}})$, then from $\widetilde{\varepsilon} \leq \widetilde{\varepsilon'} \leq \frac{\varepsilon}{3}$ $((\overline{\gamma},\widetilde{\varepsilon}) \geq (\gamma_\varepsilon,\widetilde{\varepsilon'}))$, we have

$$d(x_\theta, x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}})}) < d(x_\theta, x_{\rho_0(k^*_{\overline{\gamma}})}) + d(x_{\rho_0(k^*_{\overline{\gamma}})}), x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}})}) < \frac{2\varepsilon}{3}. \tag{53}$$

Otherwise, if $\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}}) \geq \rho_0(k^*_{\overline{\gamma}})$, since $(x_{\rho(k^*_{\overline{\gamma}})})_{\rho(k^*_{\overline{\gamma}})\in P_{k^*_{\overline{\gamma}}}}$ then

$$d(x_\theta, x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}})}) < \frac{\varepsilon}{3} < \frac{2\varepsilon}{3}. \tag{54}$$

Therefore, by (52), (53) and (54) we have

$$d(x_\theta, x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}})}) < \frac{2\varepsilon}{3} \text{ whenever } \theta \geq \theta_0. \tag{55}$$

Let now $(x_\nu)_{\nu\in N}$ be a right S-net of $\mathcal{A}_{\phi(x_{(\overline{\gamma},\widetilde{\varepsilon})})}$. Then, since $(x_{(\overline{\gamma},\widetilde{\varepsilon})}, x_\nu) \to 0$, there exists $\nu_0 \in N$ such that for each $\nu \geq \nu_o$ we have

$$d(x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}}))}, x_\nu) < \frac{\varepsilon}{3} \text{ whenever } \nu \geq \nu_0. \tag{56}$$

By (55) and (56) we have that

$$d(x_\theta, x_\nu) \leq d(x_\theta, x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}}))}) + d(x_{\widetilde{\rho}_{\widetilde{\varepsilon}}(k^*_{\overline{\gamma}}))}, x_\nu) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{whenever } \nu \geq \nu_0 \text{ and } \theta \geq \theta_0. \tag{57}$$

It follows that $\widehat{d}(\xi_\gamma, \phi(x_{(\overline{\gamma},\widetilde{\varepsilon})}) < \varepsilon$ whenever $\gamma \geq \gamma^*_{(\overline{\gamma},\widetilde{\varepsilon})}$. Therefore, $(\phi(x_{(\gamma,\epsilon)}))_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ is right $\widehat{d}$-cofinal to $(\xi_\gamma)_{\gamma\in\widehat{\Gamma}}$. Hence, the required net $(t_\sigma)_{\sigma\in\Sigma}$ of the hypothesis is the net $(x_{(\gamma,\varepsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$. □

If the cardinality of index set A in Proposition 3.16 equals to the cardinality $\aleph_0$ of the set of all natural numbers, then we have the following corollary (see also [2, Proposition 25]).

Corollary 3.17. *Let (X, d) be a quasi-pseudometric space and let $(\xi_n)_{n\in\mathbb{N}}$ be a non-constant right K-sequence in $(\widehat{X}, \widehat{d})$ without last element. Then, there exists a right K-sequence $(t_\nu)_{\nu\in\mathbb{N}}$ in (X, d) such that the sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\phi(t_\nu))_{\nu\in\mathbb{N}}$ are right $\widehat{d}$-cofinal sequences.*

P r o o f. Let $(\xi_n)_{n\in\mathbb{N}}$ be a non-constant right K-sequence in $(\widehat{X}, \widehat{d})$ without last element and let $(x_{(\gamma,\varepsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}}$ be as in Proposition 3.16. Since $(\xi_n)_{n\in\mathbb{N}}$ is a sequence, we have that $(x_{(\gamma,\varepsilon)})_{(\gamma,\epsilon)\in\widetilde{\Gamma}} = (x_{(\gamma(n),\varepsilon_n)})_{(\gamma(n),\epsilon_n)\in\widetilde{\mathbb{N}}}$, where $\widetilde{\Gamma} = \widetilde{\mathbb{N}} \subseteq \mathbb{N}$ and $\varepsilon_n = \max\left\{\frac{1}{n} | \frac{1}{n} < \varepsilon\right\}$. By Proposition 3.16, we have that $(\xi_n)_{n\in\mathbb{N}}$ and $(\phi(x_{(\gamma(n),\varepsilon_n)}))_{(\gamma(n),\epsilon_n)\in\widetilde{\mathbb{N}}}$ are right $\widehat{d}$-cofinal sequences. Therefore, the required sequence $(t_\nu)_{\nu\in N}$ of the hypothesis is the sequence $(x_{(\gamma(n),\varepsilon_n)})_{(\gamma(n),\epsilon_n)\in\widetilde{\mathbb{N}}}$. □

Proposition 3.18. *Let (X, d) be a quasi-pseudometric space and let $(\xi_a)_{a\in A}$ be a non-constant right S-net in $(\widehat{X}, \widehat{d})$ without last element. Then, there exists a right S-net $(t_\sigma)_{\sigma\in\Sigma}$ in (X, d) such that the nets $(\xi_a)_{a\in A}$ and $(\phi(t_\sigma))_{\sigma\in\Sigma}$ are right $\widehat{d}$-cofinal nets.*

P r o o f. Let $(\widehat{X}, \widehat{d})$ and $(\xi_a)_{a\in A}$ be as in the assumptions of the Proposition. Without loss of generality, we can assume that for each $a, a' \in A$, $a \neq a'$, we have $\mathcal{A}_{\xi_a} \neq \mathcal{A}_{\xi_{a'}}$. Pick $a_0 \in A$ such that $\widehat{d}(\xi_a, \xi_{a'}) \leq \frac{1}{2^0} = 1$ for $a \geq a_0$,

$a' \geq a_0$ and $a' \not\geq a$. Given $a_n \in A$ such that $\widehat{d}(\xi_a, \xi_{a'}) \leq \frac{1}{2^n}$ whenever $a \geq a_n$, $a' \geq a_n$ and $a' \not\geq a$, choose $a_{n+1} \in A$ such that $a_{n+1} \geq a_n$ and $\widehat{d}(\xi_a, \xi_{a'}) \leq \frac{1}{2^{n+1}}$ whenever $a \geq a_{n+1}$, $a' \geq a_{n+1}$ and $a' \not\geq a$. Then, the sequence $(\xi_{a_n})_{n\in\mathbb{N}}$ is a right K-sequence in $(\widehat{X}, \widehat{d})$. We have two cases to consider; (i) There exists $a^* \in A$ such that $a^* > a_n$ for each $n \in \mathbb{N}$; (ii) For each $a \in A$ there exists $n \in \mathbb{N}$ and $a_n \in \mathbb{N}$ such that $a \not> a_n$. In case (i), for each $a, a' \geq a^* \geq a_n, n \in \mathbb{N}$ and $a' \not\geq a$, we have $\widehat{d}(\xi_a, \xi_{a'}) \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Hence, $\widehat{d}(\xi_a, \xi_{a'}) = 0$. Thus, Proposition 3.16 ensures the existence of a right d_S-net $(t_\sigma)_{\sigma\in\Sigma}$ in (X, d) such that the nets $(\xi_a)_{a\in A}$ and $(\phi(t_\sigma))_{\sigma\in\Sigma}$ are right $\widehat{d}$-cofinal nets. In case (ii), since $(\xi_{a_n})_{n\in\mathbb{N}}$ is a subnet of $(\xi_a)_{a\in A}$, Proposition 3.5 implies that $(\xi_{a_n})_{n\in\mathbb{N}}$ and $(\xi_a)_{a\in A}$ are right $\widehat{d}$-cofinal. On the other hand, Corollary 3.17 implies that there is right K-net $(t_\nu)_{\nu\in\mathbb{N}}$ in (X, d) such that the nets $(\xi_{a_n})_{n\in\mathbb{N}}$ and $(\phi(t_\nu))_{\nu\in\mathbb{N}}$ are right $\widehat{d}$-cofinal nets. It follows that $(\xi_a)_{a\in A}$ and $(\phi(t_\nu))_{\nu\in\mathbb{N}}$ are right $\widehat{d}$-cofinal nets. □

Because a net is a left S-net in (X, d) if and only if it is a right S-net in (X, d^{-1}), we may prove the following claim in the same way we proved the preceding one.

Proposition 3.19. *Let (X, d) be a quasi-pseudometric space and let $(\eta_\beta)_{\beta\in B}$ be a non-constant left S-net in $(\widehat{X}, \widehat{d})$ without last element. Then, there exists a left S-net $(\mathsf{t}_\tau)_{\tau\in T}$ in (X, d) such that the nets $(\eta_\beta)_{\beta\in B}$ and $(\phi(\mathsf{t}_\tau))_{\tau\in T}$ are left $\widehat{d}$-cofinal nets.*

Theorem 3.20. *Every quasi-pseudometric space (X, d) has a δ-completion.*

P r o o f. Let $(\xi_a)_{a\in A}$ be a δ-Cauchy net in the space $(\widehat{X}, \widehat{d})$. Then, by definition 3.8, there exists a δ-cut $\widehat{\xi} \in \widehat{X}$ such that $(\xi_a)_{a\in A} \in \mathcal{A}_{\widehat{\xi}}$. Let

$$\widehat{\xi} = (\mathcal{A}_{\widehat{\xi}}, \mathcal{B}_{\widehat{\xi}}) \text{ where} \mathcal{A}_{\widehat{\xi}} = \{(\xi^i_k)_{k\in K_i} | i \in I\} \text{ and } \mathcal{B}_{\widehat{\xi}} = \{(\eta^j_\lambda)_{\lambda\in\Lambda_j} | j \in J\}. \tag{58}$$

We demonstrate the existence of a δ-cut ξ in (X, d) such that $(\xi_a)_{a\in A}$ converges to ξ.

We define $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$, where

$$\mathcal{A}_\xi = \{(x_\sigma)_{\sigma\in\Sigma} \mid (x_\sigma)_{\sigma\in\Sigma} \text{ is a right } S-\text{net in } (X, d) \text{ such that } (\phi(x_\sigma)_{\sigma\in\Sigma} \in \mathcal{A}_{\widehat{\xi}}\} \tag{59}$$

and

$$(60)\quad \mathcal{B}_{\xi} = \{(y_{\rho})_{\rho\in P} \mid (y_{\rho})_{\rho\in P} \text{ is a left } S-\text{net in } (X,d) \text{ such that } (\phi(y_{\rho})_{\rho\in P} \in \mathcal{B}_{\widehat{\xi}}\}.$$

The classes $\mathcal{A}_{\xi}$ and $\mathcal{B}_{\xi}$ are non-void according to Propositions 3.18 and 3.19. We begin by confirming that $\xi = (\mathcal{A}_{\xi}, \mathcal{B}_{\xi})$ is a δ-cut in (X,d). To do so, we must demonstrate that the pair $(\mathcal{A}_{\xi}, \mathcal{B}_{\xi})$ meets the conditions of Definition 3.6. We first prove the validity of Condition (i) of Definition 3.6. Let $(x_{\sigma})_{\sigma\in\Sigma} \in \mathcal{A}_{\xi}$ and $(y_{\rho})_{\rho\in P} \in \mathcal{B}_{\xi}$. Then, by construction of $\mathcal{A}_{\xi}$ and $\mathcal{B}_{\xi}$, we have $\lim\limits_{\rho,\sigma} \widehat{d}(\phi(y_{\rho}), \phi(x_{\sigma})) = 0$. Hence, Proposition 3.14 implies that $\lim\limits_{\sigma,\rho} d(y_{\sigma}, x_{\rho}) = 0$. To demonstrate that ξ meets the second condition of Definition 3.6, consider $(x_{\sigma})_{\sigma\in\Sigma}$, $(x_{\rho})_{\rho\in P}$ to be two right d_S-Cauchy nets of $\mathcal{A}_{\xi}$. Because $(\phi(x_{\sigma}))_{\sigma\in\Sigma}$, $(\phi(x_{\rho}))_{\rho\in P}$ belong to $\mathcal{A}_{\widehat{\xi}}$, they are right $\widehat{d}$-cofinal according to the definition of ξ. As a result, Proposition 3.14 comes to the conclusion that $(x_{\sigma})_{\sigma\in\Sigma}$ and $(x_{\rho})_{\rho\in P}$ are right d-cofinal. Finally, condition (iii) of Definition 3.6 follows directly from the maximality of $\mathcal{A}_{\xi}$ and $\mathcal{B}_{\xi}$, respectively. Similarly, we can show conditions (ii) and (iii) for left d_S-Cauchy nets, which are members of $\mathcal{B}_{\xi}$.

We now prove that $(\xi_a)_{a\in A}$ converges to ξ. Indeed, Proposition 3.18 states that there exists a right d_S-Cauchy net $(x_{\sigma})_{\sigma\in\Sigma}$ in (X,d) such that the nets $(\xi_a)_{a\in A}$ and $(\phi(x_{\sigma}))_{\sigma\in\Sigma}$ are right $\widehat{d}$-cofinal. According to Proposition 3.15, $\phi(x_{\sigma}) \longrightarrow \xi$. Since $(\phi(x_{\sigma}))_{\sigma\in\Sigma}$ and $(\xi_a)_{a\in A}$ are right $\widehat{d}$-cofinal, Proposition 3.8 implies that $\xi_a \longrightarrow \xi$. It follows that (X,d) is δ-complete. □

Theorem 3.21. *Let (X,d) be a quasi-pseudometric space and let $\widehat{\xi}$ be a δ-cut in $(\widehat{X}, \widehat{d})$. Then, there exists a ξ in $(\widehat{X}, \widehat{d})$ such that each $(\xi_a)_{a\in A} \in \mathcal{A}_{\widehat{\xi}}$ converges to ξ with respect to $\tau_{\widehat{d}}$ and each $(\eta_{\beta})_{\beta\in B} \in \mathcal{B}_{\widehat{\xi}}$ converges to ξ with respect to $\tau_{\widehat{d}}$.*

Proof. Let $\widehat{\xi} = (\mathcal{A}_{\widehat{\xi}}, \mathcal{B}_{\widehat{\xi}}) \in \widehat{X}$ be a δ-cut in (X,d) and $(\xi_a)_{a\in A} \in \mathcal{A}_{\widehat{\xi}}$.

Then, according to Definition 3.8 and Theorem 3.20, $(\xi_a)_{a\in A}$ will converge to a δ-cut $\xi = (\mathcal{A}_{\xi}, \mathcal{B}_{\xi})$ in X, where

$$\mathcal{A}_{\xi} = \{(x_{\sigma})_{\sigma\in\Sigma} \mid (x_{\sigma})_{\sigma\in\Sigma} \text{ is a right } S\text{-net in } (X,d) \text{ such that } (\phi(x_{\sigma})_{\sigma\in\Sigma} \in \mathcal{A}_{\widehat{\xi}}\}$$
$$\mathcal{B}_{\xi} = \{(y_{\rho})_{\rho\in P} \mid (y_{\rho})_{\rho\in P} \text{ is a left } S\text{-net in } (X,d) \text{ such that } (\phi(y_{\rho})_{\rho\in P} \in \mathcal{B}_{\widehat{\xi}}\}.$$

If $(\xi_{\gamma})_{\gamma\in\Gamma} \in \mathcal{A}_{\widehat{\xi}}$ is an arbitrary right S-net of $\mathcal{A}_{\widehat{\xi}}$, then by Corollary 3.8, since $(\xi_a)_{a\in A}$ and $(\xi_{\gamma})_{\gamma\in\Gamma}$ are right S-cofinal we have that $(\xi_{\gamma})_{\gamma\in\Gamma}$ converges to ξ with respect to $\tau(\widehat{d})$. Similarly, if $(\eta_{\beta})_{\beta\in B}$ is a left S-net in $\mathcal{B}_{\widehat{\xi}}$, then by Proposition

3.19, there exists a left S-net $(\mathfrak{t}_{\tau})_{\tau\in T}$ in (X,d) such that the nets $(\eta_{\beta})_{\beta\in B}$ and $(\phi(\mathfrak{t}_{\tau}))_{\tau\in T}$ are left $\widehat{d}$-cofinal nets. By Proposition 3.7, $(\phi(\mathfrak{t}_{\tau}))_{\tau\in T}$ belongs to $\mathcal{B}_{\widehat{\xi}}$. Hence, $(\mathfrak{t}_{\tau})_{\tau\in T}$ belongs to $\mathcal{B}_{\xi}$. By Proposition 3.15, $(\phi(\mathfrak{t}_{\tau}))_{\tau\in T}$ converges to ξ with respect to $\tau(\widehat{d^{-1}})$. Therefore, since $(\eta_{\beta})_{\beta\in B}$ and $(\phi(\mathfrak{t}_{\tau}))_{\tau\in T}$ are left $\widehat{d}$-cofinal, by 3.8 we have that $(\eta_{\beta})_{\beta\in B}$ converges to ξ with respect to $\tau(\widehat{d^{-1}})$. □

4. The $\mathcal{U}$-completion. We now pass from quasi-pseudometric to quasi-uniform completion theory.

Definition 4.1. *A net $(x_a)_{a\in A}$ in a quasi-uniform space $(X,\mathcal{U})$ is said to be* right *(resp.* left*) $\mathcal{U}$-cofinal to a net $(x_{\beta})_{\beta\in B}$ on X, if for each $U\in\mathcal{U}$ there exists $a_U\in A$ satisfying the following property: For each $a\geq a_U$ there exists $\beta_a\in B$ such that $(x_{\beta},x_a)\in U$ (resp. $(x_a,x_{\beta})\in U$) whenever $\beta\geq\beta_a$. The nets $(x_a)_{a\in A}$ and $(x_{\beta})_{\beta\in B}$ are* right *(resp.* left*) $\mathcal{U}$*-cofinal *if $(x_a)_{a\in A}$ is right (resp. left) $\mathcal{U}$-cofinal to $(x_{\beta})_{\beta\in B}$ and vice versa.*

Definition 4.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space. We call $\mathcal{U}$-cut in X an ordered pair $\xi=(\mathcal{A}_{\xi},\mathcal{B}_{\xi})$ of families of right S-nets and left S-nets, respectively, with the following properties:*

(i) Any $(x_a)_{a\in A}\in\mathcal{A}_{\xi}$ has as conet any $(y_{\beta})_{\beta\in B}\in\mathcal{B}_{\xi}$;

(ii) Any two members of the family $\mathcal{A}_{\xi}$ (resp. $\mathcal{B}_{\xi}$) are right (resp. left) $\mathcal{U}$-cofinal.

(iii) The classes $\mathcal{A}_{\xi}$ and $\mathcal{B}_{\xi}$ are maximal with respect to set inclusion.

Definition 4.3. *To every $x\in X$ we correspond the $\mathcal{U}$-cut $\phi(x)=(\mathcal{A}_{\phi(x)},\mathcal{B}_{\phi(x)})$ satisfying the requirements (i)–(iii) of Definition 4.2 with the additional condition (iv): The members of $\mathcal{A}_{\phi(x)}$ converge to x with respect to $\tau_{\mathcal{U}}$ and the members of $\mathcal{B}_{\phi(x)}$ converge to x with respect to $\tau_{\mathcal{U}^{-1}}$.*

According to Definition 4.3, the net $(x)=x,x,x,\ldots$ itself, belongs to both of the classes $\mathcal{A}_{\phi(x)}$ and $\mathcal{B}_{\phi(x)}$.

Definition 4.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space. We call $\mathcal{U}$-*Cauchy net *any right S-net in $(X,\mathcal{U})$ member of the first class of a $\mathcal{U}$-cut. The space $(X,\mathcal{U})$ is $\mathcal{U}$-*complete *if and only if each $\mathcal{U}$-Cauchy net converges in X.*

Two $\mathcal{U}$-Cauchy nets $(x_a)_{a\in A}$ and $(x_{\beta})_{\beta\in B}$ in a quasi-uniform space $(X,\mathcal{U})$ are said to be $\mathcal{U}$-*equivalent* if any conet to $(x_a)_{a\in A}$ is a conet to $(x_{\beta})_{\beta\in B}$ and vice versa.

It is easy to see that $\mathcal{U}$-equivalence defines an equivalence relation on $(X,\mathcal{U})$. Hence, $\mathcal{A}$ is the equivalence class of the $\mathcal{U}$-Cauchy nets that are considered equivalent by this equivalence relation.

In uniform spaces the classes $\mathcal{A}$ and $\mathcal{B}$ coincide, resulting in a class of Cauchy nets, which is the known used in uniform completion

Proposition 4.1. *Let (X,d) be a quasi-pseudometric space and let $\mathcal{U}_d$ be the quasi-uniformity induced on X by d. Then, $(X,\mathcal{U}_d)$ is $\mathcal{U}_d$-complete if and only if (X,d) is δ-complete.*

Proof. By definition, we have that each quasi-pseudometric d on X generates a quasi-uniformity $\mathcal{U}_d$ with base the family $\{U_\varepsilon|\varepsilon > 0\}$ where $U_\varepsilon = \{(x,y) \in X \times X | d(x,y) < \epsilon\}$. Therefore, a net $(x_a)_{a\in A}$ is δ-Cauchy in (X,d) if and only if it is a $\mathcal{U}_d$-Cauchy in $(X,\mathcal{U}_d)$. The Proposition follows directly from Definitions 3.8 and 4.4. □

Lemma 4.2. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be any quasi-uniform spaces and let $f : (X,\mathcal{U}) \longrightarrow (X,\mathcal{V})$ be a quasi-uniformly continuous mapping. If $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$ where $\mathcal{A}_\xi = \{(x^i_a)_{a\in A} | i \in I\}$ and $\mathcal{B}_\xi = \{(y^j_\beta)_{\beta\in B} | j \in J\}$ is a $\mathcal{U}$-cut in $(X,\mathcal{U})$, then $f(\xi) = (f(\mathcal{A}_\xi), f(\mathcal{B}_\xi))$ where $f(\mathcal{A}_\xi) = \{(f(x^i_a))_{a\in A} | i \in I\}$ and $f(\mathcal{B}_\xi) = \{(f(y^j_\beta))_{\beta\in B} | j \in J\}$ is a $\mathcal{V}$-cut in $(X,\mathcal{V})$.*

Proof. Since f is quasi-uniform continuous, if $(x_a)_{a\in A}$ is a right S-net in $(X,\mathcal{U})$ with a left S-net $(y_\beta)_{\beta\in B}$ as conet, then $(f(x_a))_{a\in A}$ is a right S-net in $(Y,\mathcal{V})$ with $(f(y_\beta))_{\beta\in B}$ a left S-conet. The rest is self-evident. □

Proposition 4.3 (see[32, Theorem 1.7])**.** *Each quasi-uniform space $(X,\mathcal{U})$ can be embedded in a product of quasi-pseudometric spaces.*

Proposition 4.4. *If $\{(X_i,\mathcal{U}_i) \mid i \in I\}$ is a family of $\mathcal{U}_i$-complete quasi-uniform spaces, then the product space $\left(\widetilde{X},\widetilde{\mathcal{U}}\right) = \left(\prod_{i\in I} X_i, \prod_{i\in I} \mathcal{U}_i\right)$ is $\widetilde{\mathcal{U}}$-complete.*

Proof. Let $(\mathfrak{x}_a)_{a\in A} = \left(\prod_{i\in I} x^i_a\right)_{a\in A}$ be a $\widetilde{\mathcal{U}}$-Cauchy net in $(\widetilde{X},\widetilde{\mathcal{U}})$. Then, there exists a $\widetilde{\mathcal{U}}$-cut $\Xi = (\mathcal{A}_\Xi, \mathcal{B}_\Xi)$ where

$$\mathcal{A}_\Xi = \left\{\left(\prod_{i\in I} x^{(i,k)}_a\right)_{a\in A} | k \in K\right\} \text{ and } \mathcal{B}_\Xi = \left\{\left(\prod_{j\in J} y^{(j,\lambda)}_\beta\right)_{\beta\in B} | \lambda \in \Lambda\right\}$$

(K and Λ are counter sets of the right S-nets and left S-nets of $\mathcal{A}_\Xi$ and $\mathcal{B}_\Xi$, respectively) such that $(\mathfrak{x}_a)_{a\in A} \in \mathcal{A}_\Xi$. To each $i \in I$ we correspond the families $\mathcal{A}_{\xi_i(\Xi)} = \{(x^{(i,k)}_a)_{a\in A} | k \in K\}$ and $\mathcal{B}_{\xi_i(\Xi)} = \{(y^{(i,\lambda)}_\beta)_{\beta\in B} | \lambda \in \Lambda\}$. Because of product topology, for each $i \in I$, the pair $(\mathcal{A}_{\xi_i(\Xi)}, \mathcal{B}_{\xi_i(\Xi)}) = \xi_{i(\Xi)}$ is a $\mathcal{U}$-cut in $(X_i, \mathcal{U}_i)$. Therefore, $(x_a^i)_{a \in A} = (x_a^{(i,k^*)})_{a \in A}$ for some $k^* \in K$ is a $\mathcal{U}_i$-Cauchy net in $(X_i, \mathcal{U}_i)$. Thus, $(x_a^i)_{a \in A}$ converges to a point $x_i \in X_i$. Let $\widetilde{x} = \{x_i | i \in I\} \in \prod_{i \in I} X_i$. Then, it is easy to verify that $\mathfrak{x}_a \longrightarrow \widetilde{x}$. □

Proposition 4.5. *A closed subspace of a $\mathcal{U}$-complete quasi-uniform space $(X, \mathcal{U})$ is $\mathcal{U}$-complete.*

Proof. Let $(X, \mathcal{U})$ be $\mathcal{U}$-complete quasi-uniform space and let F be a closed subset of X. We show that $(F, \mathcal{U}_F)$ is a $\mathcal{U}$-complete quasi-uniform space. Indeed, let $(x_a)_{a \in A}$ be a $\mathcal{U}$-Cauchy net in $(F, \mathcal{U}_F)$. Therefore, there exists a $\mathcal{U}$-cut $\xi_F = (\mathcal{A}_{\xi_F}, \mathcal{B}_{\xi_F}) \in (F, \mathcal{U}_F)$ such that $(x_a)_{a \in A} \in \mathcal{A}_{\xi_F}$. But then, there is a $\mathcal{U}$-cut $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi) \in (X, \mathcal{U})$ such that $\mathcal{A}_\xi \supseteq \mathcal{A}_{\xi_F}$ and $\mathcal{B}_\xi \supseteq \mathcal{B}_{\xi_F}$. Since $(x_a)_{a \in A} \in \mathcal{A}_\xi$ we have that $(x_a)_{a \in A}$ is a $\mathcal{U}$-Cauchy net in $(X, \mathcal{U})$ and thus $(x_a)_{a \in A}$ converges to some $x \in X$. However, as F is closed, x has to be in F. Therefore, $(F, \mathcal{U}_F)$ is $\mathcal{U}$-complete. □

Theorem 4.6. *Any quasi-unifrom space $(X, \mathcal{U})$ has a $\mathcal{U}$-completion.*

Proof. By Proposition 4.3, we have that $(X, \mathcal{U})$ can be embedded in a product of quasi-pseudometric spaces $\prod_{i \in I}(X_i, d_i)$. Without loss of generality, we may assume that $d_i(x_i, y_i) \leq 1$ for all $x_i, y_i \in X_i$ and for all $i \in I$. By Theorem 3.20 each space (X_i, d_i) has a δ_i-completion $(\widehat{X}_i, \widehat{d}_i)$. Therefore, $(X, \mathcal{U})$ can be embedded in $\prod_{i \in I}(\widehat{X}_i, \widehat{d}_i)$. That is, there exists a quasi-uniformly continuous mapping $\widehat{\phi} : (X, \mathcal{U}) \longrightarrow \prod_{i \in I}(\widehat{X}_i, \widehat{d}_i)$ such that $\widehat{\phi}(X) \subseteq \prod_{i \in I}(\widehat{X}_i, \widehat{d}_i)$. Let $\overline{\widehat{\phi}(X)}$ be the closure of $\widehat{\phi}(X)$ in $\prod_{i \in I}(\widehat{X}_i, \widehat{d}_i)$ and let

$$\mathcal{U}_{\overline{\widehat{\phi}(X)}} = \left\{ U \bigcap (\overline{\widehat{\phi}(X)} \times \overline{\widehat{\phi}(X)}) | U \text{ is a member of the product quasi-uniformity for } \prod_{i \in I}(\widehat{X}_i, \widehat{d}_i) \right\}.$$

It follows from Propositions 4.4 and 4.5 that the space $\left(\overline{\widehat{\phi}(X)}, \mathcal{U}_{\overline{\widehat{\phi}(X)}}\right)$ is a $\mathcal{U}$-completion for $(X, \mathcal{U})$. □

5. The relationship between the $\mathcal{U}$-completion theory and other completion theories of quasi-uniform spaces. Many attempts have been made in the past to embed many structures into a common topological superstructure. For example, Császár [9] embeds topological, uniform, and proximity spaces into syntopogeneous spaces, and Doitchinov [15] embed generalized topological spaces into supertopological spaces. In fact, Császár [8] defined Syntopogeneous Structure utilizing the notion of order. By imposing sufficient requirements on order syntopogeneus structures, topological, proximity, and uniform structures can be derived as special examples of syntopogeneous structures. In his monograph [9], Császár obtains many of the usual theorems of general topology in this more general setting. Among them is the completeness of syntopogeneous spaces. Doitchinov [17], in his completeness theory of quasi-uniform space, adheres closely to Csazar's concept of a Cauchy object (see [9], [10]). In the present paper, starting with the work of Császár, cited above, we give a completion theory by using three key elements: ($\mathfrak{a}$) A kind of proximity between the elements of a Cauchy object; ($\mathfrak{b}$) The notion of left (resp. right) cofinality of two sequences (resp. nets) (see Definitions 3.1, 3.5 and 4.1); and ($\mathfrak{c}$) The notion of the cuts of sequences (resp. nets) which generalizes the idea of Doitchinov for a completeness theory for quasi-metric and quasi-uniform spaces and satisfies all the nice properties of bicompletion. Despite bicompletion being a very useful and satisfactory completion of a quasi-uniform space, its underlying idea depends heavily on symmetry and it sometimes yields undesirable results (see [25]). Our completion, on the other hand, is asymmetrical and perfectly compatible with quasi-uniform space characteristics. Many examples from the literature confirm the above allegations. For instance, the set $\mathbb{Q}$ of the rationals, equipped with the so-called Sorgenfrey T_0-quasi-metric $d(x, y) = 1$ if $x > y$ and $d(x, y) = y - x$ if $y \geq x$, is bicomplete, since d^* is equal to the discrete metric. In this simple example, the bicompletion of the set of rational numbers yields a trivial outcome. Our completion, on the other hand, yields the expected collection of the real numbers. Such examples have spurred various attempts in the literature to develop a completion theory for T_0-quasi-uniform spaces that is based on a more asymmetric approach. In [26, Page 879] it is pointed out that Stoltenberg [32], [33], was the first to study thoroughly the non-symmetric completion problem in quasi-uniform spaces. After Stoltenberg, many mathematicians developed completion theories for quasi-uniformities, with the only constraint that when the space is uniform, the proposed completion theory has to coincide with the classical completion theory. Doitchinov adopted and expanded on this constraint, and as a result of his efforts, we have the requirements (i)–(iv) described in the introduction.

The most talked-about requirement for a successful quasi-uniform completion is the requirement (i). In the review of Császár's book, regarding the development of the completion theory of Császár for topogeneous spaces, Isbell [22] wrote: "*An unfortunate side effect is that not every convergent filter is Cauchy*". The same problem with Császár's definition remains with all definitions described in [29] except that of Sieber-Pervin one. The definition of Sieber-Pervin has been used by many authors in order to be generalized the notion of Cauchy net in uniform spaces. Before the work of Doitchinov [17], the definition of Cauchyness of Sieber and Pervin was the only definition in the literature that met requirement (i), but, according to Doitchinov, this definition has a serious flaw. Namely, a Cauchy sequence (net) depends not only on its terms but also on some other points which need not belong to it. To support this view he gives the following example [16, Example 2]:

Example 5.1. Let $(X,\mathcal{U}_d)$ be a quasi-unifrorm space where $X = \{x_n|n = 0,1,2,\dots\}$ and $\mathcal{U}_d = \{U_\varepsilon|\varepsilon > 0\}$ with $U_\varepsilon = \{(x,y) \in X \times X|d(x,y) < \varepsilon\}$ and

$$d(x_m,y_n) = \begin{cases} 1 & \text{for } m > 0 \text{ and } m \neq n, \\ \dfrac{1}{n} & \text{for } m = 0 \text{ and } n \neq 0, \\ 0, & \text{for } m = n. \end{cases}$$

Clearly, the net $\{x_n|n = 0,1,2,\dots\}$ converges to x_0 and is thus a Cauchy net according to Sieber and Pervin's definition. If we remove the point x_0 from the space X, we get a discrete metric space in which the same net is not Cauchy. But as we can easily see, based on this example, the same goes for Doitchinov's definition.

As shown in example 5.1 above, the requirement that the definition of a Cauchy net in quasi-uniform spaces must satisfy condition (i) is incompatible with the nature of the asymmetry because this attribute is related to a symmetry property. More precisely, in a uniform space $(X,\mathcal{U})$, if a net $(x_a)_{a\in A}$ converges to a point $x_0 \in X$, then $(x_0,x_a) \to 0$ and $(x_a,x_0) \to 0$ implies that the elements of $(x_a)_{a\in A}$ become arbitrarily close to each other as the net progresses. The fact that this can not happen in topological spaces with asymmetric topology also shows the difficulty of finding a completion theory for quasi-uniform spaces to satisfy the requirement (i). According to Kunzi [24], for there to be a compatibility of requirements (i), (ii), and (vi), not all the convergent nets must be Cauchy but a subfamily of those that constitute an equivalence relation with respect to a proximity relation. The solution to this problem is to restrict the family of nets that satisfy requirement (i) to those nets that are Cauchy respect to the definition of Stoltenberg which we call from now on *S-nets*. Although the definition of Stoltenberg does not satisfy requirements (i), (v) and (vi), it satisfies a proximity relation between its elements which plays an important role for the definition of Cauchynnes that has been introduced in this paper.

According to what we said above, we replace the requirement (i) as follows:

(i') Every convergent S-net in $(X,\mathcal{U})$ is a Cauchy net.

Clearly, the concepts of convergent net and convergent S-net coincide in uniform spaces. Therefore, by restricting requirement (i) to requirement (i'), we eliminate the side effect described by Isbell for Császár's definition of Cauchyness without sacrificing generality. We now add two more conditions for a nice quasi-uniform completion as stated earlier in the introduction. It is well known that in metric spaces, the completeness via sequences and the completeness via nets agree. Further, the completeness of a metric space is equivalent to the completeness of the associated uniform space. For this reason, one of the most obvious requirements of an acceptable theory of completeness in quasi-metric spaces is the following:

(v) In quasi-metric spaces, the sequential completeness and completeness via nets agree.

On the other hand, the problem of completion in uniform spaces can be approached either with equivalence classes of nets or nets themselves. In both cases we have the same result. For example, real numbers can be defined as a quotient of Cauchy sequences of rationals. Similarly, we define the completion of a uniform space $(X,\mathcal{U})$ as the quotient of the Cauchy sequences in X and then we used it to prove that Cauchy sequences in the quotient have limits in the quotient. This procedure proceeds in three steps: we first define the Cauchy sequences and then define an equivalence relation between the Cauchy sequences and finally the quotient space of the Cauchy sequences. Alternatively, because of symmetry, the quotient step can be replaced by working with Cauchy sequences. Therefore, one last requirement for a nice quasi-uniform completion is the following:

(vi) The completion of a quasi-uniform space $(X,\mathcal{U})$ as the quotient of equivalence classes of Cauchy nets can be replaced by working with Cauchy nets themselves.

We'll take a look at a slightly different set of rules for completing quasi-uniform spaces. More specifically, it would be desirable to have a concept of Cauchy net that satisfies requirements (i′) and (*ii*), as well as a constructed completion that satisfies requirements (*iii*), (*iv*), (*v*), and (*vi*).

Summarizing what has been said so far, a natural definition of a Cauchy net in a quasi-uniform space $(X,\mathcal{U})$ has to be defined in such a manner that the following requirements are fulfilled:

($\mathfrak{i}$) Every convergent S-net is a Cauchy net;

($\mathfrak{ii}$) In the uniform case (i.e. when $(X,\mathcal{U})$ is a uniform space) the Cauchy nets are the usual ones.

Further, a standard construction of a completion $(\widehat{X},\widehat{\mathcal{U}})$ of any quasi-uniform space $(X,\mathcal{U})$ should be possible such that:

($\mathfrak{iii}$) In the case when $(X,\mathcal{U})$ is a uniform space, $(\widehat{X},\widehat{\mathcal{U}})$ is nothing but the usual uniform completion of $(X,\mathcal{U})$.

($\mathfrak{iv}$) The completion of a quasi-uniform space $(X,\mathcal{U})$ as the quotient of equivalence classes of Cauchy nets can be replaced by working with Cauchy nets themselves.

($\mathfrak{v}$) In quasi-metric spaces the sequential completeness and completeness by nets agree.

We will now look at the relationship between the $\mathcal{U}$-completion and other quasi-uniform space completions. All known completions are divided into two categories: Those that meet requirements ($\mathfrak{i}$)–($\mathfrak{v}$) and those that do not. Doitchinov's D-completion is the only one in the first category, whereas Stoltenberg's $\mathcal{U}_S$ completion is in the second. The present new completion theory for quasi-uniform spaces extends and removes the weaknesses of non-symmetric completion theories of Stoltenberg [17] and Doichinov [32]. As a result, we opt to employ these two theories to investigate their differences with $\mathcal{U}$-completion. The majority of the differences between all the other completions and $\mathcal{U}$-completion are the same as those mentioned in [16] and [17] between them and D-completion.

5.1. $\mathcal{U}$-completion and D-completion. Doitchinov's completeness theory for quiet spaces is very well behaved and extends the completion theory of uniform spaces in a natural way. But, Fletcher and Hunsanker in [18] have proved that a totally bounded quiet quasi-uniform space is already a uniformity. This means that the category of all quiet spaces is in a certain sense very small. In contrast, $\mathcal{U}$-completion theory can be applied to any quasi-uniform space.

We now show that Doitchinov completion as well as $\mathcal{U}$-completion satisfy requirements ($\mathfrak{i}$)-($\mathfrak{v}$).

The following definitions are due to Doitchinov (see [17, Condition (Q) and Definition 11]) and inspired us to define the notion of cuts of nets. More precisely:

Doitchinov [16] developed an interesting completion theory for quiet quasi-uniform spaces, and he gives the following definition for Cauchy net:

Definition 5.1. *A net $(x_a)_{a\in A}$ in a quasi-uniform space is D-Cauchy if there exists another net $(y_\beta)_{\beta\in B}$ such that for each $U \in \mathcal{U}$, there exist $a_U \in A$, $\beta_U \in B$ satisfying $(y_\beta, x_a) \in U$ whenever $a \geq a_U \in A$ and $\beta \geq \beta_U \in B$. The space $(X,\mathcal{U})$ is D-complete if every D-Cauchy net in X is convergent.*

Definition 5.2. *A quasi-uniform space $(X,\mathcal{U})$ is called* quiet *provided that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that, if $x', x'' \in X$ and $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ are two nets in X, then from $(x, x_a) \in V$ for $a \in A$, $(x_\beta, y) \in V$ for $\beta \in B$ and $(x_\beta, x_a) \to 0$ it follows that $(x, y) \in U$. We say that V is Q-*subordinated *to U.*

Definition 5.3. *Two D-Cauchy nets $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ are called equivalent if every conet of $(x_a)_{a\in A}$ is a conet of $(x_\beta)_{\beta\in B}$ and vice versa.*

Proposition 5.2 (see [17, Proposition 12])**.** *If two D-Cauchy nets in a quiet quasi-uniform space $(X,\mathcal{U})$ have a common conet, then they are equivalent.*

In the quasi-metric case, the notion of D-completeness is defined by sequences as follows:

Definition 5.4 (see [16, Definition 1])**.** *Let (X, d) be a quasi-pseudometric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence if for every natural number k there are a $y_k \in X$ and an $N_k \in \mathbb{N}$ such that $d(y_m, x_n) < \frac{1}{k}$ when $m, n > N_k$.*

The concept of D-Cauchy net (sequence) proposed by Doitchinov enables him to realize the program $(\mathfrak{i})$–$(\mathfrak{iii})$ outlined above under the assumption of quietness (see [17, Propositions 2, 4, 28].

Moreover, D-completeness satisfies requirement $(\mathfrak{iv})$(see [17, Proposition 12]). The following proposition shows that D-completeness satisfies requirement $(\mathfrak{v})$ as well.

Proposition 5.3. *In balanced quasi-metric spaces the notions of D-completeness by sequences and by nets agree.*

P r o o f. If a quasi-metric space is D-complete, then each D-Cauchy net converges in X, and thus each D-Cauchy sequence converges in X. Conversely, we prove that the sequential D-completeness implies that every D-Cauchy net in X is convergent. Let (X, d) be a balanced quasi-metric space in which any D-Cauchy sequence converges and let $(x_a)_{a\in A}$ be a D-Cauchy net in (X, d). Then, $(x_a)_{a\in A}$ is a D-Cauchy net in the quasi-uniform space $\mathcal{U}_d$ generated by d. By [17, Page 208], $\mathcal{U}_d = \{U_\epsilon | \epsilon > 0\}$ where $U_\epsilon = \{(x, y) \in X \times X | d(x, y) < \epsilon\}$ is a quiet quasi-uniform space. Since $(x_a)_{a\in A}$ is D-Cauchy there exists a net $(y_\beta)_{\beta\in B}$ and for any $U_\epsilon \in \mathcal{U}_d$ there are $a_{U_\epsilon} \in A$ and $\beta_{U_\epsilon} \in B$ such that $(y_\beta, x_a) \in U$ whenever $a \geq a_{U_\epsilon}$, $\beta \geq \beta_{U_\epsilon}$ or equivalently $\lim_{a,\beta} d(y_\beta, x_a) = 0$. Therefore, for each $n \in \mathbb{N}$, there exists $a_n \in A$, $\beta_n \in B$ such that $(y_{\beta_n}, x_{a_n}) \in U_n = \left\{(x, y) \in X \times X | d(x, y) < \dfrac{1}{n}\right\}$. It follows that $(x_{a_n})_{n\in\mathbb{N}}$ is a D-Cauchy sequence and thus it converges to a $x \in X$. Therefore, $\lim_n (y_{\beta_n}, x) = 0$ ([17, Lemma 15]). Let $\epsilon > 0$. Then, since $\mathcal{U}_d$ is quiet there exists $\epsilon' > 0$ such that $U_{\epsilon'}$ is Q-subordinated to U_ϵ. Let $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \epsilon'$. Then, from $(y_{\beta_n}, x_a) \in U_{\epsilon'}$, $(x, x) \in U_{\epsilon'}$ and $\lim_n (y_{\beta_n}, x) = 0$ we conclude that $(x, x_a) \in U_\epsilon$. It follows that $(x_a)_{a\in A}$ converges to x. □

By Proposition 5.2, to each D-Cauchy net $(x_a)_{a\in A}$, it corresponds a set of pairs of nets-conets which lead to the notion of cut of nets, that is, a pair $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ contains all equivalent nets of $(x_a)_{a\in A}$ and $\mathcal{D}$ contains all of their conets (see [16, Propositions 12 and 13]). The notion of $\mathcal{U}$-cut defined in this paper is a cut of nets $(\mathcal{A}, \mathcal{B})$ where the members of $\mathcal{A}$ contains right S-nets and $\mathcal{B}$ contains left S-nets as they are defined by Stoltenberg.

According what we said above, it is clear that all the requirements $(\mathfrak{i})$-$(\mathfrak{iii})$ are satisfied for the proposed $\mathcal{U}$-completion. The requirements $(\mathfrak{iv})$ and $(\mathfrak{v})$ are also satisfied as shows the following Propositions.

Proposition 5.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then, the $\mathcal{U}$-completion as the quotient of equivalence classes of Cauchy nets can be replaced by working with Cauchy nets themselves.*

Proof. It is an immediate consequence of Corollary 3.8. □

Proposition 5.5. *A quasi-metric space (X, d) is δ-complete if and only if every right K-Cauchy sequence converges to a point of (X, d).*

Proof. If a quasi-metric space is δ-complete, then each δ-Cauchy net converges in X. Therefore, each right K-Cauchy sequence converges in X.

Conversely, suppose that (X, d) is a quasi-metric space in which every right K-Cauchy sequence converges to a point of X. Let $(x_a)_{a\in A}$ be a δ-Cauchy net in X. Then, by Proposition 3.18 we have two cases to consider: $(\mathfrak{a})$ $d(x_a, x_{a'}) = 0$, where $a, a' \geq a_0$ for some $a_0 \in A$ and $a' \not\geq a$; $(\mathfrak{b})$ There exists a subsequence $(x_{a_n})_{n\in\mathbb{N}}$ of $(x_a)_{a\in A}$ such that $(x_a)_{a\in A}$ and $(x_{a_n})_{n\in\mathbb{N}}$ are right d-cofinal. In case $(\mathfrak{a})$ we have $x_a = x_{a_0}$ for all $a \in A$ with $a_0 \not\geq a$. Therefore, $(x_a)_{a\in A}$ converges to x_{a_0}. In case $(\mathfrak{b})$, $(x_{a_n})_{n\in\mathbb{N}}$ converges to a point $l \in X$. By Corollary 3.8 we have that $(x_a)_{a\in A}$ converges to l as well. □

5.2. $\mathcal{U}$-completion and $\mathcal{U}_S$-completion. The $\mathcal{U}_S$-completion of Stoltenberg is the first non-symmetric quasi-uniform completion in literature. This completion does not satisfy requirements $(\mathfrak{iii})$–$(\mathfrak{v})$. But the most important weakness of this completion theory is the requirement $(\mathfrak{v})$, which plays a central role in the Stoltenbergs' completion theory since the constructed $\mathcal{U}_S$-completion is a subset of $\prod_{i\in I}(\widehat{X}_i, \widehat{d}_i)$ ($(\widehat{X}_i, \widehat{d}_i)$ is the $(d_i)_S$-completion of the T_0 quasi-pseudometric spaces (X_i, d_i), $i \in I$).

The validity of requirement $(\mathfrak{v})$ does not hold for most completion theories. Such an example is Kelly's classic definition of Cauchyness, whome net version is the following (see [32, Definition 2.1]: *A net $(x_a)_{a\in A}$ in a quasi-uniform space $(X,\mathcal{U})$ is Cauchy if and only if for each U in $\mathcal{U}$ there exists $a_0 \in A$ such that $(x_a, x_{a'}) \in U$ whenever $a, a' \in A$ and $a \geq a' \geq a_0$.* As Stoltenberg remarked, with this definition, there exists a right K-sequentially complete quasi-metric space (X, d) which is not right $\mathcal{U}_S$-complete. He support this claim by offering the following example [32, Example 2.4]: Let $\mathcal{A}$ be the family of all countable subsets of the closed interval $\left[0, \frac{1}{3}\right]$ and let for every $A \in \mathcal{A}$ and $k \in \mathbb{N}$, $X^A_{k+1} = A \cup \left\{\frac{1}{2}, \frac{3}{4}, \ldots, \frac{2^k-1}{2^k}\right\}$ and $X^A_\infty = \bigcup_{k\geq 1} X^A_k$.

Define

$$\mathcal{X}_A = \{X^A_k | k = 1, 2, \ldots, \infty\} \quad \text{and} \quad \mathfrak{J} = \left\{\bigcup\{\mathcal{X}_A | A \in \mathcal{A}\right\}.$$

Define $d : \mathfrak{J} \times \mathfrak{J} \to [0, \infty)$ by

$$d(X^A_k, X^B_j) = \begin{cases} 0 & \text{if } A = B,\ k = j, A, B \in \mathcal{A} \text{ and } k, j \in \mathbb{N} \cup \infty, \\ \frac{1}{2^j} & \text{if } X^B_j \subset X^A_k,\ A, B \in \mathcal{A},\ k \in \mathbb{N} \cup \infty \text{ and } j \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Stoltenberg proves that $(\mathfrak{J}, d)$ is right K-sequentially complete and not d_S-complete.

To overcome the weaknesses of this definition, Stoltenberg uses Definition 3.1 and gives the following results (see [32, Theorem 2.5]): *A quasi-metric space*

(X,d) is d_S-complete if and only if every right K-Cauchy sequence in (X,d) converges with respect to τ_d in X. Gregori and Ferrer [21] showed that Stoltenberg's result based on Definition 3.1 is not valid in general. To show this, they use the space $(\mathfrak{J},d)$ above. This space is right K-sequentially complete as shown in Lemmas (a),(b) of [32, example 2.4]. In contrary to Theorem 2.5 of stoltenberg [32], Gregori and Ferrer prove that $(\mathfrak{J},d)$ is not d_S-complete. To show this, the authors define a net Φ in $(\mathfrak{J},d)$ as follows: Let $D=\mathbb{N}\cup\{a,b\}$, where $\mathbb{N}$ is the set of natural with the usual order and $a,b\notin\mathbb{N}$, $a\neq b$ and $a\geq k$ and $b\geq k$ for $k\in\mathbb{N}$, $a\geq b$, $b\geq a$, $a\geq a$ and $b\geq b$. Clearly, D is a directed set. Consider two sets $A,A\in\mathcal{A}$ with $A\subset B$and $\Phi:D\to\mathfrak{J}$ be given by

$$\Phi(k)=X_k^A \ \text{ for } \ k\in\mathbb{N},\ \Phi(a)=X_\infty^A,\ \Phi(b)=X_\infty^B.$$

Then, Φ is a right d_S-Cauchy net. Indeed, let $0<\varepsilon<1$ and $k_0\in\mathbb{N}$ be such that $\frac{1}{2^{k_0}}<\varepsilon$. We have $\lambda\leq a$, $\lambda\leq b$, $\lambda\geq k_0$ for all $\lambda\in\mathbb{N}$, $k_0\leq a\leq b$ and $k_0\leq b\leq a$. Therefore, the condition $\mu\not\geq\lambda$ can hold for some $\lambda,\mu\in D$, $\lambda,\mu\geq k_0$ in the following cases:

(α) $\mu,\lambda\in\mathbb{N}$, $\mu,\lambda\geq k_0$, $\lambda>\mu$.
(β) $\lambda=a$, $\mu\in\mathbb{N}$, $\mu\geq k_0$.
(γ) $\lambda=b$, $\mu\in\mathbb{N}$, $\mu\geq k_0$.

In the case (α), we have that $X_\mu^A\subset X_\lambda^A$ and thus

$$d(\Phi(\lambda),\Phi(\mu))=d(x_\lambda^A,x_\mu^A)=\frac{1}{2^\mu}<\frac{1}{2^{k_0}}<\varepsilon.$$

In the case (β), we have that $X_\mu^A\subset X_\infty^A$ and thus

$$d(\Phi(a),\Phi(\mu))=d(x_\infty^A,\ x_\mu^A)=\frac{1}{2^\mu}<\frac{1}{2^{k_0}}<\varepsilon.$$

The case (γ) is similar to (β). To show that Φ is not convergent in $(\mathfrak{J},d)$, let $X\in\mathfrak{J}\setminus X_\infty^B$. Then, for each $\lambda\in I$, $b\geq\lambda$ holds and thus

$$d(X,\varphi(b))=d(X,X_\infty^B)=1.$$

It concludes that for each $\varepsilon>0$, $X_\infty^B\notin B(X,\varepsilon)$ holds. Therefore, for any $\varepsilon>0$ no final segment of Φ is contained in $B(X,\varepsilon)$ which implies that Φ does not converge to X.

Similarly, if $X=X_\infty^B$, then $a\geq\lambda$ for each $\lambda\in D$ and $d(X_\infty^B,X_\infty^A)=1$.

Hence the problem that arises here is that the net Φ is a $\mathcal{U}_S$-Cauchy net but in view of the topology on $(\mathfrak{J}, d)$, it seems very inconvenient to regard this net as potentially convergent.

At this point we will look at Doitchinov's view on this problem starting with the Sieber-Pervin's definition of Cauchyness which is usually accepted as the most appropriate way of generalizing the notion of Cauchy net in uniform spaces. More precisely:

Definition 5.5 (see [16, Definition, Page 129])**.** *Let (X, d) be a quasi-pseudometric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence if for every natural number k there are a $y_k \in X$ and an $N_k \in \mathbb{N}$ such that $d(y_k, x_n) < \frac{1}{k}$ when $n > N_k$.*

According to Doitchinov, a nonformal objection to this definition of Cauchyness is illustrated by the following example.

Example 5.6 (Sorgenfrey line)**.** Let $\mathbb{R}$ be the real line equipped with the quasi-metric

$$d(x,y) = \begin{cases} y - x & \text{if } x \leq y, \\ 1 & \text{if } x > y. \end{cases}$$

For each $x \in X$, the collection $\{[x, x + r)|r > 0\}$ form a local base at the point x for the topology generated by d in X.

According to Doitchinov [16, Page 130]: *The sequence $-\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence but non-convergent in the sense of Definition 5.5. However, in view of the special character of the topology on the space $(\mathbb{R}, d)$, it seems very inconvenient to regard this sequence as a potentially convergent one, i.e. as one that could be made convergent by completing the space.*

As we describe above, in order to avoid this unwanted phenomenon, Doitchinov has been introduced the D-completeness (see Definition 5.4) which lead us to the notion of cuts of nets. In our proposed completion theory which satisfies Doitchinov's requirements for a good definition of Cauchyness, the net Φ is not a $\mathcal{U}$-Cauchy net in $(\mathfrak{J}, d)$. Indeed, since for each $X \in \mathfrak{J}$ we have that $d(X, X_\infty^A) = 1$ or $d(X, X_\infty^B) = 1$ and X_∞^A, X_∞^B belong to each final segment of Φ none of nets of $(\mathfrak{J}, d)$ could be a conet of Φ. Therefore, Φ cannot be a $\mathcal{U}$-Cauchy net. The net Φ on $(X, \mathfrak{J})$ is an example of a net which is $\mathcal{U}_S$-Cauchy net but not a $\mathcal{U}$-Cauchy net.

It is noteworthy that, Gregori and Ferrer [21] also proposed a new definition of a right $\mathcal{U}_S$-Cauchy net, for which the requirement $(\mathfrak{v})$ holds.

Definition 5.6. *A net* $(x_a)_{a\in A}$ *in a quasi-metric space* (X,d) *is called* GF*-Cauchy if one of the following conditions holds:*

(i) For every maximal element $a^* \in A$ *the net* $(x_a)_{a\in A}$ *converges to* x_{a^*}*;*

(ii) A *has no maximal elements and* $(x_a)_{a\in A}$ *converges in* X*;*

(iii) A *has no maximal elements and* $(x_a)_{a\in A}$ *satisfies Definition* 2.1 *of* [32].

Cobzas [7] has recently given new results to Stoltenberg's completion theory. To avoid the shortcomings of the preorder relation, such as those demonstrated by Gregori and Ferrer, he provides a new definition for Cauchy net in a quasi-metric space, where the sequential completeness and completeness by nets agree. This definition is the following:

Definition 5.7. *A net* $(x_a)_{a\in A}$ *in a quasi-metric space* (X,d) *is called* strongly Stoltenberg-Cauchy *if for every* $\epsilon > 0$ *there exists* $a_\epsilon \in A$ *such that, for all* $a, \beta \geq a_\epsilon$, $\beta \leq a$ *or* $a \nsim b$ *implies that* $d(x_a, x_\beta) < \epsilon$ *where* $a \nsim b$ *means that* a, β *are incomparable (that is, no one of the relations* $a \leq \beta$ *or* $\beta \leq a$ *holds).*

Question 5.1. How $\mathcal{U}$-Completion changes when the members of a $\mathcal{U}$-cuts' classes are replaced by Gregori and Ferrer Cauchy nets or Cobzas Cauchy nets, respectively.

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