Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **48** (2022), 41–68

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A NEWTON–SECANT METHOD FOR DIFFERENTIABLE SET-VALUED MAPS IN NACHI–PENOT SENSE*

Michaël Gaydu, Olguine Yacinthe, Silvere Paul Nuiro,
Alain Pietrus

Communicated by J. Revalski

Abstract. This paper is concerned with the solving of variational inclusions of the form $0 \in f(x) + g(x) + F(x) - K$, where g is a function which is differentiable at a solution x^* of the inclusion but may be not differentiable in a neighborhood of x^*. The function f and the set-valued mapping F are differentiable in the sense of Nachi–Penot [14] and K is a nonempty closed convex cone.

We introduce a Newton-Secant method to solve our problem and the sequence associated is semilocally convergent to x^* with an order equal to $\dfrac{1+\sqrt{5}}{2}$. Finally, some numerical results are also given to illustrate the convergence of the proposed method.

2020 *Mathematics Subject Classification:* 49J53, 49J40, 90C48, 65K10.
Key words: Variational inclusions, set-valued maps, divided differences, generalized differentiation of set-valued maps, normed convex processes, majorizing sequences.
*Research of the authors was supported by Contract EA4540 (France).

1. Introduction. This paper deals with the solving of variational inclusions of the form

$$0 \in f(x) + g(x) + F(x) - K, \tag{1.1}$$

where $f : X \to Y$ is a smooth function from a reflexive Banach space X into a Banach space Y, g is a function from X into Y admitting divided differences, $F : X \rightrightarrows Y$ is a set-valued mapping differentiable in some sense which will be specified in the sequel and K stands for a closed convex cone containing the origin of Y. It is well known that the generalized equations

$$0 \in f(x) + G(x), \tag{1.2}$$

when f is function while G is a set-valued mapping, are abstract models for various problems encountered in fields such as mathematical programming, engineering, optimal control, economy and transport theory. For more informations on the wide field of modelized phenomena by these generalized equations, readers may consult the monograph of Facchinei and Pang [5] and the article of Ferris and Pang [6].

Many authors proposed and analysed several iterative methods for solving the inclusion (1.2). In 1996, A. Dontchev [4] proposed (when f is smooth) the following extension of Newton's method:

$$\begin{cases} x_0 \text{ is a given starting point which is close to a solution } \bar{x}, \\ 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + G(x_{k+1}), \text{ for } k = 0, 1, 2, \ldots \end{cases}$$

which is based on a partial linearization of the function f and leaving the set-valued part unchanged. Subsequently, using the same technique and tools as Dontchev, many authors have proposed numerous methods of resolution of (1.2), all of them based on the notion of metric regularity and a generalization of fixed point lemma.

Inspired by Dontchev's method, the authors in [8] proposed a method which is defined by

$$\begin{cases} x_0 \text{ is a given starting point which is close to a solution } \bar{x}, \\ 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + g(x_k) + G(x_{k+1}), \text{ for } k = 0, 1, 2, \ldots \end{cases}$$

where f is smooth, g is Lispchitz function (nondifferentiable) and $\bar{x}$ is solution of the problem

$$0 \in f(x) + g(x) + G(x).$$

They proved, under some metric regularity condition and a fixed point theorem for set-valued maps, the semilocal and superlinear convergence of their method.

We remark that the above iterative method is similar to the classical Zincensko [23] method when the set-valued map G is the set $\{0\}$.

Lately, in the case where the univoque part of (1.2) is a sum of a smooth function f with a function g differentiable in a solution x^* but may be not differentiable in a neighborhood of a x^*, Piétrus and Geoffroy in [9] associated to (1.2) the inclusion

$$\begin{cases} x_0 \text{ is a given starting point which is close to a solution } \bar{x}, \\ 0 \in f(x_k) + g(x_k) + (\nabla f(x_k) + [x_{k-1}, x_k; g])(x_{k+1} - x_k) + G(x_{k+1}), \\ \text{for } k = 0, 1, 2, \ldots \end{cases}$$

where $[x_{k-1}, x_k; g]$ is an operator called divided difference of g at the points x_{k-1} and x_k. A similar method has been considered by Cãtinas [2] for solving nonlinear equations when G is the set $\{0\}$. To do this, he proposed a combination of Newton's method (applied to f) with the secant's one (applied to g). He proved also that the order of convergence is $p = \dfrac{1+\sqrt{5}}{2}$.

Nevertheless, these methods present two major disadvantages. The first is that one obtains only a local convergence results under metric regularity assumption which depends strongly of the unknown solution that we wish to determine. The second is the use of fixed point lemma, which not implies unicity of an iterate x_{k+1} when starting with x_k. Thus, the numerical treatment is not easy.

To overcome the disadvantages, Robinson in [18] used the concept of convex processes to solve inclusion (1.2) when $-G$ is a closed convex cone. The Newton type method introduced is quadratically semilocally convergent.

Following the Robinson contributions, the authors in [16] extended to variational inclusions a method introduced by Cãtinas [2] in the case where the set-valued $-G$ is a non empty closed convex cone in the Banach space Y and X is a reflexive Banach space. They proved that the order of convergence of this algorithm is $p = \dfrac{1+\sqrt{5}}{2}$ under classical conditions. We note that the metric-regularity concept is not used in their contribution. In the same spirit a perturbated version of inclusion (1.2) has been studied in [15] in the case of a Lipschitz perturbation g which was not supposed to possess divided differences. We want to underline this recent interesting extension of the Newton-type method of Robinson's by G.N. Silva and al. in [22]

We also note that Robinson's approach is limited to a reflexive space X to make the algorithmic construction feasible. Indeed, from the consideration of an adequate problem of convex minimization at each step, it is possible to build each iterate thanks to a projection mean. Thus, the numerical treatment seems be more confortable in this case. Let us quote that in [16, 15, 18], only the case where the set valued is a constant cone has been studied.

In the present paper, unlike the results obtained by Jean-Alexis, Pietrus and Robinson [16, 15, 18], we do not assume that the set-valued part is constant, but we are interested in the case when G is decomposed into the sum of a cone K with a set-valued mapping F enjoying adequate properties of convexity, compactness and of generalized differentiation. So, trying to overcome the weaknesses of the above methods, we combine the ideas of [18], [16] and [14] for proposing a new algorithm which has semilocal convergence to a solution $\bar{x}$ of (1.1). This idea is new and the present algorithm is based on a partial linearization of $f + F$ in the Nachi-Penot sense and the use of divided differences for the function g. This paper is a continuation of a recent work [7] in which we introduced a Newton method in the context of set-valued maps differentiable in the Nachi-Penot sense.

In the section below, we present the background material while Section 3 for its part is devoted to the presentation and the study of convergence of our method. In Section 4, the performance of the proposed algorithm is illustrated by several numerical examples.

2. Background material. Throughout this paper, X is a reflexive Banach space, Y is a Banach space, both endowed with their respective norms and $K \subset Y$ is a closed convex cone containing 0. We note $F : X \rightrightarrows Y$ the set-valued mapping F defined from X into subsets of Y, while $f : X \to Y$ stands for a classical function from X toward Y. For a given set-valued map $F : X \rightrightarrows Y$, we define respectively the domain of F, noted $\mathrm{dom}(F)$ and the range of F, noted $\mathrm{rg}(F)$ by $\mathrm{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$ and $\mathrm{rg}(F) := \cup_{x\in\mathrm{dom}(F)} F(x) = \{y \in F(x) \mid x \in \mathrm{dom}(F)\}$. We denote by $\mathcal{L}(X, Y)$ the set of continuous linear maps from X to Y. The distance from a point x to a subset C is given by $d(x, C) := \inf\limits_{y\in C} \|x - y\|$. The closed unit ball of Y is denoted by $\mathbb{B}_Y$, while $\mathbb{B}_r(a) := \{x \in X \mid \|x - a\| \leq r\}$ stands for the closed ball of radius r centered at a. For a set-valued map F, its inverse, denoted by $F^{-1} : Y \rightrightarrows X$ still exists and is defined as $x \in F^{-1}(y) \iff y \in F(x)$, while its graph is given by $\mathrm{gph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$. Given two subsets A and B of Y, the excess from A to B is defined by $e(A, B) := \sup\limits_{x\in A} d(x, B)$.

Let us begin by recalling the definitions of the first and second divided operators.

Definition 2.1. *Let X and Y be two Banach spaces. A linear continuous operator acting from X into Y is called a first order divided difference of the continuous function $g : X \to Y$ on the points x_0, y_0, denoted by $[x_0, y_0; g]$, if the following property holds:*

$$[x_0, y_0; g](y_0 - x_0) = g(y_0) - g(x_0), \qquad x_0 \neq y_0.$$

In particular, if g is Fréchet différentiable at $x_0 \in X$ then $[x_0, x_0; g] = \nabla g(x_0)$.

Definition 2.2. *Let X and Y be two Banach spaces. A linear operator acting from X into $\mathcal{L}(X, Y)$ is called a second order divided of the operator $g : X \to Y$ on the points x_0, y_0 and z_0 denoted by $[x_0, y_0, z_0; g]$, if the following property holds:*

$$[x_0, y_0, z_0; g](z_0 - x_0) = [y_0, z_0; g] - [x_0, y_0; g], \qquad x_0, y_0, z_0, \text{ distinct.}$$

In particular, if g is twice Fréchet différentiable et $x_0 \in X$ then $[x_0, x_0, x_0; g] = \frac{1}{2} g''(x_0)$.

These operators has been used by different authors in various works, the reader could be referred to [2, 10] and the references therein.

We now recall the concept of convex processes which has been introduced by Rockafellar [20] and then studied in depth by Robinson [19].

Definition 2.3. *A set-valued mapping $T : X \rightrightarrows Y$ is said to be convex process if and only if all following assertions are satisfied.*

a) $0 \in T(0)$;

b) $T(\lambda x) = \lambda T(x)$ *for all* $x \in X$ *and* $\lambda > 0$;

c) $T(x) + T(x') \subset T(x + x')$ *for all* $x, x' \in X$.

Let us mention that the domain and the range of convex process are both convex cone containing 0 and that its inverse which always exists is itself a convex processes. For more details on convex processes, the reader could refer to [1, 3, 20, 21].

To perform computations involving convex processes we need some fundamental tools such as the inner norm which is denoted by $\| \cdot \|^-$ and is defined as follow.

Definition 2.4. *Given a convex process $T : X \rightrightarrows Y$, its inner norm is*

$$\|T\|^{-} := \sup_{\|x\|\leq 1} \inf_{y\in T(x)} \|y\|. \tag{2.1}$$

Let us note that (2.1) could be rewritten in this way

$$\|T\|^{-} = \inf\{\kappa > 0 \mid H(x) \cap \kappa I\!B_Y \neq \emptyset, \text{ for all } x \in I\!B_X\}.$$

In order to avoid possible confusion in this paper, we choose to note the inner norm for a convex precess T by $\|T\|$ instead of $\|T\|^{-}$ for being more short. We will say that a convex process T is bounded if and only if its inner norm, $\|T\| < \infty$.

The following result will be of interest in the next section. It provides us with an estimation of the inner norm from [19] (see Theorem 5A.8).

Theorem 2.5. *Let T and Δ be convex processes from X into Y. Assume that T, T^{-1} and Δ are normed and that $\|T^{-1}\|\|\Delta\| \leq 1$. Suppose further that* $\mathrm{dom}(T) \subset \mathrm{dom}(\Delta)$, $\Delta(\mathrm{dom}(T)) \subset \mathrm{rg}(T)$, $\mathrm{dom}(T)$ *is closed and* $(T - \Delta)(x)$ *is closed for each* $x \in \mathrm{dom}(T)$. *Then the convex process $T - \Delta$ has the following properties:*

i) $\mathrm{rg}(T) \subset \mathrm{rg}(T - \Delta)$,

ii) $(T-\Delta)^{-1}_{\mathrm{rg}(T)}$ *is a normed convex process, and* $\|(T-\Delta)^{-1}_{\mathrm{rg}(T)}\| \leq \dfrac{\|T^{-1}\|}{1 - \|T^{-1}\|\|\Delta\|}$,

where $(T - \Delta)^{-1}_{\mathrm{rg}(T)}$ *denote the restriction of mapping* $(T - \Delta)^{-1}$ *to the set* $\mathrm{rg}(T)$.

Now, we briefly give below, the definition of majorizing sequences which has been introduced by Kantorovich in [11].

Definition 2.6. *Consider a sequence $(x_k) \in X$. Then the real non negative sequence (t_k) is said to majorize (x_k) if*

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, fork = 0, 1, 2 \ldots .$$

Obviously, (t_k) is a nondecreasing sequence and for all $m > k \geq 0$, we have

$$\|x_m - x_k\| \leq \sum_{j=k}^{m-1} \|x_{j+1} - x_j\| \leq \sum_{j=k}^{m-1} (t_{j+1} - t_j) = t_m - t_k.$$

Hence, when $\lim_{k\to\infty} t_k = t^* < \infty$, (x_k) is a Cauchy sequence of X which converges toward a element $x^* \in X$ as soon as X is complete. Furthermore, the error estimation is immediately given by $\|x_k - x^*\| \leq t^* - t_k$ for $k = 0, 1, 2, \ldots$

The concept of generalized differentiation we are dealing with has been introduced by Nachi and Penot [14] in order to extend the classical inverse mapping theorem to correspondences relying on a fixed point theorem as in the classical case instead of use the Ekeland variational principle. From now on, given a set-valued mapping, we call of N-P-differentiation instead of differentiation in the sense of Nachi-Penot for being more short. Let us recall that many concepts of generalized differentiation for correspondences has been developed in the literature, for more details reader can refer to [1, 13, 17] and the references therein.

Definition 2.7. *Consider X_0 an open subset of X, a set valued-map $F : X_0 \rightrightarrows Y$ with $X_0 \subset \mathrm{dom}(F)$, and $x_0 \in X_0$. F is said to be N-P-differentiable at x_0 if F is lower semi-continuous at x_0 and if there exists some $A \in \mathcal{L}(X, Y)$ such that the function o given by*

$$e(F(x_0 + x), F(x_0) + A(x)) = o(x) \textit{is a remainder.} \tag{2.2}$$

Then, $A \in \mathcal{L}(X, Y)$ is called a N-P-derivative of F at x_0 and the set of N-P-derivatives of F at x_0 is denoted by $\mathcal{D}F(x_0)$.

Recall that here, we say that a function $o : X \to \mathbb{R}$ is a remainder if $\lim_{x\to 0, x\neq 0} \frac{o(x)}{\|x\|} = 0$, or equivalently, if there exists a modulus function $\mu : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$, continuous at 0 with $\mu(0) = 0$ and such that $\|o(x)\| \leq \mu(\|x\|)\|x\|$; without loss of generality, one may assume that μ is nondecreasing. Hence, an useful and equivalent formulation of (2.2) is given by

$$F(x_0 + x) \subset F(x_0) + A(x) + \mu(\|x\|)\|x\|\mathbb{B}_Y.$$

Let us mention that F will said to be N-P-differentiable on X_0 if and only if, it is N-P-differentiable at every element $x_0 \in X_0$. It is easy to see that for a single-valued function, the N-P derivative when it exists coincides with the classical Fréchet-derivative.

3. Description and convergence of the method.

3.1. Description of the algorithm. To begin with, consider a subset $X_0 \subset X$, a set-valued mapping $F : X \rightrightarrows Y$, $f : X \to Y$ a function and assume that F and f are differentiable on X_0. Now, for any fixed v and $w \in X_0$, we define a set-valued mapping $\tilde{T}_{(v,w)} : X \rightrightarrows Y$ by

$$\tilde{T}_{(v,w)}(x) := \big(\nabla f(w) + A(w) + [v, w; g]\big)x - K \text{ for } x \in X,$$

where, $\nabla f(w)$ and $A(w)$ denote respectively the Fréchet-derivative of f and the N-P-derivative of F at w and $[v,w;g]$ is the first order divided difference of the operator g.

From the Definition 2.3, it is easy to see that $\tilde{T}_{(v,w)}$ is a convex process and that its inverse which is also a convex process is defined by

$$\tilde{T}^{-1}_{(v,w)}(y) := \{z \in X \mid \big(\nabla f(w) + A(w) + [v,w;g]\big)z \in y + K\}.$$

Given two starting points x_0 and $x_1 \in X_0$ such that $\tilde{T}^{-1}_{(x_0,x_1)}(-f(x_1) - g(x_1) - F(x_1)) \neq \emptyset$, the point x_2 is defined as the sum of x_1 with a projection of the origin of X on the set $\tilde{T}^{-1}_{(x_0,x_1)}(-f(x_1) - g(x_1) - F(x_1))$. Then for building x_3 we repeat the procedure using x_1 and x_2 as starting points. Hence, at the k^{th} step, we have x_{k-1}, x_k and we define x_{k+1} as the sum of x_k with a projection of the origin in X on the set $\tilde{T}^{-1}_{(x_{k-1},x_k)}(-f(x_k) - g(x_k) - F(x_k))$.

An equivalent way to rewrite the above process is when x_k is computed, the point x_{k+1} appears to be any solution of the following minimization problem:

$$\text{Minimize}\Big\{\|x - x_k\| \mid (f+g)(x_k) + \big(\nabla f(x_k) + A(x_k) + [x_{k-1}, x_k; g]\big)(x - x_k) \in K - F(x_k)\Big\}. \tag{3.1}$$

Remark 3.1. The continuity of the linear map $\nabla f(x_k) + A(x_k) + [x_{k-1}, x_k; g]$ and the fact that K is a closed convex set, imply that the feasible set of (3.1) is a closed convex set for all $k \in \mathbb{N}^*$. Then, the existence of a feasible point $\tilde{x}$ implies that any solution of (3.1) must lies in the intersection of the feasible set of (3.1) with the closed ball centered at x_k with radius $\|\tilde{x} - x_k\|$. Since the function $x \mapsto \|x - x_k\|$ is weakly lower semicontinuous, a solution of (3.1) exists, see [12]. Furthermore, it is clear that if (3.1) is feasible then it is solvable and the convexity ensures that any local solution will be global.

3.2. Convergence analysis. For the convergence analysis of our algorithm, we need to prove the following proposition.

Proposition 3.2. *Consider the sequence (t_n) defined by*

(i) $t_0 = 0,\ t_1 = \alpha,\ t_2 = \beta,$

(ii) $t_{n+1} - t_n = M\Big(\frac{L}{2}(t_n - t_{n-1})^2 + K_1(t_n - t_{n-1})(t_n - t_{n-2}) + K_2(t_n - t_{n-1})\Big)$
where α, β, M, L, K_1 and K_2 are positive constants.

If the conditions

(*iii*) $\alpha \leq M$, $\beta \leq 2\alpha$,

(*iv*) $q + K_2 M < 1$, *with* $q = M^2 \left(\frac{L}{2} + 2K_1 \right)$,

(*v*) $q^{u_{l-1}} + K_2 M \leq q^{u_{l-2}}$, *where* (u_l) *is Fibonacci's sequence defined by* $u_0 = u_1 = 1$ *and* $u_{l+1} = u_l + u_{l-1}$ *with* $l \geq 1$,

are fulfilled, then the sequence (t_n) *is such that all its terms belong to* $\mathbb{B}_s(t_1)$ *where* $s = \frac{M}{q} \sum_l^{\infty} q^{u_l}$. *Moreover, the sequence* (t_k) *is convergent and one has the following error estimate*

$$t^* - t_n \leq \frac{M}{q\left(1 - q^{\frac{p^n(p-1)}{\sqrt{5}}}\right)} q^{\frac{p^n}{\sqrt{5}}}$$

where $t^* = \lim t_n$ *and* $p = \frac{1+\sqrt{5}}{2}$.

Proof. We follow the sketch of the proof given in [2] which consists to show first by induction (for any $n \geq 2$) that

$$t_n \in \mathbb{B}_s(t_1), \tag{3.2}$$

$$t_n - t_{n-1} \leq t_{n-1} - t_{n-2}, \tag{3.3}$$

$$t_n - t_{n-1} \leq q^{u_{n-1}-1} M. \tag{3.4}$$

For $k \geq 2$, let us suppose that (3.2), (3.3), (3.4) has been verified for $n \leq k$.

Since

$$t_{k+1} - t_k = M\Big[\frac{L}{2}(t_k - t_{k-1})^2 + K_1(t_k - t_{k-2})(t_k - t_{k-1}) + K_2(t_k - t_{k-1})\Big]$$

and

$$t_k - t_{k-2} = t_k - t_{k-1} + t_{k-1} - t_{k-2},$$

we obtain

$$t_{k+1}-t_k = M\Big[\Big(\frac{L}{2}+K_1\Big)(t_k-t_{k-1})(t_k-t_{k-1}) + K_1(t_{k-1}-t_{k-2})(t_k-t_{k-1}) + K_2(t_k-t_{k-1})\Big].$$

Using assumption (3.3) of the induction, one has

$$t_{k+1}-t_k \le M\Big[\Big(\frac{L}{2}+2K_1\Big)(t_{k-1}-t_{k-2})(t_k-t_{k-1}) + K_2(t_k-t_{k-1})\Big].$$

Applying the assumption (3.4), and observing that the indices $k-2,\ k-1<k$, it follows that

$$t_{k+1}-t_k \le M\Big[\Big(\frac{L}{2}+2K_1\Big)q^{u_{k-2}-1}M(t_k-t_{k-1}) + K_2(t_k-t_{k-1})\Big],$$

i.e.,

$$\text{(3.5)}\quad t_{k+1}-t_k \le \Big(M^2\Big(\frac{L}{2}+2K_1\Big)q^{u_{k-2}-1} + K_2M\Big)(t_k-t_{k-1}) \le (q^{u_{k-2}} + K_2M)(t_k-t_{k-1})$$

since $q = M^2\Big(\frac{L}{2}+2K_1\Big)$.

Using the assuption (iv), we obtain then

$$t_{k+1}-t_k \le t_k - t_{k-1}$$

and (3.3) is checked for $n=k+1$.

Now, let us show that

$$\forall n\ge 1,\ t_n - t_{n-1} \le q^{u_{n-1}-1}M.$$

For $k\ge 2$, suppose that (3.4) holds for $n\le k$.

According to the inequality (3.5), we have

$$t_{k+1}-t_k \le (q^{u_{k-1}} + K_2M)(t_k-t_{k-1}).$$

With the induction assumption and the condition (v), we get:

$$t_{k+1}-t_k \le (q^{u_{k-1}}+K_2M)q^{u_{k-1}-1}M \le q^{u_{k-2}+u_{k-1}-1}M = q^{u_k-1}M.$$

This proves that (3.4) for $n=k+1$.

Moreover, one has

$$t_{k+1} - t_k = (t_2 - t_1) + (t_3 - t_2) + \cdots + (t_k - t_{k-1}) + (t_{k+1} - t_k).$$

Then (3.5) yields

$$\begin{aligned}(3.6)\quad t_{k+1} - t_1 &\leq q^{u_1 - 1}M + q^{u_2 - 1}M + \cdots + q^{u_{k-1}-1} + q^{u_k - 1}M \\ &= \frac{M}{q}(q^{u_1} + q^{u_2} + \cdots + q^{u_{k-1}} + q^{u_k}).\end{aligned}$$

By using this assumption (3.5), we may write

$$t_{k+1} - t_1 \leq q^{u_1 - 1}M + q^{u_2 - 1}M + \cdots + q^{u_{k-1}-1} + q^{u_k - 1}M.$$

Invoking this equality $s = \dfrac{M}{q}\sum_{l=1}^{\infty} q^{u_l}$, and $q^{u_l} > 0$, then $t_{k+1} - t_1 < s$. (3.4) is checked for $n = k + 1$.

This, combined with the inequality $t_2 - t_1 = \beta - \alpha \leq 2\alpha - \alpha = \alpha = t_1$ (from hypothesis (iii)), implies that (3.4) is verified for all n.

Now, let us show that (t_k) is a Cauchy sequence of real numbers.

For any $k \geq 1, m \geq 1$, we have

$$t_{k+m} - t_k \leq \frac{M}{q}(q^{u_k} + q^{u_{k+1}} + \cdots + q^{u_{k+m-2}} + q^{u_{k+m-1}}).$$

Since $u_l \geq \dfrac{p^l}{\sqrt{5}}, \quad \forall l \geq 1$, we deduce that

$$t_{k+m} - t_k \leq \frac{M}{q} q^{\frac{p^k}{\sqrt{5}}}(1 + q^{\frac{p^{k+1} - p^k}{\sqrt{5}}} + q^{\frac{p^{k+2} - p^k}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-2} - p^k}{\sqrt{5}}} + q^{\frac{p^{k+m-1} - p^k}{\sqrt{5}}}).$$

This inequality, together with the fact $p^{k+s} - p^k = p^k(p^s - 1)$ implies that

$$t_{k+m} - t_k \leq \frac{M}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k 2(p-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(m-2)(p-1)}{\sqrt{5}}} + q^{\frac{p^k(m-1)(p-1)}{\sqrt{5}}}\Big).$$

The sum of m terms $1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k 2(p-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(m-2)(p-1)}{\sqrt{5}}} + q^{\frac{p^k(m-1)(p-1)}{\sqrt{5}}}$ is a geometrical sequence of reason $q^{\frac{p^k(p-1)}{\sqrt{5}}} < 1$, and equals $\dfrac{1 - q^{\frac{p^k(p-1)m}{\sqrt{5}}}}{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}}$ allow us to write

$$t_{k+m} - t_k \leq \frac{M q^{\frac{p^k}{\sqrt{5}}}\big(1 - q^{\frac{p^k(p-1)m}{\sqrt{5}}}\big)}{q\big(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}\big)}.$$

The right-hand of the last inequality converges to zero as $k \to \infty$, regardless of m. Thus, (t_n) is a Cauchy sequence of real numbers and we set $t^* = \lim t_n$. The error estimate is obtained with $m \to \infty$. □

We shall now give the main result of our paper.

Theorem 3.3. *Consider an open subset $X_0 \subset X$, $T : X \rightrightarrows Y$ a convex process and a function $g : X \to Y$. Let $f : X \to Y$ and $F : X \rightrightarrows Y$ be respectively Fréchet-differentiable and N-P-differentiable on X_0, and whose derivatives at $x_0 \in X$ are respectively $\nabla f(x)$ and $A(x)$. Assume that F is compact convex valued with closed graph, and further that exist positive numbers B, L, M_1, K_1, K_2, α and β satisfying*

(i) $\|\tilde{T}^{-1}_{(x_0,x_1)}\| \leq B$,

(ii) $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$, *for all* $x, y \in X_0$,

(iii) *g is continuous on X_0 and admits divided differences of first and second order on X_0,*

(iv) *For all x_{k+1} solution of* (3.1), $F(x_k) \subset F(x_{k+1})$, $k = 1, 2, \dots$,

(v) $BM_1 < 1$ *and* $\alpha \leq \beta \leq 2\alpha$,

(vi) $\|x_1 - x_0\| \leq \alpha \leq \dfrac{B}{1 - BM_1}$, $\|x_2 - x_1\| \leq \beta$; *where x_2 is any point obtained from x_0 and x_1 by our algorithm (such a point exists by the remark in the previous section),*

(vii) *For any distinct points x, y, and z in X_0,* $\|[x, y, z; g]\| \leq K_1$, $\|A(x)\| \leq K_2$, $\|\nabla f(y) + A(y) + [x, y; g] - (\nabla f(x_1) + A(x_1)) - [x_0, x_1; g]\| \leq M_1$;

(viii) $q = \dfrac{B^2}{(1 - BM_1)^2}(\dfrac{L}{2} + 2K_1) < 1$ *and* $q + \dfrac{B}{1 - BM_1}K_2 < 1$,

(ix) $q^{u_{l-1}} + \dfrac{B}{1 - BM_1}K_2 \leq q^{u_{l-2}}$, *where (u_l) is a Fibonacci's sequence defined by $u_0 = u_1 = 1$ and $u_{l+1} = u_l + u_{l-1}$ with $l \geq 1$,*

(x) $\mathrm{gph}(K - F)$ *is closed.*

Then, there are $M > 0$ and $t^* > 0$ such that for all initial guess x_0 satisfying $\mathbb{B}_{t^*}(x_0) \subset X_0$, there exists at least a sequence $\{x_k\}$ generated by algorithm (3.1) *such that* $\forall k \geq 0, x_k$ *remain in* $\mathbb{B}_{t^*}(x_0)$ *and converges to some* x^* *solution of* (1.1). *Moreover, the following error estimate holds*

$$\|x^* - x_n\| \leq \frac{M}{q\Big(1 - q^{\frac{p^n(p-1)}{\sqrt{5}}}\Big)} q^{\frac{p^n}{\sqrt{5}}}$$

where $p = \dfrac{1+\sqrt{5}}{2}$.

Proof. First of all, using assumptions (iii) and (x), let us observe that if a sequence x_k satisfies (3.1) and converges to an element $x^* \in X$ then necessarily x^* is a solution of (1.1).

Second of all, we will built by induction both a sequence x_k satisfies (3.1) whose elements remain in $\mathbb{B}_{t^*}(x_0)$ and a nondecreasing sequence $\{t_k\}$ satisfying

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \text{for all } k \in \mathbb{N} \dots . \tag{3.7}$$

Let us set, $t_0 = 0$, $t_1 = \alpha$ and $t_2 = \beta$. From assumption (vi) and (vii), we have

$$\|x_1 - x_0\| \leq \alpha = t_1 - t_0, \ \|x_2 - x_1\| \leq \beta \leq \beta - \alpha = t_2 - t_1.$$

Thus, $\|x_2 - x_1\| \leq \beta - \alpha \leq 2\alpha - \alpha = \alpha = t_1$, and $x_1, \ x_2 \in \mathbb{B}_{t^*}(x_0)$.

This means that our result is checked for $k = 0$ and $k = 1$.

Now, let us suppose that we have built $x_1, x_2, \dots, x_j$, from (3.1) and $t_0, t_1, \dots, t_j$ such that

$$\|x_k - x_{k-1}\| \leq t_k - t_{k-1} \text{and} x_k \in \mathbb{B}_{t^*}(x_0) \text{ for } k = 1, \dots, j.$$

Thanks to (i) and $(viii)$ we obtain $\|\tilde{T}^{-1}_{(x_0,x_1)}\| \ \|\Delta_j\| < 1$ with

$$\Delta_j = -\big(\nabla f(x_j) - \nabla f(x_1) + A(x_j) - A(x_1) + [x_{j-1}, x_j; g] - [x_0, x_1; g]\big)$$

We also observe that the convex process

$$\tilde{T}_{(x_{j-1}, \ x_j)}(x) = (\nabla f(x_j) + A(x_j) + [x_{j-1}, x_j; g])x - K$$

could be rewritten

$$\tilde{T}_{(x_{j-1}, \ x_j)}(x) = \big(\nabla f(x_1) + A(x_1) + [x_0, x_1; g]\big)x - K$$
$$+\big(\nabla f(x_j) - \nabla f(x_1) + A(x_j) - A(x_1) + [x_{j-1}, x_j; g] - [x_0, x_1; g]\big)x = (\tilde{T}_{(x_0, \ x_1)} - \Delta_j)x$$

with

$$\Delta_j(x) = -\big(\nabla f(x_j) + A(x_j) - (\nabla f(x_1) + A(x_1)) + [x_{j-1}, x_j; g] - [x_0, x_1; g]\big)x.$$

Thus, using Theorem 2.5, together with assumption (i), (v), (vii) and the previous inequality, we obtain that $\tilde{T}_{(x_{j-1},\, x_j)}$ carries X onto Y, $\tilde{T}^{-1}_{(x_{j-1},\, x_j)}$ is normed and its norm satisfies

$$\|\tilde{T}^{-1}_{(x_{j-1},\, x_j)}\| \leq \frac{\|\tilde{T}^{-1}_{(x_0,\, x_1)}\|}{1 - \|\tilde{T}^{-1}_{(x_0,\, x_1)}\| \; \|\Delta_j\|} \leq \frac{B}{1 - BM_1} \quad \forall j = 2, \dots k.$$

Since $\tilde{T}_{(x_{j-1},\, x_j)}$ carries X onto Y then (3.1) is feasible, hence solvable for $k = j$ and successively we obtain the existence of x_{j+1} as solution of (3.1).

Now, let us consider the following problem which consists to find an x which is a solution of

(3.8)
$$\begin{aligned} f(x_j) + g(x_j) + \big(\nabla f(x_j) + A(x_j) + [x_{j-1}, x_j; g]\big)(x - x_j) &\in f(x_{j-1}) + g(x_{j-1}) \\ &+ (\nabla f(x_{j-1}) + A(x_{j-1}) + [x_{j-2}, x_{j-1}; g])(x_j - x_{j-1}) + K. \end{aligned}$$

Since x_j is a solution of (3.1) (for $k = j - 1$) and using assumption (iv), the below inclusion becomes

$$\begin{aligned} f(x_{j-1}) + g(x_{j-1}) + \big(\nabla f(x_{j-1}) + A(x_{j-1}) + [x_{j-2}, x_{j-1}; g]\big)(x_j - x_{j-1}) + K \\ \subset K - F(x_{j-1}) \subset K - F(x_j), \end{aligned}$$

and we deduce that any x satisfying (3.8) is necessarily feasible for (3.1).

We can rewrite (3.8) in the following way

$$\begin{aligned} (3.9) \quad x - x_j \in \tilde{T}^{-1}_{(x_{j-1},\, x_j)}\Big(&- f(x_j) - g(x_j) + f(x_{j-1}) + g(x_{j-1}) \\ &+ (\nabla f(x_{j-1}) + A(x_{j-1}) + [x_{j-2}, x_{j-1}; g])(x_j - x_{j-1}))\Big). \end{aligned}$$

Since the right-hand of (3.9) is a nonempty closed convex is contained in the reflexive Banach space X, there exists an element $\tilde{x}$ solution of (3.9), thus of (3.1).

By triangle inequality of the norm,

(3.10)
$$\|\tilde{x} - x_j\| \leq \|T^{-1}(x_{j-1}, x_j)\| \Big(\| - f(x_j) + f(x_{j-1}) + \nabla f(x_{j-1})(x_j - x_{j-1})\|$$

$$+\|g(x_{j-1})-g(x_j)+[x_{j-2},x_{j-1};g](x_j-x_{j-1})\|+\|A(x_{j-1})\|\|(x_j-x_{j-1}))\|\Big).$$

A small calculation gives us

$$\|-f(x_j)+f(x_{j-1})+\nabla f(x_{j-1})(x_j-x_{j-1})\|\leq \frac{L}{2}\|x_j-x_{j-1}\|^2,$$

$$\|A(x_{j-1})(x_j-x_{j-1})\|\leq\|A(x_{j-1})\|\|x_j-x_{j-1}\|\leq K_2\|x_j-x_{j-1}\|.$$

And by the definitions of divided differences, we have

$$\begin{aligned}&\|g(x_{j-1})-g(x_j)+[x_{j-2},x_{j-1};g](x_j-x_{j-1})\|\\ &=\|([x_{j-1},x_j;g]-[x_{j-2},x_{j-1};g])(x_j-x_{j-1})\|\\ &=\|[x_{j-2},x_{j-1},x_j;g](x_j-x_{j-2})(x_j-x_{j-1})\|\\ &\leq K_1\|(x_j-x_{j-2})(x_j-x_{j-1})\|.\end{aligned}$$

We infer that

$$\|\tilde{x}-x_j\|\leq M\Big(\frac{L}{2}\|x_j-x_{j-1}\|^2+K_1\|(x_j-x_{j-2})(x_j-x_{j-1})\|+K_2\|x_j-x_{j-1}\|\Big)$$

with $M=\dfrac{B}{1-BM_1}$.

Then by inequality $\|x_{k+1}-x_k\|\leq t_{j+1}-t_j, \text{for} k=0,1,2,\ldots,j$, we deduce

$$\|\tilde{x}-x_j\|\leq M\Big(\frac{L}{2}(t_j-t_{j-1})^2+K_1(t_j-t_{j-2})(t_j-t_{j-1})+K_2(t_j-t_{j-1})\Big).$$

Let us consider the following sequence (t_k) defined by

$$t_{k+1}-t_k=M\Big(\frac{L}{2}(t_k-t_{k-1})^2+K_1(t_k-t_{k-2})(t_k-t_{k-1})+K_2(t_k-t_{k-1})\Big)$$

with $t_1=\alpha$, $t_2=\beta$ and $t_0=0$.

We can apply the Proposition 3.2 with the theorem's assumptions, we conclude that the sequence (t_k) is strictly increasing and converges to a number t^*.

Since $\tilde{x}$ is also feasible for (3.1) with $k=j$, we have

$$\|x_{j+1}-x_j\|\leq\|\tilde{x}-x_j\|\leq t_{j+1}-t_j;$$

thus $x_{j+1}\in \mathbb{B}_{t^*}(x_0)$, and the induction is complete. The error estimate come from the application of the last proposition and the fact that the sequence (x_n) is majorized by the sequence (t_n) completes the proof. □

4. Numerical simulations. In this section, our algorithm is tested for several finite dimensional cases. All the experiments are implemented in Matlab 2020.2 and performed on Apple Mc Book Air 10.1 with Apple M1 chip (8-core CPU with 4 high-performance cores and 4 energy-efficient cores, GPU up to 8 cores, Neural Engine 16 cores) and RAM 8.00 GB.

Example 1. Let us consider the inclusion:

$$0 \in |x|(x-1)(x-2) + x^4(x-1)(x-2)[10^{-200}, 1] - K, \tag{4.1}$$

where $x \in X = Y := \mathbb{R}$, and $K = \mathbb{R}_-$. This problem is somewhat inspired by [14] and we can prove that the set of solutions is $\mathcal{S} = \{0\} \cup [1, 2]$.

The inclusion (4.1) has been treated by the method introduced in the present paper, taking:

$$f(x) := 0, \quad g(x) = |x|(x-1)(x-2) \quad \text{and} \quad F(x) := x^4(x-1)(x-2)[10^{-200}, 1].$$

For this example, our method relies in

$$\text{Minimize}\{\|x - x_k\|_2 / (A(x_k) + [x_{k-1}, x_k; g])(x - x_k) \in K - F(x_k)\}, \tag{4.2}$$

where $A(x) \in (4x^3(x-1)(x-2) + x^4(2x-3))[10^{-200}, 1]$ is the Nachi-Penot derivative of F. We assume that the choice of the used $A(x)$ is made with a uniformly distributed probability law, that is to say, for each step k, we generate a random number $\alpha_k \in [10^{-200}, 1]$ and set

$$A(x_k) = (4x_k^3(x_k-1)(x_k-2) + x_k^4(2x_k-3))\alpha_k.$$

In (4.6), we proceed similarly for $F(x)$, so, for each step k, we generate a random number $\beta_k \in [10^{-200}, 1]$ and replace $F(x_k)$ by

$$x_k^4(x_k-1)(x_k-2)\beta_k.$$

An easy computation give the first order divided difference of the operator g on the points $\{x_{k-1}, x_k\}$:

$$[x_{k-1}, x_k; g] := \frac{|x_{k-1}|(x_{k-1}-1)(x_{k-1}-2) - |x_k|(x_k-1)(x_k-2)}{x_{k-1} - x_k}.$$

By using the algorithm (4.6) with the guess points $x_0 = 2.20$ and $x_1 = 2.19$, we find:

Table 1. Solution of (4.1) starting from $x_0 = 2.20$, $x_1 = 2.19$ and using (4.6); cpu time = 0.104 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	2.20		
1	2.19		
2	2.12642321	0.00404201	0.06357679
3	2.04619376	0.00643676	0.08022945
4	1.97764926	0.00469835	0.06854450
5	1.97764926	0.00000000	$1.00000000 \times 10^{-199}$

where M_k still denoted the value of the previous minimization problem at the step k.

Another implementation with the same previous guess points leads to

Table 2. Solution of (4.1) still starting from $x_0 = 2.20$, $x_1 = 2.19$ and using (4.6); cpu time = 0.098 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	2.20		
1	2.19		
2	2.08892974	0.01021520	0.10107026
3	2.05520062	0.00113765	0.03372912
4	1.99996429	0.00305105	0.05523632
5	1.99996429	0.00000000	$1.00000000 \times 10^{-199}$

In the last two tables, the solutions estimations reached are $\overline{x} = 1.97764926$ and $\overline{\overline{x}} = 1.99996429$. These computations illustrate the remark that one might not reach the same approximation of the solution while starting with the same guess points. These tables also illustrate fast convergence situations.

To finish, two examples of computations starting from different couples of guess points:

Table 3. Solution of (4.1) starting from $x_0 = -5 \times 10^{-6}$, $x_1 = 3 \times 10^{-2}$, and using (4.6); cpu time = 4.441 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	-5×10^{-6}		
1	3×10^{-2}		
2	0.02999958	$1.74103035 \times 10^{-13}$	$4.17256557 \times 10^{-7}$
12	0.02999309	$4.70708548 \times 10^{-13}$	$6.86082027 \times 10^{-7}$
22	0.02998877	$4.23457980 \times 10^{-13}$	$6.50736490 \times 10^{-7}$
32	0.02998423	$2.23451706 \times 10^{-13}$	$4.72706786 \times 10^{-7}$
42	0.02997978	$3.98313965 \times 10^{-13}$	$6.31121197 \times 10^{-7}$
52	0.02997574	$3.33859892 \times 10^{-14}$	$1.82718333 \times 10^{-7}$
144	0.02993712	$7.26296901 \times 10^{-15}$	$8.52230544 \times 10^{-8}$

Table 4. Solution of (4.1) starting from $x_0 = 0.85$, $x_0 = 3.63$, and using (4.6); cpu time = 0.207 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	0.85		
1	3.63		
2	2.98091104	0.42131647	0.64908896
3	2.08910641	0.79531551	0.89180464
4	2.06854368	0.00042283	0.27649380
5	2.00697227	0.00379104	0.06157141
6	2.00068577	0.00003952	0.00628650
7	2.00062077	$4.22456992 \times 10^{-9}$	0.00006500
8	1.99904941	$2.46919050 \times 10^{-6}$	0.00157137
9	1.99904941	0.00000000	$1.00000000 \times 10^{-199}$

In Table 3, we remark a chaotic behavior in the neighborhood of 0, probably due to the fact that it is an isolated solution. The Table 4 allows to see that the code does not necessarily converge toward the solution closer to the guess points.

Example 2. Let us consider the system of inclusions:

$$\begin{cases} x_1^2 + x_2^2 - |x_1 - 0.5| - 1 \in \mathbb{R}_-, \\ x_1^2 + (x_2 - 1)^2 - |x_1 - 0.5| - 1 \in \mathbb{R}_-, \\ (x_1 - 1)^2 + (x_2 - 1)^2 - 1 \in [0, 10^{-2}]. \end{cases} \tag{4.3}$$

where $x = (x_1, x_2) \in X := \mathbb{R}^2$. The previous system is somewhat a modification of the Robinson's one ([18]). The first two inclusions define a surface obtained by intersecting unions of disks of $\mathbb{R}^2$. The first inclusion is a parameterization of the union of the two blue disks, and, the second, of the union of the two red disks.

One can remark that the points

$$x^* = \Big(\frac{1}{2}, 1 - \frac{1}{2}\sqrt{3}\Big) = (0.5000000000,\ 0.1339745962)$$

and

$$x^{**} = \Big(\frac{11}{26} - \frac{3}{13}\sqrt{3}, \frac{8}{13} + \frac{9}{26}\sqrt{3}\Big) = (0.0233728905,\ 1.2149406641)$$

are two solutions of the system. The set of solutions (see Fig. 1) is the arc of the crown (green color) of center $(1, 1)$ with radiuses $r_1 = 1$ and $r_2 = 1.01$, located between the arcs of the unit radius circles of center $(\frac{1}{2}, 1)$ passing by the point x^* (red color) and of center $(-\frac{1}{2}, 0)$ passing by the point x^{**} (blue color).

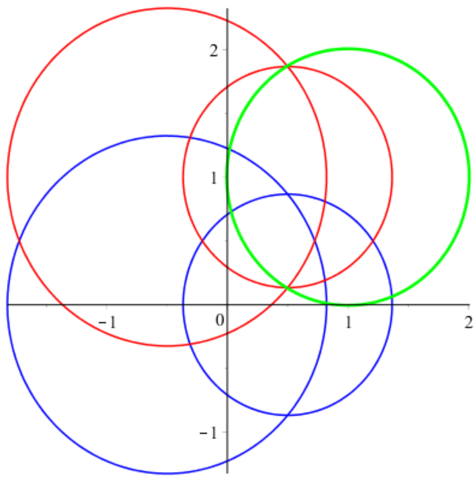

Fig. 1. Set of solutions of the system (4.3)

The method introduced in this paper is now used in order to solve the system (4.3), taking $Y = \mathbb{R}^3$, $K = \mathbb{R}^2_- \times \{0\}$, $g(x) := -|x_1 - 0.5|(1, 1, 0)$,

$$f(x) := (x_1^2 + x_2^2)(1, 1, 1) - 2x_1(0, 0, 1) - 2x_2(0, 1, 1) - (1, 0, -1),$$

and

$$F(x) := (0, 0, -10^{-2}) + (10^{-2})\{0\} \times \{0\} \times [0, 1],$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. It is useful to recall that our method consists in

$$\text{(4.4)} \quad \text{Minimize}\{\|x - x_k\|_2/f(x_k) + (A(x_k) + f'(x_k) + [x_{k-1}, x_k; g])(x - x_k) \in K - F(x_k)\},$$

where the Nachi-Penot derivative of F is $A = \{0\}$, and the first order divided difference of the operator g on the points $\{x_{k-1}, x_k\}$ is

$$[x_{k-1}, x_k; g] := \frac{|x_{k,1} - 0.5| - |x_{k-1,1} - 0.5|}{x_{k-1,1} - x_{k,1}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

for $x_{k-1} = (x_{k-1,1}, x_{k-1,2})$ and $x_k = (x_{k,1}, x_{k,2})$.

We use a random stategy governed by a uniform probability law for the choice of the used element for $F(x)$, that is to say, for each step k, a random number $\beta_k \in [0, 1]$ is generated in order to substitute

$$F_k := (0, 0, 10^{-2}(\beta_k - 1)) \in F(x_k)$$

for $F(x_k)$ in the algorithm.

Using the algorithm (4.4) with $x_0 = (0.55,\ 0.10)$ and $x_1 = (0.52,\ 0.14)$ as guess points and M_k the value of the previous minimization problem at the step k, we find:

Table 5. Solution of (4.3) starting from $x_0 = (0.55,\ 0.10)$, $x_1 = (0.52,\ 0.14)$ and using (4.4); cpu time = 0.103 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.55, 0.10)		
1	(0.52, 0.14)		
2	(0.51986486, 0.14581081)	0.00003378	0.00594595
3	(0.51986486, 0.14581081)	$2.00000000 \times 10^{-200}$	$2.00000000 \times 10^{-200}$

Starting with $x_0 = (0.51,\ 0.13)$ and $x_1 = (0.505,\ 0.137)$ which is a little modification of the previous guess points, we find

Table 6. Solution of (4.3) starting from $x_0 = (0.51,\ 0.13)$, $x_1 = (0.505,\ 0.137)$ and using (4.4); cpu time = 0.123 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.51, 0.13)		
1	(0.505, 0.137)		
2	(0.50656461, 0.13972779)	$9.88884483 \times 10^{-6}$	0.00429240
3	(0.50656461, 0.13972779)	$3.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$

Here, the reached approximate solutions are $\overline{x} = (0.51986486,\ 0.14581081)$ and $\overline{\overline{x}} = (0.50656461,\ 0.13972779)$. One can remark that we did not reach the same solution approximation, regarding the two previous computations, while the second guess point is closer to the first reached one.

Example 3. Let us consider the system of inclusions:

$$
\text{(4.5)} \qquad \begin{cases} x_1^2 + x_2^2 - |x_1 - 0.5| - 1 \in \mathbb{R}_-, \\ x_1^2 + (x_2 - 1)^2 - |x_1 - 0.5| - 1 \in \mathbb{R}_-, \\ 0 \in \mathbb{R}_- - p(x_1, x_2)[10^{-200}, 1], \end{cases}
$$

where $(x_1, x_2) \in X = \mathbb{R}^2$ and

$$
p(x_1, x_2) := ((x_1 - 1)^2 + (x_2 - 1)^2 - 1)((x_1 - 1)^2 + (x_2 - 1)^2 - 2).
$$

In order to treat the system (4.5) with our method, we complete the framework by considering $Y = \mathbb{R}^3$, $K = \mathbb{R}^3_-$, $g(x) := -|x_1 - 0.5|(1, 1, 0)$, $F(x) := \{0\} \times \{0\} \times p(x_1, x_2)[10^{-200}, 1]$, and

$$
f(x) := (x_1^2 + x_2^2)(1, 1, 0) - 2x_2(0, 1, 0) - (1, 0, 0).
$$

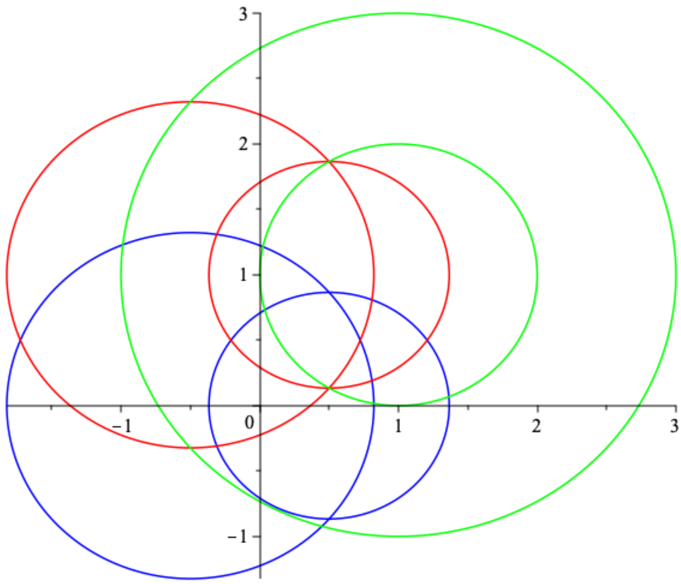

Fig. 2. Set of solutions of the system (4.5)

We can remark that the points

$$x^* = \Big(\frac{1}{2}, 1 - \frac{1}{2}\sqrt{3}\Big) = (0.5000000000, 0.1339745962),$$

$$x^{**} = \Big(-\frac{1}{2}\sqrt{3}, -\frac{1}{2}\Big) = (-0.1339745962, -0.5000000000)$$

and $x^{***} = (0,1)$ are three solutions of the system. In fact, the set of solutions (see Fig. 2) is the part of the crown (green color) of center $(1,1)$ with radiuses $r_1 = 1$ and $r_2 = 2$, located between the arc of the unit radius circles of center $(0,1)$ passing by the points x^* and x^{**} (red color), and the other of center $(0,0)$ passing by the point x^{**} and x^{***} (blue color).

As previously, our method consists in

$$\text{(4.6)} \quad \text{Minimize}\{\|x - x_k\|_2 / f(x_k) + (A(x_k) + f'(x_k) + [x_{k-1}, x_k; g])(x - x_k) \in K - F(x_k)\},$$

where A still denoted the Nachi-Penot derivative of F, and the first order divided difference of the operator g on the points $\{x_{k-1}, x_k\}$ is

$$[x_{k-1}, x_k; g] := \frac{|x_{k,1} - 0.5| - |x_{k-1,1} - 0.5|}{x_{k-1,1} - x_{k,1}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

We adopted the same strategy as in the previous example, that is to say, for each step k, we generate two random numbers $\alpha_{k,1}, \alpha_{k,2}, \beta_k \in [10^{-200}, 1]$, which permits to set

$$A(x_k) = 2(2(x_{k,1}-1)^2 + 2(x_{k,2}-1)^2 - 3)\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha_{k,1}(x_{k,1}-1) & \alpha_{k,2}(x_{k,2}-1) \end{pmatrix}$$

for $x_k = (x_{k,1}, x_{k,2})$, and we replace $F(x_k)$ by $(0, 0, p(x_{k,1}, x_{k,2})\beta_k)$ in (4.6).

Using the algorithm (4.6) with $x_0 = (0.51,\ 0.13)$ and $x_1 = (0.49,\ 0.12)$ as guess points, we find:

Table 7. Solution of (4.5) starting from $x_0 = (0.51,\ 0.13)$, $x_1 = (0.49,\ 0.12)$ and using (4.6); cpu time = 0.439 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.51, 0.13)		
1	(0.49, 0.12)		
2	(0.48649828, 0.12628881)	0.00005181	0.00979054
3	(0.48648356, 0.12630185)	$3.86456502 \times 10^{-10}$	0.00002775
4	(0.48647938, 0.12630555)	$3.11813150 \times 10^{-11}$	$7.88252588 \times 10^{-6}$
5	(0.48647819, 0.12630660)	$2.51579694 \times 10^{-12}$	$2.23900975 \times 10^{-6}$
6	(0.48647785, 0.12630690)	$2.02979954 \times 10^{-13}$	$6.35981917 \times 10^{-7}$
7	(0.48647776, 0.12630698)	$1.63768245 \times 10^{-14}$	$1.80647951 \times 10^{-7}$
8	(0.48647773, 0.12630701)	$1.32131373 \times 10^{-15}$	$5.13122628 \times 10^{-8}$
9	(0.48647772, 0.12630701)	$1.06606116 \times 10^{-16}$	$1.45750231 \times 10^{-8}$
10	(0.48647772, 0.12630702)	$8.60118464 \times 10^{-18}$	$4.13997127 \times 10^{-9}$
11	(0.48647772, 0.12630702)	$6.93959959 \times 10^{-19}$	$1.17594064 \times 10^{-9}$

Starting with $x_0 = (0.50,\ 0.12)$ and $x_1 = (0.51,\ 0.13)$, which is a little modification of the previous guess points, we find

Table 8. Solution of (4.5) starting from $x_0 = (0.51000000,\ 0.13000000)$ and using (4.6); cpu time = 0.069 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.50, 0.12)		
1	(0.51, 0.13)		
2	(0.49012668, 0.13954169)	0.00048599	0.02941501
3	(0.49012668, 0.13954169)	$1.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$

For these two sets of computations, we reach two different approximate solutions, although the two couple of guess points are close. For the test of Table 8, we observe a very fast convergence.

Starting with $x_0 = (0.151,\ 0.980)$ and $x_1 = (-0.394,\ -0.870)$, we find

Table 9. Solution of (4.5) starting from $x_0 = (0.151, 0.980)$, $x_1 = (-0.394, -0.870)$, and using (4.6); cpu time = 0.141 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.151, 0.980)		
1	(−0.394, −0.870)		
2	(−0.43406778, −0.16314395)	0.50125090	0.74692383
3	(−0.39351013, 0.07185138)	0.05686773	0.27555298
4	(−0.32013598, 0.12072230)	0.00777213	0.12224506
5	(0.11765970, 0.41231649)	0.27669223	0.72938988
6	(0.11765970, 0.41231649)	0.00000000	$3.00000000 \times 10^{-200}$
7	(0.11765970, 0.41231649)	$1.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$
8	(0.11765970, 0.41231649)	0.00000000	$2.00000000 \times 10^{-200}$

This is an illustration of convergence with a couple of guess points not close to the reached approximate solution.

Example 4. Let us consider the system of inclusions:

$$\begin{cases} x_1^2 + x_2^2 - |x_1 - 0.5| - 1 \in \mathbb{R}_- - x_2^2[10^{-200}, 1], \\ x_1^2 + (x_2 - 1)^2 - |x_2 - 0.5| - 1 \in \mathbb{R}_- - x_1^2[10^{-200}, 1], \\ 0 \in \mathbb{R}_- - p(x_1, x_2)[10^{-200}, 1], \end{cases} \tag{4.7}$$

where $(x_1, x_2) \in X = \mathbb{R}^2$ and

$$p(x_1, x_2) := 5.10^{-1}((x_1 - 1)^2 + (x_2 - 1)^2 - 1)((x_1 - 1)^2 + (x_2 - 1)^2 - 2).$$

Contrary to the previous examples, we will not try to draw or determine the set of solutions of this system (4.7). But some tedious calculations, which are not transcribed here, allow to check the existence of solutions.

To numerically solve the system (4.7) with our method, we adjust the framework by setting $Y = \mathbb{R}^3$, $K = (\mathbb{R}_-)^3$, $g(x) := -(|x_1 - 0.5|, |x_2 - 0.5|, 0)$, $F(x) := (x_2^2, x_1^2, p(x_1, x_2))[10^{-200}, 1]$, and

$$f(x) := (x_1^2 + x_2^2)(1, 1, 0) - 2x_2(0, 1, 0) - (1, 0, 0).$$

As previously, the most important step of our method is to

$$\text{Minimize}\{\|x - x_k\|_2 / f(x_k) + (A(x_k) + f'(x_k) + [x_{k-1}, x_k; g])(x - x_k) \in K - F(x_k)\}, \tag{4.8}$$

where A still denoted the Nachi-Penot derivative of F, and the first order divided difference of the operator g on the points $\{x_{k-1}, x_k\}$ is

$$[x_{k-1}, x_k; g] := \begin{pmatrix} \dfrac{|x_{k,1}-0.5|-|x_{k-1,1}-0.5|}{x_{k-1,1}-x_{k,1}} & 0 \\ 0 & \dfrac{|x_{k,2}-0.5|-|x_{k-1,2}-0.5|}{x_{k-1,2}-x_{k,2}} \\ 0 & 0 \end{pmatrix},$$

for $x_{k-1} = (x_{k-1,1}, x_{k-1,2})$ and $x_k = (x_{k,1}, x_{k,2})$. As in the previous examples, for each step k, we generate two random vectors $\alpha_k, \beta_k \in [10^{-200}, 1]^3$, which permits to set

$$A(x_k) = 2\begin{pmatrix} 0 & \alpha_{k,1}x_{k,2} \\ \alpha_{k,2}x_{k,1} & 0 \\ 5.10^{-1}\alpha_{k,3}(x_{k,1}-1)q(x_{k,1},x_{k,2}) & 5.10^{-1}\alpha_{k,3}(x_{k,2}-1)q(x_{k,1},x_{k,2}) \end{pmatrix}$$

with $q(x_{k,1}, x_{k,2}) := 2(x_{k,1}-1)^2 + 2(x_{k,2}-1)^2 - 3$, and, in (4.8), we replace $F(x_k)$ by

$$(\beta_{k,1}x_{k,2}^2, \beta_{k,2}x_{k,1}^2, \beta_{k,3}p(x_{k,1}, x_{k,2})).$$

Using the algorithm (4.8) with three different couples of guess points, we find:

Table 10. Solution of (4.7) starting from $x_0 = (0.55, 0.13)$, $x_1 = (0.49, 0.12)$ and using (4.8); cpu time = 0.282 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.55, 0.13)		
1	(0.49, 0.12)		
2	(0.47518780, 0.12952888)	0.00031020	0.02434108
3	(0.41831801, 0.15834569)	0.00406458	0.08568660
4	(0.41831801, 0.15834569)	$1.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$
5	(0.38955969, 0.17266485)	0.00103208	0.04307748
6	(0.38955969, 0.17266485)	0.00000000	$2.00000000 \times 10^{-200}$

Table 11. Solution of (4.7) starting from $x_0 = (0.69, 0.12)$, $x_1 = (0.67, 0.13)$ and using (4.8); cpu time = 0.079 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.69, 0.12)		
1	(0.67, 0.13)		
2	(0.40260603, 0.12912794)	0.07150030	0.26826603
3	(0.40260603, 0.12912794)	0.00000000	$2.00000000 \times 10^{-200}$

Table 12. Solution of (4.7) starting from $x_0 = (0.69, 0.42)$, $x_1 = (0.67, 0.13)$ and using (4.8); cpu time = 0.23 s.

step k	x_k	M_k	$\|x_k - x_{k-1}\|_1$
0	(0.69, 0.42)		
1	(0.67, 0.13)		
2	(0.46868846, 0.18307167)	0.04334294	0.25438321
3	(0.45851014, 0.16742179)	0.00034852	0.02582820
4	(0.45567702, 0.16306568)	0.00002700	0.00718923
5	(0.42058744, 0.17949830)	0.00150131	0.05152221
6	(0.42058744, 0.17949830)	$1.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$
7	(0.42058744, 0.17949830)	$1.00000000 \times 10^{-200}$	$3.00000000 \times 10^{-200}$

The calculations in Table 10 show one of the effects of randomly choosing the values of F and A at each iteration. Indeed, we see that the points

$$x_4 = (0.41831801,\ 0.15834569) \quad \text{and} \quad x_6 = (0.38955969,\ 0.17266485)$$

are two distinct approximate solutions of the system, but with a significant difference. The last two tables permit to find other solutions by changing the couples of guess points.

5. Concluding remarks. A new mathematical method has been introduced in this paper for solving variational inclusions. The algorithm is based on a partial linearization of $f + F$ in the Nachi-Penot sense and the use of divided differences for the function g. With the help of the concept of convex processes we obtained semilocal convergence to a solution of our variational inclusion. The numerical part allows us to underline that the use of random numbers may not reach the same solution or even require the same number of iterations to reach the same solution, if we run the same code twice. Thus, the random part of the code makes it impossible to predict which solution will be reached. Regarding the random part of the implemented algorithm, it will be interesting to define optimal choices for $F(x_k)$ and $A(x_k)$, which will increase the speed of convergence. In the present paper we did not examine the method with other kind of derivatives for set-valued maps. This could be studied in future works.

Acknowledgements The authors are grateful to the anonymous referees and editor for all the suggestions and comments that allowed us to improve the quality of the paper.

REFERENCES

[1] J.-P. Aubin, H. Frankowska. Set-valued analysis. Systems & Control: Foundations & Applications, 2. Boston, MA, Birkhäuser Boston, Inc.,1990.

[2] E. Cătinaş. On some iterative methods for solving nonlinear equations. *Rev. Anal. Numer. Theor. Approx.* **23**, *1* (1994), 47–53.

[3] A. L. Dontchev, R. T. Rockafellar. Implicit functions and solution mappings. A view from variational analysis. Springer Monographs in Mathematics. Dordrecht, Springer, 2009.

[4] A. L. Dontchev. Local convergence of the Newton method for generalized equation. *C. R. Acad. Sci. Paris Ser. I Math.* **322**, *4* (1996), 327–331.

[5] F. Facchinei, J. S. Pang. Finite-dimensional variational inequalities and complementarity problems, Vol. II. Springer Series in Operations Research, New York, Springer, 2003.

[6] M. C. Ferris, M. C. Pang. Engineering and economic applications of complementarity problems. SIAM Rev. **39**, *4* (1977), 669-713.

[7] M. Gaydu, P. Nuiro, A. Pétrus, O. Yacinthe. A Newton method for differentiable set-valued maps in Nachi-Penot sense (submitted).

[8] M. H. Geoffroy, A. Piétrus. An iterative method for perturbed generalized equations, C. R. Acad. Bulgare Sci., 57 (2004), no. 11, 7 - 12.

[9] M. H. Geoffroy, A. Piétrus. Local convergence of some iterative methods for generalized equations. *J. Math. Anal. Appl.* **290**, *2* (2004), 497–505.

[10] M. Grau-Sánchez, M. Noguera, S. Amat. On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods. *J. Comput. Appl. Math.* **237**, *1* (2013), 363–372.

[11] L. V. Kantorovich. On Newton's method. *Trudy Mat. Inst. Steklov* **28** (1949), 104–144 (in Russian).

[12] A.S. Levitin, B. T. Polyak. Constrained minimization methods. *Zh. Vychisl. Mat. Mat. Fiz.* **6** (1966), 787–823 (in Russian); English translation in *USSR Comput. Math. Math. Phys.* **6**, *5* (1966), 1–50 (1968).

[13] B. S. Mordukhovich. Variational analysis and generalized differentiation. I. Basic Theory. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **330**. Berlin, Springer-Verlag, 2006.

[14] K. Nachi, J.-P. Penot. Inversion of multifunctions and differential inclusions. *Control Cybernet.* **34**, *3* (2005), 871–901.

[15] A. Piétrus, Non differentiable perturbed Newton's method for functions with values in a cone. *Investigación Oper.* **35**, *1* (2014), 58–67.

[16] A. Piétrus, C. Jean-Alexis. Newton-Secant Method for functions with values in a cone. *Serdica Math. J.* **39**, *3–4* (2013), 217–286.

[17] C. H. J. Pang. Generalized differentiation with positive homogeneous maps: applications in set-valued analysis and metric regularity. *Math. Oper. Res.* **36**, *3* (2011), 377–397.

[18] S. M. Robinson. Extension of Newton's method to nonlinear functions with values in a cone. *Numer. Math.* **19** (1972), 341–347.

[19] S. M. Robinson. Normed convex processes. *Trans. Amer. Math. Soc.* **174** (1972), 127–140.

[20] R.T. Rockafellar. Convex Analysis. Princeton Mathematical Series, No. **28**. Princeton, N.J., Princeton University Press, 1970.

[21] R. T. Rockafellar, J. B. West. Variational analysis. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **317**. Berlin, Springer-Verlag, 1998.

[22] G. N. Silva, P. S. M. Santos, S. S. Souza. Extended Newton-type method for nonlinear functions with values in a cone. Comput. Appl. Math. **37**, *4* (2018), 5082–5097.

[23] A. I. Zinčensko. Some approximate method of solving equations with non-differentiable operators. Dopovidi Akad. Nauk. Ukraïne RSR (1963), 156–161 (in Russian).

LAMIA
Département de Mathématiques
Université Antilles
F-97159 Pointe-à-Pitre, Guadeloupe, France
e-mail: michael.gaydu@univ-antille.fr *(Michaël Gaydu)*
yolguine@yahoo.fr *(Olguine Yacinthe)*
paul.nuiro@univ-antilles.fr *(Silvere Paul Nuiro)*
alain.pietrus@univ-antilles.fr *(Alain Pietrus)*

Received April 29, 2022
Accepted December 22, 2022