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GENERALIZED WEINGARTEN HYPERSURFACES OF RADIAL SUPPORT TYPE

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Abstract. In this paper we introduce a class of Weingarten hypersurfaces called generalized Weingarten hypersurfaces of radial support type (in short, RSGW hypersurfaces). These hypersurfaces are parameterized using a new technique consisting in obtain locally any hypersurface in Euclidean space as an envelope of a sphere congruence wherein the other envelope is contained in a sphere. We extend the definition of the classical Appell surfaces to hypersurfaces and we characterize both Appell and RSGW hypersurfaces in terms of a same harmonic function in the sphere. For two-dimensional case, we provide a Weierstrass-type representation for RSGW surfaces and from this we get a Weierstrass-type representation for the classical Appell surfaces. These representations depend on two holomorphic functions in such a way that a same pair of functions provides examples in each class. Using it, we give a classification for the rotational cases for both classes. Furthermore is provided a necessary and sufficient condition on the holomorphic data of these classes so that they are parameterized by lines of curvature. We also prove that a compact, complete RSGW surface is a sphere.

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1. Introduction. The Weingarten surfaces form a very large and important class of surfaces in $\mathbb{R}^3$. They are surfaces that present a differentiable relation W between its Gaussian curvature K and its mean curvature H such that $W(H,K) \equiv 0$. Examples are those surfaces with a constant Gaussian curvature or a constant mean curvature, in particular those with a zero mean curvature, i.e., minimal surfaces.

A great advantage in studying Weingarten surfaces is in their potential applications in CAGD, particularly in surface design. Grant and Van-Brunt investigate in [14] general properties of Weingarten surfaces which make them of possible interest to problems in surface shape investigation. In the sequel to this paper, the authors continue this discussion dealing with the construction of Weingarten surfaces of interest in CAGD and focusing on a certain class described locally by partial differential equations.

The general classification of Weingarten surfaces is still an open question. Classification of certain classes of Weingarten surfaces has been reported in the literature and a great number of them arise in many situations.

Papantoniou [10] classified the Weingarten surfaces of revolution whose principal curvatures k_i satisfy a linear relation $Ak_1 + Bk_2 = 0$, with A and B not simultaneously null. Schief [12] studied *generalized Weingarten surfaces*, which accept the relation

$$1 + \mu H + (\mu^2 \pm \rho^2)K = 0,$$

where the functions ρ and μ are harmonics in a certain sense. These surfaces have been shown to be integrable.

There are also some examples of Weingarten hypersurfaces as those named Weingarten hypersurfaces of spherical-type. Machado [8] defines them as hypersurfaces M^n for which the r-th mean curvatures H_r satisfy the relation

$$\sum_{r=1}^{n} (-1)^{r+1} r f^{r-1} H_r \binom{n}{r} = 0,$$

for some function $f \in C^\infty(M, \mathbb{R})$.

In [4], Dias introduced a class of oriented Weingarten surfaces in $\mathbb{R}^3$ that satisfies a relation of the form

$$A(\Psi_\nu, \Lambda_\nu) + B(\Psi_\nu, \Lambda_\nu)H + C(\Psi_\nu, \Lambda_\nu)K = 0,$$

where $A, B, C : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions that depend on the support function Ψ_ν and the quadratic distance function Λ_ν, for some fixed point $\nu \in \mathbb{R}^3$.

These surfaces are referred to as *generalized Weingarten surfaces that depend on the support and distance functions* (in short, DSGW surfaces). Some of the main classes of Weingarten surfaces studied in the literature are classes of DSGW surfaces, such as the linear Weingarten, Appell and Tzitzéica's surfaces.

In 1888, Appell [1] studied a class of oriented surfaces in $\mathbb{R}^3$ associated with area preserving transformation in the sphere. Later, Ferreira and Roitman [7] found that these surfaces satisfy the Weingarten relation $H + \Psi_\nu K = 0$, for a fixed point $\nu \in \mathbb{R}^3$.

Another classical example are the Tzitzéica's surfaces (see [13]). These are hyperbolic surfaces where $K + c^2\Psi_\nu^4 = 0$, for some fixed point $\nu \in \mathbb{R}^3$ and a nonzero constant $c \in \mathbb{R}$.

Martínez and Roitman [9] presented what seems to be the first example found for a problem posed by Élie Cartan . Such surfaces are DSGW surfaces that obey the Weingarten relation

$$2\Psi H + (1 + \Lambda)K = 0,$$

where Ψ and Λ mean Ψ_0 and Λ_0, respectively.

A typical method to characterize the classes of Weingarten surfaces is to provide them with a Weierstrass-type representation by which they can be parameterized in terms of holomorphic functions. Dias studied a special class of DSGW surfaces satisfying a relation of the form $2\Psi H + \Lambda K = 0$. When $K \neq 0$, he provided these surfaces with a Weierstrass-type representation that depended on two holomorphic functions.

Corro [2] considered immersed surfaces $X : M \to \mathbb{H}^3$ for which the Gauss hyperbolic application G defines a sphere congruence, where $X(M)$ and $G(X(M))$ are envelopes and the radius of each sphere defines a differentiable function R. He introduced *generalized Weingarten surfaces of Bryant type* (in short, BGW surfaces) as those whose mean curvature H, Gaussian curvature K_I, and radius function R satisfy the relation

$$2acR^{\frac{2(c-1)}{c}}(H - 1) + (a + b - acR^{\frac{2(c-1)}{c}})K_I = 0,$$

where $a, b, c \in \mathbb{R}$; $a + b \neq 0$; and $c \neq 0$. Holomorphic data were obtained for this class depending on two holomorphic functions.

Fernandes [3] studied surfaces M in the hyperbolic space $\mathbb{H}^3$ whose mean curvature H and Gaussian curvature K_I satisfy

$$2(H - 1)e^{2\mu} + K_I(1 - e^{2\mu}) = 0.$$

Here μ is a harmonic function with respect to the quadratic form $\sigma = -K_I I + 2(H-1)II$, where I and II are the first and second fundamental forms of M, respectively. She called these surfaces *generalized Weingarten surfaces of harmonic type* and obtained for them a Weierstrass-type representation that depends on three holomorphic functions.

In the works [4], [2] and [3] mentioned earlier, the surfaces are parameterized as envelopes of a sphere congruence in which the other envelope is cointained in a plane. In [8], Machado generalizes this technique and characterizes the hypersurfaces in the Euclidean space that are envelopes of a sphere congruence in which the another one is cointained in a hyperplane. More recently, in [11], a method was developed to locally obtain hypersurfaces in the Euclidean space as envelopes of a sphere congruence in which the other envelope is contained in a unit sphere. In all these cases, the radius function R of the congruence is a geometric invariant, meaning it doesn't depend on the parameterization of the hypersurface. Hence, we can consider classes of Weingarten hypersurfaces satisfying a differentiable relation $W(k_i, \Psi, R) \equiv 0$.

Motivated by the works of Pereira [11] and Dias [4], we consider hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ that are envelopes of a sphere congruence in which the other envelope is contained in $\mathbb{S}^n$. For such a Σ, we introduce its *spherical radial curvatures s_i associated to $\mathbb{S}^n$* and *spherical mean curvature H_S associated to $\mathbb{S}^n$* as follows:

$$s_i = \frac{1+k_i}{1-k_i R}, \qquad H_S = \frac{1}{n}\sum_{i=1}^{n} s_i, \tag{1.1}$$

where k_i are the principal curvatures of Σ, $1 \leqslant i \leqslant n$, and R is the radius function of the congruence. Thus, we call *generalized Weingarten hypersurfaces of radial support type* (in short, RSGW hypersurfaces) the hypersurfaces Σ satisfying the Weingarten relation

$$1-\Psi = \big(1+R\big)H_S,$$

which depends on the r-th mean curvatures of Σ.

It has been proved that the radius function for the aforementioned hypersurface Σ is given by

$$R = \frac{1-\Lambda}{2\big(\Psi-1\big)},$$

so that, for the two-dimensional case, the aforesaid Weingarten relation becomes

$$\big[4\Lambda(\Psi-1)^2-(1-\Lambda)^2\big]H+\big[(\Lambda-1)(\Psi-2\Lambda+\Lambda\Psi)\big]K+2(\Psi-1)(1-\Lambda)+4\Psi(\Psi-1)^2 = 0,$$

which shows that the RSGW surfaces are a particular class of DSGW surfaces.

We have just noted that Appell surfaces are also DSGW surfaces with a very simple Weingarten relation given by $H/K = -\Psi$. The interesting point is that any Appell surface can be transformed in a RSGW surface by a transformation involving inversion, parallelism and dilatation. In order to establish a link between these classes in the n-dimensional case, we extend the definition of classical Appell surfaces, calling a hypersurface $S \subset \mathbb{R}^{n+1}$ an *Appell hypersurface* if it satisfies the relation $H_{n-1}(p)/H_n(p) = -\Psi(p)$, for all $p \in S$, where H_r, $1 \le r \le n$, denotes the r-th mean curvature of S. It follows that RSGW hypersurfaces and Appell hypersurfaces are related in terms of a harmonic function in the sphere as described below.

It is proved in [11] that the envelopes of a sphere congruence associated to a unit sphere are locally parameterized by

$$X = Y - 2\left(\frac{h+c}{S}\right)\eta, \tag{1.2}$$

where h is a differentiable function nonzero everwhere, c is a nonzero real constant, Y is a orthogonal local parameterization of $\mathbb{S}^n$ and

$$\eta = \nabla_L h + hY, \qquad S = \langle \eta, \eta \rangle = \left|\nabla_L h\right|^2 + h^2, \tag{1.3}$$

with $L_{ij} = \langle Y_{,i}, Y_{,j} \rangle$. We show that when h is harmonic in the metric L, then X defines a RSGW hypersurface and η defines an Appell hypersurface.

For two-dimensional RSGW hypersurfaces, the characterization above allow us to construct a Weierstrass-type representation for them that depends on two holomorphic functions. In addition, we present a classification when they are surfaces of rotation.

For the application, we provide a Weierstrass-type representation for Appell surfaces from that found for RSGW surfaces, such that the same holomorphic data yield examples of surfaces in each of these classes. We also give a necessary and sufficient condition on the holomorphic data in a way to construct examples of surfaces parameterized by lines of curvature in both classes. Futhermore, using its holomorphic representation, we classify Appell surfaces of rotation. The exposition made here about Appell surfaces is not presented elsewhere and the approach made in our work is a big step toward understanding them.

The rest of the paper is organized as follows. Section 2 is devoted to certain classical definitions and theorems in Differential Geometry and Complex

Analysis and to the presentation of the results concerning the sphere congruence used in the text. In Section 3, we define and discuss RSGW and Appell hypersurfaces and we establish a link between them. We also show that in the two-dimensional case a compact RSGW hypersurface is cointained in a sphere. In Section 4, we provide Weierstrass-type representations for the RSGW and Appell surfaces, giving a necessary and sufficient condition for them to be parameterized by lines of curvature. Finally, we present a classification when they are rotational.

2. Preliminaries.

In this section we give the notation and the main classical results in the literature that will be used in the work.

2.1. Hypersurfaces in the Euclidean Space. Throughout this paper $f_{,i}$ indicates the partial derivative of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to i-th variable, U denotes an open subset of $\mathbb{R}^n$ and Σ a hypersurface in $\mathbb{R}^{n+1}$ with normal Gauss map N. In this sense, if $X : U \to \Sigma$ is a local parameterization of Σ, the matrix $W = (W_{ij})$ such that

$$N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j}, \quad 1 \leq i \leq n,$$

is called the *Weingarten matrix* of Σ. The vector $X_{,ij}$, $1 \leqslant i, j \leqslant n$, can be written as

$$X_{,ij} = \sum_{k=1}^{n} \Gamma_{ij}^{k} X_{,k} + b_{ij} N, \tag{2.1}$$

and if the parameterization X is such that the metric $g_{ij} = \langle X_{,i},\ X_{,j} \rangle$ is diagonal, the Christoffel symbols satisfy

$$\begin{aligned} \Gamma_{ij}^{k} &= 0, \quad \text{for } i, j, k \text{ distincts}, \\ \Gamma_{ij}^{j} &= \frac{g_{jj,i}}{2g_{jj}}, \quad \text{for all } i, j; \\ \Gamma_{ii}^{j} &= -\frac{g_{ii,j}}{2g_{jj}} = -\frac{g_{ii}}{g_{jj}} \Gamma_{ji}^{i}, \quad \text{for } i \neq j. \end{aligned} \tag{2.2}$$

The first fundamental form I of Σ is the standard scalar product of $\mathbb{R}^{n+1}$ restricted to the tangent hyperplanes $T_p\Sigma$, whereas the second form of Σ, denoted by II, is defined as

$$II_p\big(w_1, w_2\big) = \langle -dN_p(w_1), w_2\rangle\,, \qquad w_1, w_2 \in T_p\Sigma,$$

where $p \in \Sigma$ and dN_p is the differential of the normal Gauss map in p.

Take Σ oriented by its normal Gauss map N. Given $\nu \in \mathbb{R}^n$, the functions $\Psi_\nu, \Lambda_\nu : \Sigma \to \mathbb{R}$ given by

$$\Psi_\nu(p) = \big\langle p - \nu, N(p)\big\rangle, \qquad \Lambda_\nu(p) = \big|p - \nu\big|^2, \qquad p \in \Sigma, \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$, are called the *support function* and *quadratic distance function* with respect to $\nu \in \mathbb{R}^n$, respectively. Geometrically, $\Psi_\nu(p)$ measures the signed distance from ν to the tangent plane $T_p\Sigma$ and $\Lambda_\nu(p)$ measures the square of the distance from p to ν. If ν is the origin, we write $\Psi_0 = \Psi$ and $\Lambda_0 = \Lambda$.

Appell surfaces are given by the surfaces S such that $H(p)/K(p) = -\Psi(p)$, for all $p \in S$, where H and K denote the mean and Gaussian curvatures of S, respectively. In what follows, we generalize this notion for hypersurfaces in $\mathbb{R}^{n+1}$.

Definition 2.1. *A hypersurface S in $\mathbb{R}^{n+1}$ is called an Appell hypersurface if satisfy*

$$\frac{H_{n-1}(p)}{H_n(p)} = -\Psi(p),$$

for all $p \in S$, where H_r, $1 \le r \le n$, denotes the r-th mean curvature of S defined by

$$H_r = \binom{n}{r}^{-1} \sum_{j_1<\cdots<j_r} k_{j_1}\cdots k_{j_r}$$

Note that $H_1 = H$ and $H_n = K$. When $n = 2$, this definition of Appell hypersurfaces coincides with the classical one.

2.2. Sphere congruence. A *sphere congruence* is a n-parameter family of spheres whose centers lie on a hypersurface Σ_0 contained in $\mathbb{R}^{n+1}$ with a differentiable radius function. In other words, if we consider Σ_0 locally parameterized by $X_0 : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, then for each point $u \in U$ there exists a sphere centered at $X_0(u)$ with radius $R(u)$, where R is a differentiable real function, named *radius function*.

An *envelope of a sphere congruence* is a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ such that each point of Σ is tangent to a sphere of the sphere congruence. Two hypersurfaces Σ and $\widetilde{\Sigma}$ are said to be (*locally*)*associated by a sphere congruence* if there is a (local) diffeomorphism $\varphi : \Sigma \to \widetilde{\Sigma}$ such that at corresponding points p and $\varphi(p)$ the hypersurfaces are tangent to the same sphere of the sphere congruence. It follows that the normal lines at corresponding points intersect at an equidistant point on the hypersurface Σ_0. If, moreover, the diffeomorphism φ preserves lines of curvature, we say that Σ and $\widetilde{\Sigma}$ are associated by a *Ribaucour transformation.*

In the work [11] is established that for a hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying $\langle p-\nu, N(p)\rangle \neq 1$, for all $p \in \Sigma$ and $\nu \in \mathbb{R}^{n+1}$ fixed, there exists a sphere congruence for which Σ and the unit sphere E centered in ν are envelopes. In this case, the radius function is given by the expression

$$R(p) = \frac{1-|p-\nu|^2}{2\big(\langle p-\nu, N(p)\rangle - 1\big)}, \tag{2.4}$$

which shows that R is a geometric invariant of Σ. Futhermore, it is proved that a such hypersurface Σ can be locally parameterized from a local parameterization of E in a way described below.

Theorem 2.2. *Let Σ be a hypersurface in $\mathbb{R}^{n+1}$ such that $\langle p, N(p)\rangle \neq 1$, for all $p \in \Sigma$, where N is the normal Gauss map. For each orthogonal local parameterization $Y : U \subset \mathbb{R}^n \to \mathbb{S}^n$ of $\mathbb{S}^n$, there is a differentiable function $h : U \subset \mathbb{R}^n \to \mathbb{R}$, associated to this parameterization, such that Σ can be locally parameterized by*

$$X(u) = Y(u) - 2\left(\frac{h(u)+c}{S(u)}\right)\eta(u), \qquad u \in U, \tag{2.5}$$

where the function h satisfy $h(u) \neq 0$, for all $u \in U$, c is a nonzero real constant and

$$\eta = \nabla_L h + hY, \qquad S = \langle \eta, \eta\rangle = \left|\nabla_L h\right|^2 + h^2, \tag{2.6}$$

with $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$.

In these coordinates, the Gauss map N of Σ is given by

$$N(u) = Y(u) - 2\frac{h(u)}{S(u)}\eta(u), \qquad u \in U. \tag{2.7}$$

Moreover, the Weingarten matrix W of Σ is

$$W = \left[SI - 2hV\right]\left[SI - 2\left(h+c\right)V\right]^{-1}, \tag{2.8}$$

where $V = (V_{ij})$ is given by

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \sum_{k}^{n} h_{,k}\Gamma_{ij}^{k} + hL_{ij}\delta_{ij}\right), \tag{2.9}$$

with Γ_{ij}^{k} the Christoffel symbols of the metric L and I the identity matrix $n \times n$. We also have that X is regular if and only if

$$P = \det\left[SI - 2\left(h+c\right)V\right] \neq 0, \quad \textit{for all } u \in U. \tag{2.10}$$

Conversely, if $Y : U \subset \mathbb{R}^n \to \mathbb{S}^n$ is an orthogonal local parameterization and $h : U \to \mathbb{R}$ is a differentiable function that doesn't vanish in any point and satisfies (2.10)*, then* (2.5) *defines an immersion in $\mathbb{R}^{n+1}$ with normal Gauss map N given by* (2.7)*, Weingarten matrix described by* (2.8) *and $\langle X, N\rangle \neq 1$ in all point.*

Remark 1. A hypersurface Σ in the conditions above is locally associated to $\mathbb{S}^n$ by a sphere congruence and the function h of which the theorem refers is given by

$$h(u) = -\frac{c}{R(u)+1},$$

where c is a nonzero constant and R is the radius function. Futhermore, for a such Σ, its fundamental forms I, II in local coordinates are expressed as

$$I : \langle X_{,i}, X_{,j}\rangle = \frac{4(c+h)^2}{S^2}\sum_k V_{ik}V_{jk}L_{kk} - \frac{2}{S}(c+h)(V_{ij}L_{jj} + V_{ji}L_{ii}) + L_{ij}, \tag{2.11}$$

$$II : \langle X_{,i}, N_{,j}\rangle = \frac{4h}{S^2}(c+h)\sum_k V_{ik}V_{jk}L_{kk} - \frac{2}{S}\big((h+c)V_{ij}L_{jj} + hV_{ji}L_{ii}\big) + L_{ij}. \tag{2.12}$$

In the case the matrix V is diagonal, the hypersurface Σ is parameterized by lines of curvature and it is associated to $\mathbb{S}^n$ by a Ribaucour transformation. Finally, if we take $Y = \pi_-^{-1}$, where $\pi_- : \mathbb{S}^n\backslash\{-e_{n+1}\} \to \mathbb{R}^n$ is the stereographic projection, Σ is a hypersurface of rotation if and only if the function h is radial.

Below it follows some properties of the hypersurface η given in (2.6).

Remark 2. In the conditions of Theorem 2.2, observe that Y is a unit normal vector field to the hypersurface $\eta : U \subset \mathbb{R}^n \to \eta(U)$ given by

$$\eta(u) = \nabla_L h(u) + h(u)Y(u), \qquad u \in U,$$

and $\langle \eta, Y\rangle = h$, so that h is the support function of the hypersurface η. Futhermore, η has regularity condition equal to $\det V \neq 0$ and its Weingarten matrix is V^{-1}, with V as in (2.9). Indeed, it is valid that

$$\eta_{,j} = \sum_k V_{jk} Y_{,k} \quad 1 \leq j \leq n, \tag{2.13}$$

which means that the matrix V^T is the coefficient matrix of $d\eta$ in the base $\{Y_{,i}\}$, so that η is an immersion iff $\det V^T = \det V \neq 0$. In addition, the equality (2.13) implies that

$$\sum_j^n (V^{-1})_{ij} \eta_{,j} = Y_{,i} \quad 1 \leq i \leq n,$$

which is to say that V^{-1} is the Weingarten matrix of η.

Now, we are going to show a fact that will be useful later in our discussion.

Lemma 2.3. *When Y is the inverse of stereographic projection, the hypersurface $\eta(U)$ is rotational if and only if h is a radial function.*

In fact, if $\eta(U)$ is rotational, the ortogonal sections to the axis of rotation determine on the surface $(n-1)$-dimensional spheres centered on this axis. Note that along these spheres both $|\eta|^2$ and the angle between η and Y must be constant. Since

$$\langle \eta, Y\rangle = h, \qquad \langle \eta, \eta\rangle = |\nabla_L h|^2 + h^2, \quad L_{ij} = \langle Y_{,i}, Y_{,j}\rangle\,,$$

we conclude that h and so $|\nabla_L h|^2$ are constant along these spheres. Taking Y as the inverse of stereographic projection, we get $L_{ij} = 4\delta_{ij}/(1+|u|^2)^2$, so that

$$|\nabla_L h|^2 = \left(\frac{1+|u|^2}{2}\right)^2 |\nabla h|,$$

what says that $|u|^2$ is constant as one goes around the ortogonal sections. Therefore the function h is constant along $(n-1)$-dimensional spheres in $\mathbb{R}^n$ centered in the origin and, therefore, is a radial function.

On the other hand, if we suppose that h is a radial function, we can write $h(u) = J(|u|^2), u \in U$, for some differentiable function J. We set $|u|^2 = t$ and we denote the derivative of J with respect to t as $J'(t)$. Therefore $h_{,i} = 2J'u_i$ and taking the parameterization Y as the inverse of stereographic projection, we get

$$\eta = \left(\left(J'(1-t) + \frac{2J}{1+t} \right) u, -2tJ' + J \left(\frac{(1-t)}{1+t} \right) \right).$$

If $-2tJ' + J \left(\frac{(1-t)}{1+t} \right)$ is constant, then

$$\left| \left(J'(1-t) + \frac{2J}{1+t} \right) u \right|^2 = \left(J'(1-t) + \frac{2J}{1+t} \right)^2 t,$$

which means that the ortogonal sections to the axis x_{n+1} determine on $\eta(U)$ $(n-1)$-dimensional spheres centered on this axis, so that $\eta(U)$ is rotational.

2.3. Holomorphic functions. The identification of the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ naturally induces the notion of inner product in the space of holomorphic functions. For $f, g : U \subset \mathbb{C} \to \mathbb{C}$ holomorphic functions, the inner product $\langle f, g \rangle$ is a real function defined in U, given by

$$\langle f, g \rangle = \langle 1, f \rangle \langle 1, g \rangle + \langle i, f \rangle \langle i, g \rangle ,$$

where $\langle 1, f \rangle = Re(f)$ and $\langle i, f \rangle = Im(f)$ denote the real and imaginary parts of f, respectively. Moreover, the norm of a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ is defined as

$$|f| = \sqrt{\langle f, f \rangle}.$$

This inner product satisfies the following properties for holomorphic functions f, g and h:

(1) $\langle f, g \rangle_{,1} = \langle f', g \rangle + \langle f, g' \rangle$.

(2) $\langle f, g \rangle_{,2} = \langle if', g \rangle + \langle f, ig' \rangle$.

(3) $\langle fh, g \rangle = \langle f, \bar{h}g \rangle$.

(4) $\bar{f}g = \langle f, g \rangle + i \langle if, g \rangle$.

where f' denotes the complex derivative of f. Using the notation settled in the beginning of the section, the relationship between the real and complex derivatives of a holomorphic function f is

$$f' = f_{,1} = -if_{,2}.$$

Here we present some results from the theory of holomorphic functions which later will be useful in our work.

Theorem 2.4. *Every real harmonic function defined in an open simply connected set of $\mathbb{C}$ is the real part of a holomorphic function defined in this set.*

Proposition 2.5. *Let f, g, h be holomorphic functions. Then the equality*

$$\langle 1, f\rangle + \langle g, h\rangle = 0, \tag{2.14}$$

is valid if and only if there exist real constants c_1, c_2 and a complex constant z_1 such that

$$\begin{cases} h &= ic_1 g + z_1 \\ f &= -\bar{z}_1 g + ic_2 \end{cases} \tag{2.15}$$

Proposition 2.6 (Liouville Theorem)**.** *Every harmonic function on $\mathbb{C}$ that is bounded below or above is constant.*

3. RSGW-Hypersurfaces.

In this section we introduce a class of generalized Weingarten hypersurfaces depending on support and distance functions named RSGW hypersurfaces. We give a necessary and sufficient condition – (3.3) – for an envelope of a sphere congruence, in which the other envelope is the unit sphere $\mathbb{S}^n$, to be a RSGW hypersurface. Futhermore, we show that (3.3) is also a necessary and sufficient condition for the hypersurface η in (2.6) to be an Appell hypersurface. Moreover, we present a local parameterization for RSGW hypersurfaces depending on an orthogonal local parameterization of unit sphere $\mathbb{S}^n$ and then we use such parameterization to characterize the rotational cases.

Since the radius function is a geometric invariant for a hypersurface Σ in the Euclidean space such that Σ and a unit sphere of center ν are envelopes of a sphere congruence, we can define certain curvatures for it. Thus, for such a Σ, we define its *spherical radial curvatures s_i associated to $\mathbb{S}^n$* and *spherical mean curvature H_S associated to $\mathbb{S}^n$* as

$$s_i = \frac{1 + k_i}{1 - k_i R}, \qquad H_S = \frac{1}{n}\sum_{i=1}^{n} s_i, \tag{3.1}$$

where k_i are the principal curvatures of Σ, $1 \leqslant i \leqslant n$, and R is the radius function given by (2.4).

Definition 3.1. *A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is called a generalized Weingarten hypersurface of radial support type (in short, RSGW hypersurface) if there exists a unit sphere in $\mathbb{R}^{n+1}$ of center ν with $\Psi_\nu(p) \neq 1$, for all $p \in \Sigma$, such that the relation*

$$1 - \Psi_\nu(p) = \big(1 + R(p)\big) H_S(p), \tag{3.2}$$

is valid for all $p \in \Sigma$, where $R : \Sigma \to \mathbb{R}$ is the function given in (2.4) *and H_S is the spherical mean curvature of Σ associated to $\mathbb{S}^n$.*

It follows that a RSGW hypersurface is an envelope of a sphere congruence whose other envelope is the unit sphere centered in ν.

From now on, we assume $\nu = 0$ and h always denotes the function described in Theorem 2.2.

Theorem 3.2. *Let Σ and η be hypersurfaces in the conditions of Theorem* 2.2. *Moreover consider $Y : U \subset \mathbb{R}^n \to \mathbb{S}^n$ the parameterization of unit sphere given by $Y(u) = \pi_-^{-1}(u)$, where $\pi_- : \mathbb{S}^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n$ is the stereographic projection. In this case, if*

$$\Delta_L h \equiv 0, \tag{3.3}$$

where L is the metric given by $L_{ij} = \langle Y_{,i}, Y_{,j} \rangle$, one has that

(1) *Σ is a RSGW hypersurface.*

(2) *η is an Appell hypersurface.*

On the other hand, if (3.2) *or* (3.2) *is true, then h is harmonic in the metric L.*

Proof. (1) Let Σ be a hypersurface which satisfies $\Psi(p) \neq 1$, $p \in \Sigma$. Thus, the Theorem 2.2 ensures the existence of a differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that Σ can be locally parameterized by (2.5), with Weingarten matrix $W = [SI - 2hV]\,[SI - 2(h+c)V]^{-1}$, where $V = (V_{ij})$ is the matrix (2.9). In this case,

$$trV = \sum_{j=1}^{n} V_{jj} = \sum_{j=1}^{n} \frac{1}{L_{jj}}(h_{,jj} - \sum_{k=1}^{n} h_{,k}\Gamma_{jj}^{k}) + nh. \tag{3.4}$$

Taking Y as the inverse of stereographic projection, locally we have

$$(3.5)\qquad Y(u) = \left(\frac{2u}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2}\right), \qquad L_{ij} = \langle Y_{,i}, Y_{,j}\rangle = \frac{4}{(1+|u|^2)^2}\delta_{ij}.$$

Being L a conformal metric, it follows from (3.4) the equality

$$(3.6)\qquad trV = \Delta_L h + nh,$$

and since $h = -c/(R+1)$, we can write

$$(3.7)\qquad trV = \Delta_L h - \frac{nc}{R+1},$$

where R is the radius function of the congruence.

On the other hand, since the matrix V is symmetric, its eigenvalues σ_i satisfy

$$\sigma_i = \frac{S}{2h}\left(\frac{1+k_i}{1-Rk_i}\right),$$

where $-k_i$ are the eigenvalues of W and S is the function given by (2.6). Note that from (2.5) and (2.7), the support function of Σ can be written as

$$\Psi = \frac{S+2ch}{S},$$

and using it for rewriting h in terms of the function Ψ, we get finally

$$(3.8)\qquad trV = \sum_{i=1}^{n} \sigma_i = \left(\frac{c}{\Psi-1}\right)\sum_{i=1}^{n}\left(\frac{1+k_i}{1-k_iR}\right).$$

Taken together, equalities (3.7) and (3.8) show that the hypersursurface Σ is a RSGW hypersurface if and only if $\Delta_L h \equiv 0$.

(2) From Remark 2, we know that $\widetilde{W} = V^{-1}$ is the Weingarten matrix of the hypersurface η. Therefore, if $\tilde{k}_i$ are the principal curvatures of η, it follows that $-\tilde{k}_i = 1/\sigma_i$, where σ_i are the eingenvalues of V. Thus,

$$(3.9)\qquad trV = \sum_{i=1}^{n} \sigma_i = -\frac{1}{\tilde{k}_1} - \cdots - \frac{1}{\tilde{k}_n} = -n\frac{\widetilde{H}_{n-1}}{\widetilde{K}},$$

where $\widetilde{H}_{n-1}$ and $\widetilde{K}$ are the $(n-1)$-th mean curvature and Gaussian curvature of η, respectively. Thus, if h is harmonic in the metric L, from equalities (3.6) and (3.9) we gather that

$$\frac{\widetilde{H}_{n-1}}{\widetilde{K}} = -h,$$

which shows that η is an Appell hypersurface. The converse follows immediately. $\square$

Theorem 3.2 reveals a type of association between RSGW hypersurfaces and Appell hypersurfaces, since both of them are settled by the same harmonic function h. In fact, once we have found a Weierstrass-type representation for these surfaces, we will see that the same holomorphic data provide examples of surfaces in each of these classes.

The part 2 of Theorem 3.2, for the two-dimensional case, can also be achivied from the approach used in [7].

As a corollary of Theorem 3.2 we are going to show that, for two-dimensional case, a compact RSGW hypersurface is a sphere.

Corollary 3.3. *In the conditions of Theorem* 2.2*, if* Σ *is a compact, complete RSGW surface in* $\mathbb{R}^3$*, then* Σ *is a sphere.*

Proof. Being Σ compact, there exists a sphere M centered in the origin containing Σ. Decreasing the radius of M if necessary, there will be a intersection $\Sigma \cap M$, say in a point p'. In this case, p' is the point of Σ with highest norm, i.e., $|p| \leq |p'|$, $\forall p \in \Sigma$. Moreover, $T_{p'}\Sigma = T_{p'}M$, so that $\psi(p') = |p'|$, since the support function in a point at sphere is its norm.

Therefore, for all point $p \in \Sigma$,

$$\psi(p) \leq |p| \leq |p'| = \psi(p') \tag{3.10}$$

Since Σ is a RSGW surface, then Σ is associated to the unit sphere by a sphere congruence and has a radius function R. Let's compare, for arbitray $p \in \Sigma$, the values $R(p)$ and $R(p')$. From $\Lambda(p) \leq \Lambda(p')$, $\forall p \in \Sigma$, we get

$$1 - \Lambda(p) \geq 1 - \Lambda(p'), \quad \forall p \in \Sigma \tag{3.11}$$

We can assume $\psi(p) > 1$, $\forall p \in \Sigma$, since $\psi(p) \neq 1$, whatever the point p is. So (3.10) and (3.11) imply that

$$\frac{1 - \Lambda(p)}{2(\psi(p) - 1)} \geq \frac{1 - \Lambda(p')}{2(\psi(p') - 1)}, \quad \forall p \in \Sigma$$

As $R = (1 - \Lambda)(2(\psi - 1))$, the inequality above shows that

$$R(p') \leq R(p), \quad \forall p \in \Sigma \tag{3.12}$$

Now, using that $h = -c/(1 + R)$, (3.12) gives us

$$h(p') \leq h(p), \quad \forall p \in \Sigma \tag{3.13}$$

On the other hand, consider the sphere M centered in the origin tangent to the Σ in p'. It is clear that M is also associated to the unit sphere by a sphere congruence and so it has a radius function, that we call R_1. Such function is constant given by $-(|p'|+1)/2$. Since the tangent hyperplanes $T_{p'}\Sigma$ and $T_{p'}M$ coincide, it is true that $\psi_1(p') = \psi(p')$, so that $R_1(p') = R(p')$ (subscript 1 meaning functions from M). This makes the auxiliar functions h and h_1 from Σ and M, respectively, assume the same value in p'. This fact together with (3.13) proves that, for all $p \in \Sigma$,

$$\frac{2c}{|p'|-1} = h_1(p') = h(p') \leq h(p) \tag{3.14}$$

So, without lost of generality, supposing $|p'| > 1$ and taking, for example, $l = \dfrac{2c-1}{|p'|-1}$, we have that the function h is bounded below by l. Futhermore, from Theorem 3.2, it is valid that the function h is harmonic in the metric L, given by $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$, taking $Y : U \subset \mathbb{R}^2 \to \mathbb{S}^2$ as the inverse of stereographic projection. In this case,

$$\Delta_L h = \frac{\Delta h}{L_{11}}$$

what means that being harmonic in the metric of sphere is the same as being harmonic in the Euclidean metric. Therefore, from the Liouville Theorem 2.6, we conclude that h is constant. Hence ψ is also constant, which shows that Σ is contained in a sphere. $\square$

4. RSGW surfaces. In this section we deal with the two-dimensional case of RSGW hypersurfaces, which we now call RSGW surfaces. For such surfaces, we provide a Weierstrass-type representation depending on two holomorphic functions and we characterize the RSGW surfaces of rotation. Simultaneously, the same is done for Appell surfaces.

We begin by taking a look at expression (3.2) that defines RSGW surfaces. Expanding it, we find the equation

$$\big[1 - R^2 + 2R(1-\Psi)\big]H - R\big[1 + R + R(1-\Psi)\big]K + R + \Psi = 0. \tag{4.1}$$

Replacing the function R by its expression (2.4), we get

$$\big[4\Lambda(\Psi-1)^2 - (1-\Lambda)^2\big]H + \big[(\Lambda-1)(\Psi-2\Lambda+\Lambda\Psi)\big]K + 2(\Psi-1)(1-\Lambda) + 4\Psi(\Psi-1)^2 = 0,$$

which provides a Weingarten relation depending on support and distance functions satisfied by the RSGW surfaces. Thus, every RSGW surface is a DSGW-surface.

The next proposition follows the same steps as Theorem 2.2, but now in the context of Riemann surfaces. This allows us to work with holomorphic functions, which will enable the construction of a Weierstrass-type representation for RSGW surfaces.

Proposition 4.1. *Let Σ be a Riemann surface and $X : \Sigma \to \mathbb{R}^3$ an immersion such that $\big\langle X(p), N(p)\big\rangle \neq 1$, for all $p \in \Sigma$, where N is the normal Gauss map of X. Consider also a holomorphic function $g : \mathbb{C} \to \mathbb{C}_\infty$ such that $g' \neq 0$. Then there exists a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ associated to this parameterization, such that Σ can be locally parameterized by*

$$X = \frac{1}{T}\Big(2g, 2-T\Big) - \frac{2(h+c)}{S}\eta, \tag{4.2}$$

where c is a nonzero real constant, $T = 1+|g|^2$ and

$$\eta = \left(\frac{T}{2}\frac{\nabla h}{\overline{g'}} - g\left\langle \nabla h, \frac{g}{g'}\right\rangle + \frac{2h}{T}g\ ,\ \frac{(2-T)h}{T} - \left\langle \nabla h, \frac{g}{g'}\right\rangle\right), \tag{4.3}$$

$$S = \langle \eta, \eta\rangle = \left(\frac{T}{2|g'|}\right)^2 |\nabla h|^2 + h^2, \tag{4.4}$$

with $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$.

In these coordinates, the Gauss map N of Σ is given by

$$N = \frac{1}{T}\Big(2g, 2-T\Big) - 2\frac{h}{S}\eta. \tag{4.5}$$

Moreover, the Weingarten matrix W of Σ is $W = \big[SI - 2hV\big]\big[SI - 2\big(h+c\big)V\big]^{-1}$, where the matrix V is such that

$$V_{11} = \frac{1}{L_{11}}\left[h_{,11} - \left\langle \frac{g''}{g'} - \frac{2}{T}g'\bar{g}\ ,\ \nabla h\right\rangle + hL_{11}\right], \tag{4.6}$$

$$V_{12} = \frac{1}{L_{22}}\left[h_{,12} - \left\langle i\Big(\frac{g''}{g'} - \frac{2}{T}g'\bar{g}\Big)\ ,\ \nabla h\right\rangle\right], \tag{4.7}$$

$$V_{22} = \frac{1}{L_{22}}\left[h_{,22} + \left\langle \frac{g''}{g'} - \frac{2}{T}g'\bar{g}\ ,\ \nabla h\right\rangle + hL_{22}\right]. \tag{4.8}$$

We also have that X is regular if and only if

$$P = S^2 - 2(h+c)\, S\, trV + 4(h+c)^2 \det V \neq 0. \tag{4.9}$$

Proof. Taking the parameterization $Y : U \subset \mathbb{R}^2 \to \mathbb{S}^2$ given by $Y = \pi_-^{-1} \circ g$, where $\pi_-^{-1} : \mathbb{C} \to \mathbb{S}^2 \backslash \{-e_3\}$ is the inverse of stereographic projection, we have, for $u \in U$,

$$Y(u) = \frac{1}{1 + |g(u)|^2}\Big(2g(u), 1 - |g(u)|^2\Big). \tag{4.10}$$

Thus, from Theorem 2.2, there is a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $X(\Sigma)$ can be locally parameterized by (4.2) with Gauss map (4.5), in which expression for η follows from straightforward computations, using property (3) and considering $\nabla h = h_{,1} + ih_{,2}$.

Theorem 2.2 also ensures that the Weingarten matrix of $X(\Sigma)$ is $W = \big[SI - 2hV\big]\big[SI - 2\big(h + c\big)V\big]^{-1}$, with regularity condition given by $P \neq 0$, where $P = \det\big[SI - 2\big(h + c\big)V\big]$, with V as in (2.9). For $n = 2$, this regularity condition may be rewritten as in equation (4.9).

In order to make explicit the V's entries, let us find the Christoffel symbols of the metric $L_{ij} = \langle Y_{,i}\ ,\ Y_{,j}\rangle$, $1 \le i, j \le 2$. From the expression (4.10) for Y, we have that the metric L is diagonal given by

$$L_{ij} = \frac{4|g'|^2}{T^2}\delta_{ij}, \qquad T = 1 + |g|^2, \qquad 1 \le i, j \le 2, \tag{4.11}$$

and from the equations (2.2), we have that its Christoffel symbols are

$$\Gamma_{11}^1 = \frac{T\,\langle g', g''\rangle - 2|g'|^2\,\langle g, g'\rangle}{T|g'|^2}, \qquad\qquad \Gamma_{22}^2 = \frac{T\,\langle g', ig''\rangle - 2|g'|^2\,\langle g, ig'\rangle}{T|g'|^2}.$$

From (2.9), it follows that

$$\begin{aligned} V_{11} &= \frac{1}{L_{11}}\left[h_{,11} - \left\langle \Gamma_{11}^1 + i\Gamma_{11}^2\ ,\ \nabla h\right\rangle + hL_{11}\right], \\ V_{12} &= \frac{1}{L_{22}}\left[h_{,12} - \left\langle \Gamma_{12}^1 + i\Gamma_{12}^2\ ,\ \nabla h\right\rangle\right], \\ V_{22} &= \frac{1}{L_{22}}\left[h_{,22} - \left\langle \Gamma_{22}^1 + i\Gamma_{22}^2\ ,\ \nabla h\right\rangle + hL_{22}\right]. \end{aligned} \tag{4.12}$$

Using property (3) for holomorphic functions, we have

$$\begin{aligned}\Gamma^1_{11} + i\Gamma^2_{11} &= \left\langle \frac{1}{\bar{g}'}, g'' \right\rangle - \frac{2}{T}\langle g, g'\rangle + i\left(\frac{2}{T}\langle g, ig'\rangle - \left\langle \frac{1}{\bar{g}'}, ig'' \right\rangle\right) \\ &= \left\langle 1, \frac{g''}{g'} - \frac{2}{T}g'\bar{g} \right\rangle + i\left\langle i, \frac{g''}{g'} - \frac{2}{T}g'\bar{g} \right\rangle \\ &= \frac{g''}{g'} - \frac{2}{T}g'\bar{g} \qquad (4.13)\end{aligned}$$

Similarly,

$$\Gamma^1_{12} + i\Gamma^2_{12} = i\left(\frac{g''}{g'} - \frac{2}{T}g'\bar{g}\right), \qquad \Gamma^1_{22} + i\Gamma^2_{22} = \frac{2}{T}g'\bar{g} - \frac{g''}{g'}. \tag{4.14}$$

Substituing (4.13) and (4.14) in the expressions (4.12) for the V's entries , we are done. □

Now, we are in a position to give a Weierstrass-type representation for the RSGW surfaces.

Theorem 4.2. *Let Σ be a simply connected Riemann surface and $X : \Sigma \to \mathbb{R}^3$ an immersion such that $\big\langle X(p), N(p)\big\rangle \neq 1$, for all $p \in \Sigma$, where N is the normal Gauss map of X and let $g : \mathbb{C} \to \mathbb{C}_\infty$ be a holomorphic function such that $g' \neq 0$. If $X(\Sigma)$ is a RSGW surface, there exists a holomorphic function $f : \Sigma \to \mathbb{C}$ such that $X(\Sigma)$ is locally parameterized by*

$$X = \frac{1}{T}\Big(2g, 2 - T\Big) - \frac{2(h+c)}{S}\eta, \tag{4.15}$$

where $h = \langle 1, f\rangle$, $\;T = 1 + |g|^2$ and

$$\eta = \left(\frac{T}{2}\overline{\left(\frac{f'}{g'}\right)} - g\left\langle f', \overline{\left(\frac{g}{g'}\right)}\right\rangle + \frac{2h}{T}g\,,\ \frac{(2-T)h}{T} - \left\langle f', \overline{\left(\frac{g}{g'}\right)}\right\rangle\right), \tag{4.16}$$

$$S = \left(\frac{T}{2}\left|\frac{f'}{g'}\right|\right)^2 + h^2. \tag{4.17}$$

In these coordinates, the Gauss map N of Σ is

$$N = \frac{1}{T}\Big(2g, 2 - T\Big) - 2\frac{h}{S}\eta, \tag{4.18}$$

The regularity condition is given by

$$P = \left(\left(\frac{T}{2} \left| \frac{f'}{g'} \right| \right)^2 - 2hc - h^2 \right)^2 - \left(\frac{T^2(h+c)}{2|g'|^2} \right)^2 |\xi|^2 \neq 0, \tag{4.19}$$

with

$$\xi = f' \left(\frac{g''}{g'} - \frac{2}{T} g' \bar{g} \right) - f''. \tag{4.20}$$

Conversely, let $g : \mathbb{C} \to \mathbb{C}_\infty$ *and* $f : \Sigma \to \mathbb{C}$ *be holomorphic functions satisfying* (4.19)*, such that* $g' \neq 0$ *and* $h = \langle 1, f \rangle \neq 0$ *in all point. Thus* $X : \Sigma \to \mathbb{R}^3$ *as in* (4.15) *is an immersion and* $X(\Sigma)$ *is a RSGW surface.*

Proof. It follows from Proposition 4.1 that there is a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $X(\Sigma)$ can be locally parameterized by (4.2). Now, if $X(\Sigma)$ is a RSGW surface, then Proposition 3.2 establishes that h is a harmonic function in the metric L – given by (4.11) – and so h is also harmonic in the Euclidean metric. In this case, Proposition 2.4 guarantees the existence of a holomorphic function $f : \Sigma \to \mathbb{C}$ such that $h = \langle 1, f \rangle$. Thus, it follows from the properties for holomorphic functions that $h_{,1} = \langle 1, f' \rangle$, $h_{,2} = -\langle i, f' \rangle$, so that $\nabla h = h_{,1} + i h_{,2} = \overline{f'}$. Therefore the expressions (4.3) and (4.4) for the functions η and S can be rewriten as in (4.16) and (4.17), respectively.

Proposition 4.1 still asserts that the Gauss map N of $X(\Sigma)$ can be expressed by

$$N = Y - \frac{2h}{S} \eta,$$

with Y, S and η as above. Moreover, the regularity condition is $P \neq 0$, with P given by (4.9). In order to explicit the regularity condition for this case, let us calculate the entries for the matrix V in (2.9). From $\nabla h = \overline{f'}$ and property (3), the expression (4.6) for V_{11} becomes

$$\begin{aligned} V_{11} &= \frac{1}{L_{11}} \left[\langle 1, f'' \rangle - \left\langle \frac{g''}{g'} - \frac{2}{T} g' \bar{g} \, , \bar{f}' \right\rangle + h L_{11} \right] \\ &= h + \frac{T^2}{4|g'|^2} \left\langle 1 \, , f'' - f' \left(\frac{g''}{g'} - \frac{2}{T} g' \bar{g} \right) \right\rangle \\ &= h - \frac{T^2}{4|g'|^2} \langle 1, \xi \rangle \end{aligned} \tag{4.21}$$

where ξ is the function given in (4.20). Similarly, from expressions (4.7) and (4.8), it follows

$$V_{12} = \frac{T^2}{4|g'|^2}\langle i, \xi\rangle, \qquad V_{22} = h + \frac{T^2}{4|g'|^2}\langle 1, \xi\rangle. \tag{4.22}$$

Therefore, for the matrix V is true that

$$trV = 2h, \qquad \det V = h^2 - \frac{T^4}{16|g'|^4}|\xi|^2.$$

Finally, plugging these expressions into (4.9), we get (4.19) as the regularity condition as we wished.

Conversely, let $g : \mathbb{C} \to \mathbb{C}_\infty$ and $f : \Sigma \to \mathbb{C}$ be holomorphic functions such that $g' \neq 0$ and $h = \langle 1, f\rangle \neq 0$ in all point, satisfying (4.19) and Y the parameterization described before. It follows from Theorem 2.2 that (4.15) is an immersion with Gauss map (4.18) such that $\langle X(p), N(p)\rangle \neq 1$, for all $p \in \Sigma$. Since f is holomorphic, the function h is harmonic and from Proposition 3.2 we have that $X(\Sigma)$ is a RSGW surface. □

Note that, for two-dimensional case, the definition of Appell hypersurfaces coincides with the classical one. Thus, we can conclude from Proposition 3.2 that the surface η is an Appell surface if and only if the function h is harmonic. Therefore, expression (4.16) sets up a Weierstrass-type representation for Appell surfaces. Let us write that in the next corollary.

Corollary 4.3. *Let Σ be a simply connected Riemann surface and $g : \mathbb{C} \to \mathbb{C}_\infty$, $f : \Sigma \to \mathbb{C}$ holomorphic functions such that $g' \neq 0$ and $h = \langle 1, f\rangle \neq 0$ in all point, satisfying*

$$\widetilde{P} = h^2 - \frac{T^4}{16|g'|^4}|\xi|^2 \neq 0, \tag{4.23}$$

where $T = 1 + |g|^2$, and $\xi = f'\left(\frac{g''}{g'} - \frac{2}{T}g'\bar{g}\right) - f''$. Then, $\eta : \Sigma \to \mathbb{R}^3$ given by

$$\eta = \left(\frac{T}{2}\overline{\left(\frac{f'}{g'}\right)} - g\left\langle f', \overline{\left(\frac{g}{g'}\right)}\right\rangle + \frac{2h}{T}g\,,\ \frac{(2-T)h}{T} - \left\langle f', \overline{\left(\frac{g}{g'}\right)}\right\rangle\right), \tag{4.24}$$

is an immersion and $\eta(\Sigma)$ is an Appell surface.

The way the Weierstrass representation was built allows us to obtain an example of RSGW surface and an example of Appell surface for each holomorphic data (f, g). Let us construct some of them below.

Example 4.4. Considering the holomorphic data $f(z) = z$ and $g(z) = z$, with $c = 1$ in the parameterization X, the correspondent RSGW surface and Appell surface are drawn below.

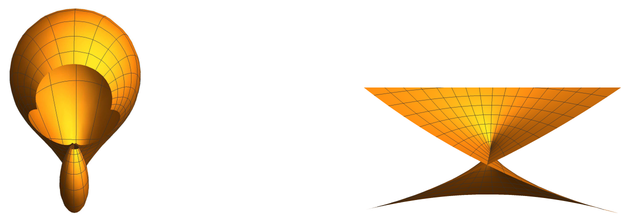


Fig. 1. On the left the RSGW surface and on the right the Appell surface, for $f(z) = z$ and $g(z) = z$

Example 4.5. For the holomorphic data $f(z) = e^z$ and $g(z) = e^z$, taking $c = 1$ in X, the correspondent RSGW surface is described by the first two figures. The last ones are from the Appell surface.

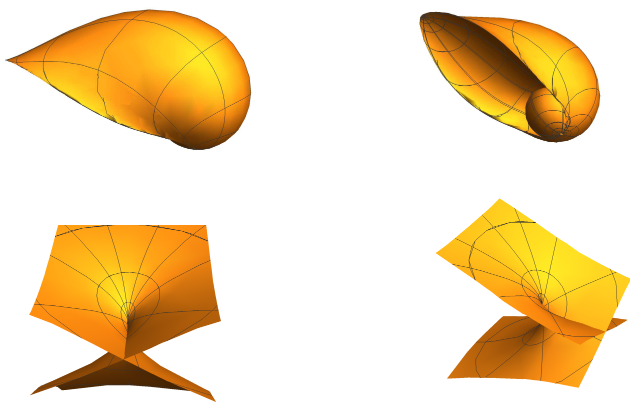


Fig. 2. The first two pictures are from the RSGW surface and the last two are from the Appell surface when $f(z) = e^z$ and $g(z) = e^z$

Example 4.6. Consider the holomorphic data $f(z) = \dfrac{4}{1-z^4}$ and $g(z) = z$. Taking $c = 10$ in the parameterization X, the first four pictures are drafts for the correspondent RSGW surface with different parameter values. The last three pictures sketch the respective Appell surface.

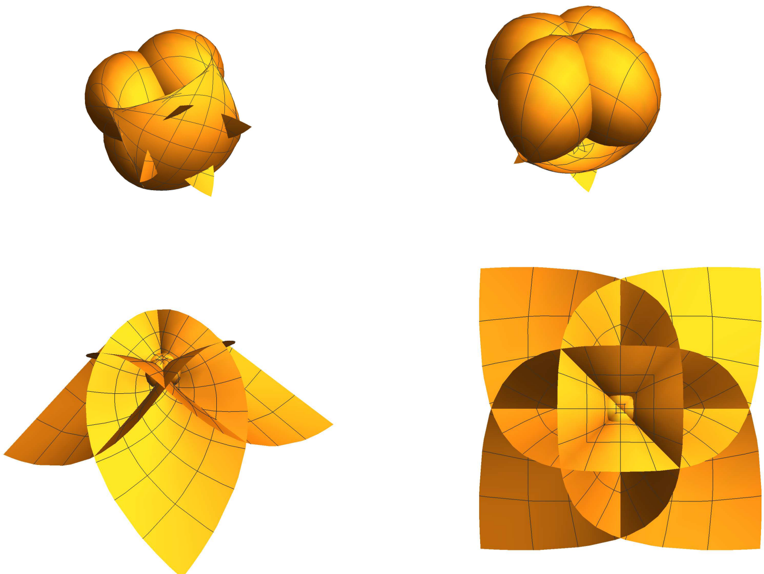


Fig. 3. Sketches for RSGW surface with $f(z) = \dfrac{4}{1-z^4}$ and $g(z) = z$

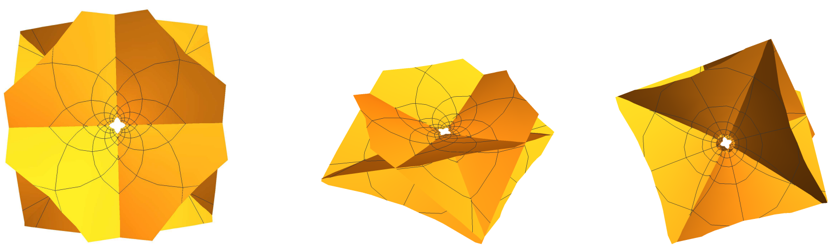


Fig. 4. Sketches for Appell surface with $f(z) = \dfrac{4}{1-z^4}$ and $g(z) = z$

Corollary 4.7. *In the conditions of Theorem* 4.2, *the coefficients F and f of the first and second fundamental forms of X, respectively, have expressions*

$$F = \langle X_{,1},\ X_{,2}\rangle = \frac{16|g'|^2}{\Delta^2}(h+c)\,\langle i,\xi\rangle \left[8h|g'|^2(h+c)-\Delta\right], \tag{4.25}$$

$$f = \langle X_{,1},\ N_{,2}\rangle = \frac{8|g'|^2}{\Delta^2}\,\langle i,\xi\rangle \left[16h^2(h+c)|g'|^2-\Delta(2h+c)\right], \tag{4.26}$$

where ξ *is given by* (4.20) *and* $\Delta = T^2|f'|^2 + 4h^2|g'|^2$.

P r o o f. Take the equations (2.11) and (2.12) and replace the entries of the matrix V by their equations (4.21) and (4.22); the function S by its expression (4.17) and the metric L as in (4.11). We get the expressions above for the coefficients after straightforward calculations. □

Corollary 4.8. *In the conditions of Theorem* 4.2*, the coefficients* $\tilde{F}$ *and* $\tilde{f}$ *of the first and second fundamental forms of* η*, respectively, have expressions*

$$\begin{aligned}\tilde{F} &= \langle \eta_{,1},\ \eta_{,2}\rangle = 2h\,\langle i,\xi\rangle\,,\\ \tilde{f} &= \langle \eta_{,1},\ Y_{,2}\rangle = \langle i,\xi\rangle\,,\end{aligned}$$

where ξ *is given by* (4.20).

P r o o f. As observed in [11], the derivative $\eta_{,i}$, $1 \le i \le 2$, can be expressed as

$$\eta_{,i} = \sum_{j=1}^{2} V_{ij} Y_{,j} \tag{4.27}$$

so that, by (4.21), (4.22) and the expression (4.11) for the metric L, we have $\tilde{F} = 2h\,\langle i,\xi\rangle$. Now using the expression (4.27) and considering the equalities (4.21), (4.22) and (4.11), we get $\tilde{f} = \langle i,\xi\rangle$. □

The next corollary brings a necessary and sufficient condition on the holomorphic data of a RSGW surface and an Appell surface so that they are parameterized by lines of curvature.

Corollary 4.9. *Let* (f,g) *be a holomorphic data of a RSGW surface* Σ_1 *and an Appel surface* Σ_2*. Then* Σ_1 *and* Σ_2 *are parameterized by lines of curvature if and only if there exist constants* $c_1, c_2 \in \mathbb{R}$ *and* $z_1 \in \mathbb{C}$ *such that*

$$\left\{\begin{aligned} if'' - i\frac{f'g''}{g'} &= -\bar{z}_1 g + ic_1\\ g\left(if'' - i\frac{f'g''}{g'} + 2i\frac{f'g'}{g}\right) &= ic_2 g + z_1\end{aligned}\right. \tag{4.28}$$

P r o o f. From the corollaries 4.7 and 4.8, we can note that the surfaces Σ_1 and Σ_2 are parameterized by lines of curvature if and only if $\langle i, \xi \rangle = 0$. On the other hand, using the property (3) for holomorphic functions, we have

$$\begin{aligned}
\langle i, \xi \rangle = 0 &\Leftrightarrow \left\langle i, f'\left(\frac{g''}{g'} - \frac{2}{T} g'\bar{g}\right) - f'' \right\rangle = 0,\\
&\Leftrightarrow T\left\langle i, \frac{f'g''}{g'} \right\rangle - 2\left\langle i, f'g'\bar{g} \right\rangle - T\left\langle i, f'' \right\rangle = 0,\\
&\Leftrightarrow -T\left\langle 1, i\frac{f'g''}{g'} \right\rangle - 2|g|^2\left\langle i, \frac{f'g'}{g} \right\rangle + T\left\langle 1, if'' \right\rangle = 0,\\
&\Leftrightarrow \left\langle 1, if'' - i\frac{f'g''}{g'} \right\rangle + |g|^2\left[\left\langle 1, -i\frac{f'g''}{g'} \right\rangle + 2\left\langle 1, i\frac{f'g'}{g} \right\rangle + \left\langle 1, if'' \right\rangle\right] = 0,\\
&\Leftrightarrow \left\langle 1, if'' - i\frac{f'g''}{g'} \right\rangle + \left\langle g, g\left(if'' - i\frac{f'g''}{g'} + 2i\frac{f'g'}{g}\right) \right\rangle = 0.
\end{aligned}$$

Thus, the result follows from Proposition 2.5. □

Example 4.10. The pair of functions $f(z) = \frac{c_1}{2} z^2 + z_2$, $g(z) = z$, with $c_1 \in \mathbb{R}$ and z_2 complex constant, is a solution for system (4.28) for $z_1 = 0$ and $c_2 = 3c_1$. Therefore, the RSGW surface and Appell surface with holomorphic data given by f and g are parameterized by lines of curvature. Below it follows their graphics when $c_1 = 1$ and $z_2 = 0$.

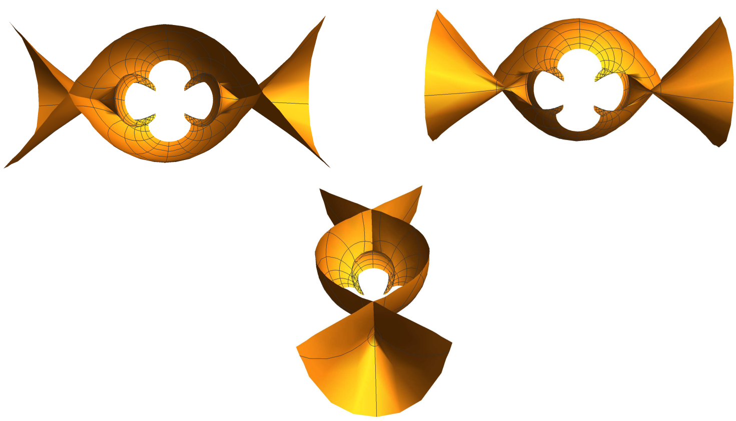


Fig. 5. RSGW surface parameterized by lines of curvature with $f(z) = \dfrac{z^2}{2}$ and $g(z) = z$, for $c = 1$

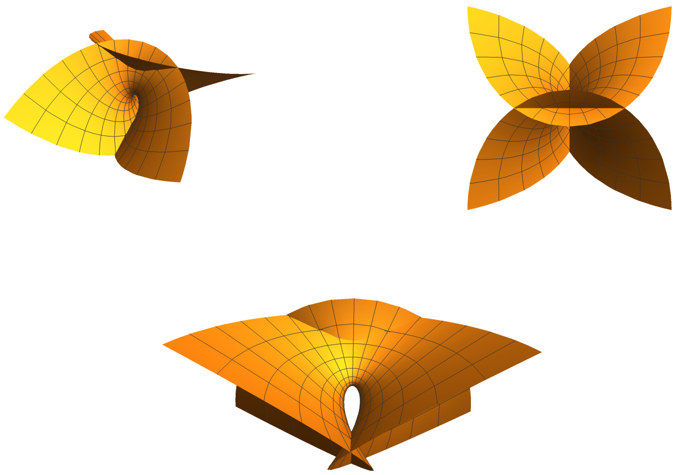


Fig. 6. Appell surface parameterized by lines of curvature with $f(z) = \dfrac{z^2}{2}$ and $g(z) = z$

Ahead, using Theorem 4.2 and Corollary (4.3), we classify the rotational cases for RSGW surfaces and Appell surfaces.

Theorem 4.11. *Let Σ_1 be a connected RSGW surface of rotation and Σ_2 a connected Appell surface of rotation. Then there exist constants $a, b \in \mathbb{R}$, not simultaneously null, such that Σ_1 and Σ_2 can be locally parameterized, respectively, by*

$$X_{ab}(u_1, u_2) = \big(M(u_1)\cos u_2,\ M(u_1)\sin u_2,\ N(u_1)\big), \tag{4.29}$$

$$\widetilde{X}_{ab}(u_1, u_2) = \big(\widetilde{M}(u_1)\cos u_2,\ \widetilde{M}(u_1)\sin u_2,\ \widetilde{N}(u_1)\big), \tag{4.30}$$

where

$$M(u_1) = \frac{2e^{u_1}\left[a^2T^2 - 4e^{2u_1}(au_1+b)(au_1+b+2c) + 2a(au_1+b+c)(e^{4u_1}-1)\right]}{T\big(a^2T^2 + 4e^{2u_1}(au_1+b)^2\big)}, \tag{4.31}$$

$$N(u_1) = \frac{(1-e^{2u_1})\left[a^2T^2 - 4e^{2u_1}(au_1+b)(au_1+b+2c)\right] + 8aTe^{2u_1}(au_1+b+c)}{T\big(a^2T^2 + 4e^{2u_1}(au_1+b)^2\big)}, \tag{4.32}$$

$$\widetilde{M}(u_1) = \frac{a(1-e^{4u_1}) + 4e^{2u_1}(au_1+b)}{2Te^{u_1}}, \tag{4.33}$$

$$\widetilde{N}(u_1) = \frac{(2-T)(au_1+b)-aT}{T}, \tag{4.34}$$

and $T = 1 + e^{2u_1}$.

Proof. Theorem 4.2 ensures that Σ_1 is locally parameterized by (4.15). We take $g(w) = w$, $w \in \mathbb{C}$, and, in this case, the Remark 1 asserts that Σ_1 is a RSGW surface of rotation if and only if h is a radial function, i.e., $h(w) = J(|w|^2)$, $w \in \mathbb{C}$, for any real differentiable function J. Making the change of parameters $w = e^z$, $z = u_1 + iu_2 \in \mathbb{C}$, we have $g(z) = e^z$ e $h_{,2} = 0$. However, from Theorem 4.2, we know that $h = \langle 1, f\rangle$ for a holomorphic function f. From Cauchy–Riemann equations and using the fact that h must be harmonic, we conclude that $f(z) = az + z_0$, $z_0 = b + id$, where $a, b, d \in \mathbb{R}$. Therefore $h(z) = au_1 + b$ and substituing all this in (4.15), we get (4.29) after straightforward calculations.

Now, from Corolary 4.3, we know that Σ_2 is locally parameterized by (4.24). Taking $g(w) = w$, $w \in \mathbb{C}$, the Remark 2 asserts that Σ_2 is an Appell surface of rotation if and only if h is a radial function. Then the argument follows as above and we get (4.30) as a local parameterization for Σ_2. □

Proposition 4.12. *Consider* $\varphi(u_1) = \big(M(u_1), N(u_1)\big)$ *and* $\tilde{\varphi}(u_1) = \big(\widetilde{M}(u_1), \widetilde{N}(u_1)\big)$ *generating curves of RSGW and Appell surfaces of rotation, respectively. Their behavior at infinity is given by*

$$\lim_{u_1\to\pm\infty} M(u_1) = 0,\ \lim_{u_1\to+\infty} N(u_1) = -1,\ \lim_{u_1\to-\infty} N(u_1) = 1,\ \textit{for } a \neq 0, \tag{4.35}$$

$$\lim_{u_1\to\pm\infty} M(u_1) = 0,\ \lim_{u_1\to+\infty} N(u_1) = \frac{b+2c}{b},\ \lim_{u_1\to-\infty} N(u_1) = -\frac{b+2c}{b},\quad \textit{for } a = 0. \tag{4.36}$$

$$\lim_{u_1\to+\infty} \widetilde{M}(u_1) = -\infty\cdot a,\ \lim_{u_1\to-\infty} \widetilde{M}(u_1) = +\infty\cdot a,\ \lim_{u_1\to\pm\infty} \widetilde{N}(u_1) = -\infty\cdot a,\quad \textit{for } a \neq 0, \tag{4.37}$$

$$\lim_{u_1\to\pm\infty} \widetilde{M}(u_1) = 0,\ \lim_{u_1\to+\infty} \widetilde{N}(u_1) = -b,\ \lim_{u_1\to-\infty} \widetilde{N}(u_1) = b,\quad \textit{for } a = 0. \tag{4.38}$$

Proof. Rewriting the curves (4.31) and (4.32) as

(4.39)
$$\begin{aligned} M(u_1) = &\big[2(a^2 - 2a^2u_1 - 2ab - 2ac)e^u - 2(2(au_1+b)^2 \\ &+ 4c(au_1+b) - a^2)e^{3u_1} + (a^2 + 2a^2u_1 + 2ab + 2ac)e^{5u_1}\big] \\ &\big[a^2 + (3a^2 + 4(au_1+b)^2)e^{2u_1} + (3a^2 + 4(au_1+b)^2)e^{4u_1} + a^2e^{6u_1}\big]^{-1}, \end{aligned}$$

$$N(u_1) = \frac{a^2 + a_1e^{2u_1} + a_2e^{4u_1} + a_3e^{6u_1}}{a^2 + a_4e^{2u_1} + a_4e^{4u_1} - a_3e^{6u_1}}, \tag{4.40}$$

where

$$\begin{aligned} a_1 &= a^2 - 4(au_1+b)(au_1+b+2c) + 8a(au_1+b+c), \\ a_2 &= -a^2 + 4(au_1+b)(au_1+b+2c) + 8a(au_1+b+c), \\ a_3 &= -a^2, \\ a_4 &= 3a^2 + 4(au_1+b)^2, \end{aligned}$$

and considering $a \neq 0$, we can get the limits in (4.35). On the other hand, if $a = 0$, with b nonzero, we have

$$M(u_1) = -2\left(\frac{b+2c}{b}\right)\frac{e^{u_1}}{1+e^{2u_1}}, \qquad N(u_1) = -\left(\frac{b+2c}{b}\right)\left(\frac{1-e^{2u_1}}{1+e^{2u_1}}\right), \tag{4.41}$$

and it follows that (4.36) is valid.

Considering now the equations (4.33) and (4.34) for $\widetilde{M}$ and $\widetilde{N}$, one has easily the limits (4.37) and (4.38). □

The next proposition indicates when a RSGW surface and an Appell surface of rotation are complete and shows that its singularities lie at circles or they are isolated.

Proposition 4.13. *Let Σ_1 be a connected RSGW surface of rotation and let Σ_2 be a connected Appell surface of rotation. Then*

(1) *The surfaces Σ_1 and Σ_2 are complete if and only if $a = 0$. In this case, Σ_1 is a sphere of radius $(b+2c)/b$ and Σ_2 is a sphere of radius b.*

(2) *For $a \neq 0$, the surfaces Σ_1 and Σ_2 has at least one circle of singularities. In addition to these, Σ_1 may present isolated singularities, while Σ_2 always have them.*

Proof. (1) Since Σ_1 and Σ_2 are surfaces of rotation, from the proof of Theorem 4.11 its holomorphic data can be taken as $f(z) = az + z_0$, $g(z) = e^z$, where z_0 is a complex constant. The function $h = \langle 1, f\rangle$ is equal to $au_1 + b$, where $z = u_1 + iu_2$, and a, b are real constants.

Thus, the regularity condition P for Σ_1, given by (4.19), may be written as

$$P(u_1) = \left(\frac{a^2(1+e^{2u_1})^2 - 4e^{2u_1}(au_1+b)(au_1+b+2c)}{4e^{2u_1}}\right)^2 - \left(\frac{a(au_1+b+c)(1-e^{4u_1})}{2e^{2u_1}}\right)^2.$$

We may factor the above expression as $P = A \cdot B$, where

$$A(u_1) = \frac{a^2(1+e^{2u_1})^2 - 4e^{2u_1}(au_1+b)(au_1+b+2c) - 2a(au_1+b+c)(e^{4u_1}-1)}{16e^{4u_1}},$$

$$B(u_1) = a^2(1+e^{2u_1})^2 - 4e^{2u_1}(au_1+b)(au_1+b+2c) + 2a(au_1+b+c)(e^{4u_1}-1).$$

In this way, we conclude that, for $a \neq 0$,

$$\lim_{u_1\to\pm\infty} A(u_1) = -\infty, \qquad \lim_{u_1\to\pm\infty} B(u_1) = +\infty. \tag{4.42}$$

Note that whatever a, b and c, with $a \neq 0$, is valid that

$$A\left(-\frac{b+c}{a}\right) = \frac{e^{4\frac{(b+c)}{a}}}{16}\left[a^2(1+e^{-2\frac{(b+c)}{a}})^2 + 4c^2e^{-2\frac{(b+c)}{a}}\right] > 0. \tag{4.43}$$

This fact, in addition with (4.42), says that A, and so P, has at least one root when $a \neq 0$. Therefore, in this case, the surface Σ_1 has at least one singularity.

Now, when $a = 0$, from the limits (4.36) we conclude that Σ_1 is cointained in a sphere of radius $(b+2c)/b$. In this case, the regularity condition P satisfies $P \equiv b^2(b+2c)^2$, which cannot be zero when $a = 0$. Therefore, this is a complete case.

On the other hand, with the holomorphic data above, the regularity condition $\widetilde{P}$ for Σ_2, given by (4.23), may be written as $\widetilde{P} = \widetilde{A}\cdot \widetilde{B}$, where

$$\widetilde{A}(u_1) = \frac{4(au_1+b)e^{2u_1} + a(1-e^{4u_1})}{4e^{2u_1}}, \qquad \widetilde{B}(u_1) = \frac{4(au_1+b)e^{2u_1} - a(1-e^{4u_1})}{4e^{2u_1}},$$

and, for $a \neq 0$,

$$\lim_{u_1\to+\infty} \widetilde{A}(u_1) = -\infty \cdot\ a, \qquad \lim_{u_1\to-\infty} \widetilde{A}(u_1) = +\infty \cdot\ a,$$

$$\lim_{u_1\to+\infty} \widetilde{B}(u_1) = +\infty \cdot\ a, \qquad \lim_{u_1\to-\infty} \widetilde{B}(u_1) = -\infty \cdot\ a$$

Thus, when $a \neq 0$, we conclude that both $\widetilde{A}$ and $\widetilde{B}$ have at least one root. Therefore the regularity condition $\widetilde{P}$ always presents roots in this case and the surface Σ_2 has at least one singularity.

Now, for $a = 0$, the limits (4.38) show that Σ_2 is cointained in a sphere of radius b. Moreover, in this case, the regularity condition $\widetilde{P}$ satisfies $\widetilde{P} \equiv b^2 \neq 0$, which shows that this is a complete case.

(2) Let $\bar{u_1}$ be a root of A. Then the set $X_{a,b}(\bar{u_1}, u_2)$, $u_2 \in \mathbb{R}$, is the circle

$$X_{a,b}(\bar{u_1}, u_2) = \left(\frac{8ae^{\bar{u_1}}(a\bar{u_1}+b+c)(e^{4\bar{u_1}}-1)(\cos u_2 + i\sin u_2)}{(1+e^{2\bar{u_1}})\big(a^2(1+e^{2\bar{u_1}})^2+4e^{2\bar{u_1}}(a\bar{u_1}+b)^2\big)},\right.$$
$$\left.\frac{(1+e^{2\bar{u_1}})(a\bar{u_1}+b+c)\big[8ae^{2\bar{u_1}}-2a(1-e^{2\bar{u_1}})^2\big]}{(1+e^{2\bar{u_1}})\big(a^2(1+e^{2\bar{u_1}})^2+4e^{2\bar{u_1}}(a\bar{u_1}+b)^2\big)}\right),$$

so that the roots of A correspond to a circle of singularities on the surface Σ_1. Considering now the expression for M in (4.31), we observe that if $\tilde{u_1}$ is a root of B, then $M(\tilde{u_1}) = 0$, so that such roots correspond to points where the generating curve of Σ_1 intercepts the Oz axis, which generates isolated singularities on the surface.

From the item (1) we know that A has at least one root for $a \neq 0$. Therefore, Σ_1 always present singularities at circles in this case. Futhermore, when B also presents roots, Σ_1 has isolated singularities as well.

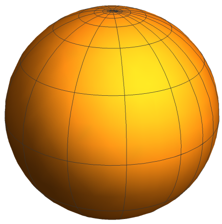
Now if $\bar{u_1}$ is a root of $\widetilde{A}$, then from the expression (4.33) for $\widetilde{M}$, we see that $\widetilde{M}(\bar{u_1}) = 0$ and such roots correspond to isolated singularities on the surface Σ_2. For roots $\tilde{u_1}$ of $\widetilde{B}$, the set $\widetilde{X}(\tilde{u_1}, u_2)$, $\quad u_2 \in \mathbb{R}$, is the circle

$$\widetilde{X}(\tilde{u_1}, u_2) = \Big(\frac{2a(1-e^{4\tilde{u_1}})}{2e^{\tilde{u_1}}}(\cos u_2 + i\sin u_2),\ (1-e^{2\tilde{u_1}})(a\tilde{u_1}+b) - a(1+e^{2\tilde{u_1}}\Big)$$

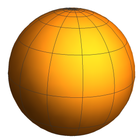
and such roots correspond to a circle of singularities on the surface Σ_2.

Since $\widetilde{A}$ and $\widetilde{B}$ always have roots when $a \neq 0$, we conclude that Σ_2 presents both singularities at circles and isolated singularities in this case. □

Example 4.14. Considering $a = 0, b = 1$ and $c = 2$, the RSGW surface Σ_1 is a sphere of radius 5 and the Appell surface Σ_2 is the unit sphere.

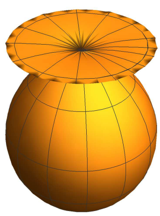


(a) RSGW surface Σ_1

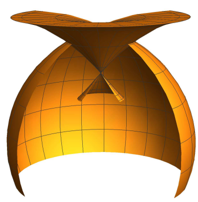


(b) Appell surface Σ_2

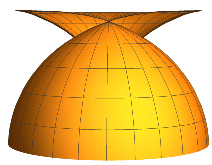
Example 4.15. For $a = 1, b = 1$ and $c = 1$, the RSGW surface Σ_1 has two circles of singularities and two isolated singularities. The Appell surface Σ_2 presents one circle of singularities and one isolated singularity.



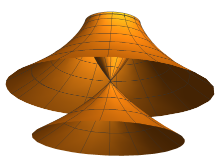
(a) RSGW surface Σ_1



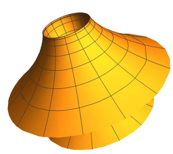
(b) RSGW surface Σ_1



(c) RSGW surface Σ_1



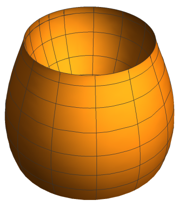
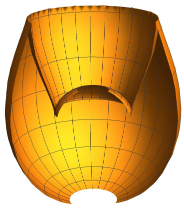
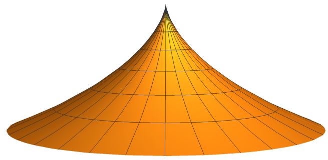
(d) Appell surface Σ_2



(e) Appell surface Σ_2

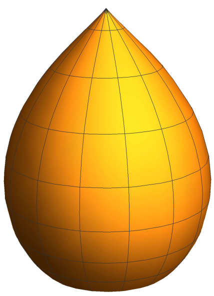
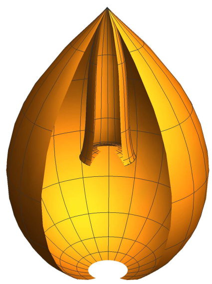
Example 4.16. For $a = 1, b = 0$ and $c = 1$, the RSGW surface Σ_1 has two circles of singularities and no isolated singularities. The Appell surface

Σ_2 presents only one isolated singularity (this is a case in which the circle of singularities degenerates at one point)

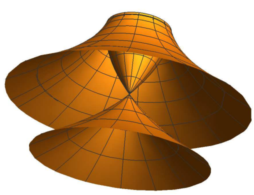




(h) RSGW surface Σ_1 (i) RSGW surface Σ_1 (j) Appell surface Σ_2

Example 4.17. When $a = 3, b = 1$ and $c = 4$, both RSGW surface Σ_1 and Appell surface Σ_2 have one circle of singularities and one isolated singularity.

(f) RSGW surface Σ_1 (g) RSGW surface Σ_1



(h) Appell surface Σ_2

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