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CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING HURWITZ-LERCH ZETA FUNCTION

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Abstract. Making use of Integral operator involving the Hurwitz-Lerch zeta function, we introduce a new subclass of analytic functions defined in the open unit disk and investigate its various characteristics. Further we obtain some usual properties of the geometric function theory such as coefficient bounds, extreme points radius of starlikness and convexity, partial sums and neighbourhood results belonging to the class.

1. Introduction. Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition

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$u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in U. A function $u \in A$ is a starlike function of the order $m, 0 \leq m < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > m, \quad z \in U. \tag{1.2}$$

We denote this class with $S^*(m)$.

A function $u \in A$ is a convex function of the order m, $0 \leq m < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > m, \quad z \in U. \tag{1.3}$$

We denote this class with $K(m)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U, respectively. For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in U). \tag{1.5}$$

Note that $f * g \in A$.

Let T denotes the class of functions analytic in U that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,\ z \in U \tag{1.6}$$

and let $T^*(m) = T \cap S^*(m)$, $C(m) = T \cap K(m)$. The class $T^*(m)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [18].

The study of operators plays an central role in the geometric function theory and its correlated fields. In the recent years, there has been an collective importance in problems concerning evaluations of various families of series linked with the Riemann zeta function and Hurwitz zeta function and their extensions and generalities such as the Hurwitz–Lerch zeta function. These functions ascend naturally in many branches of analytic function theory and their studies have plentiful important applications in mathematics and science and technology. Several interesting properties and characteristics of the Hurwitz–Lerch Zeta mapping can be found in the recent investigations by Challab et al. [4, 5], Choi [6], Ghanim et al. [7], Kim and Choi [10], Luo and Raina [12] and others see [15, 16, 2, 21, 25, 26, 27].

In [13] Mustafa and Darus have recently introduced a new generalized integral operator $\mathfrak{J}^{\alpha}_{\mu,b}u(z)$ as we show in the following:

Definition 1.1 ([22]). *A general Hurwitz- Lerch Zeta function* $\Phi(z,\mu,b)$ *defined by*

$$\Phi(z,\mu,b)=\sum_{n=0}^{\infty}\frac{z^n}{(n+b)^{\mu}},$$

where $(\mu\in\mathbb{C},\ b\in\mathbb{C}-\mathbb{Z}_0^-)$ *when* $(|z|<1)$*, and* $(\Re(b)>1)$ *when* $(|z|=1)$.

We define the function

$$\Phi^*(z,\mu,b)=(b^{\mu}z\Phi(z,\mu,b))*u(z),$$

then

$$\Phi^*(z,\mu,b)=z+\sum_{n=2}^{\infty}\frac{a_n}{(n+b-1)^{\mu}}z^n$$

Definition 1.2 ([23]). *Let the function* f *be analytic in a simply connected domain of the* z*-plane containg the orgin. The fractional derivative of* f *of order* α *is defined by*

$$D_z^{\alpha}u(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\alpha}}dt,\quad (0\le\alpha<1),$$

where the multiplycity of $(z-t)^{-\alpha}$ *is removed by requiring* $\log(z-t)$ *to be real when* $(z-t)>0$.

Using Definition 1.2 and and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [14] introduced the operator $\Omega^{\alpha}:A\to A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^{\alpha}u(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}u(z),\quad (\alpha\neq 2,3,4,\dots)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, \quad (z \in U)$$

For $\alpha \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-$, and $0 \leq \alpha < 1$, the generalized integral operator $\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} f : A \to A$, is defined by

$$\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z, \alpha, b), \quad (\alpha \neq 2, 3, 4, \dots)$$

$$(1.7) \qquad = z + \sum_{n=2}^{\infty} \Phi_n(\mu, b, \alpha)\ a_n z^n, \quad (z \in U)$$

$$(1.8) \qquad \text{where } \Phi_n(\mu, b, \alpha) = \frac{\Gamma(n+1) z^{\alpha} D_z^{\alpha} \Phi^*(z, \alpha, b)}{\Gamma(n+1-\alpha)} \left(\frac{b}{n-1+b} \right)^{\mu}.$$

Note that: $\mathfrak{J}_{\mathfrak{o},\mathfrak{b}}^{\mathfrak{o}} u(z) = u(z)$.
Special cases of this operator include :

(i). $\mathfrak{J}_{\mathfrak{o},\mathfrak{b}}^{\alpha} u(z) \equiv \Omega^{\alpha} u(z)$ is Owa and Srivastava operator [14].

(ii). $\mathfrak{J}_{\mu,\mathfrak{b}+\mathfrak{1}}^{\mathfrak{o}} u(z) \equiv J_{\mu,b} u(z)$ is the Srivastava and Attiya integral operator [24].

(iii). $\mathfrak{J}_{\mathfrak{1},\mathfrak{1}}^{\mathfrak{o}} u(z) \equiv A(f)(z)$ is the Alexander integral operator [1].

(iv). $\mathfrak{J}_{\mu+\mathfrak{1},\mathfrak{1}}^{\mathfrak{o}} u(z) \equiv L(f)(z)$ is the Libera integral operator [11].

(v). $\mathfrak{J}_{\mathfrak{1},\delta}^{\mathfrak{o}} u(z) \equiv L_{\delta}(f)(z)$ is the Bernardi integral operator [3].

(vi). $\mathfrak{J}_{\sigma,\mathfrak{2}}^{\mathfrak{o}} u(z) \equiv I^{\sigma} u(z)$ is the Jung-Kim-Kim-Srivastava integral operator [9].

Now, by making use of the Hurwitz–Lerch zeta operator $\mathfrak{J}_{\mu,b}^{\alpha} f$, we define a new subclass of functions motivated by the recent work of Thirupathi Reddy and Venkateswarlu [28] and Venkateswarlu et al. ([29], [30]).

Definition 1.3. *For $-1 \leq \upsilon < 1$, $0 \leq \tau < 1$ and $\varrho \geq 0$, we let $S(\tau, \upsilon, \varrho)$ be the subclass of A consisting of functions of the form* (1.1) *and satisfying the analytic criterion*

$$(1.9) \quad \Re\left\{ \frac{z(\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))' + \tau z^2 (\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))''}{(1-\tau)\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z) + \tau z (\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))'} - \upsilon \right\}$$

$$\geq \varrho \left| \frac{z(\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))' + \tau z^2 (\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))''}{(1-\tau)\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z) + \tau z (\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} u(z))'} - 1 \right|,$$

for $z \in U$.

The main object of the paper some usual properties of the geometric function theory such as coefficient bounds, extreme points, radii of starlikness and convexity, partial sums for the class and neighbourhood results for the class.

2. Coefficient bounds. In this sectin we obtain a necessary and sufficient condition for function $u(z)$ is in the classes $S(\tau, \upsilon, \varrho)$ and $TS(\tau, \upsilon, \varrho)$.

Theorem 2.1. *The function u defined by* (1.1) *is in the class* $S(\tau, \upsilon, \varrho)$ *if*

$$\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(1+\varrho)-(\upsilon+\varrho)]\Phi_n(\mu, b, \alpha)|a_\eta| \leq 1-\upsilon, \tag{2.1}$$

where $-1 \leq \upsilon < 1,\ 0 \leq \tau \leq 1,\ \varrho \geq 0$.

P r o o f. It suffices to show that

$$\varrho\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))' + \tau z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))''}{(1-\tau)\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z) + \tau z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))'} - 1\right| - \Re\left\{\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))' + \tau z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))''}{(1-\tau)\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z) + \tau z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))'} - 1\right\} \leq 1-\upsilon.$$

we have

$$\varrho\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))' + \tau z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))''}{(1-\tau)\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z) + \tau z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))'} - 1\right| - \Re\left\{\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))' + \tau z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))''}{(1-\tau)\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z) + \tau z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))'} - 1\right\}$$

$$\leq(1+\varrho)\left|\frac{z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))' + \tau z^2(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))''}{(1-\tau)\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z) + \tau z(\mathfrak{J}^{\alpha}_{\mu,\mathfrak{b}}u(z))'} - 1\right|$$

$$\leq \frac{(1+\varrho)\sum\limits_{\eta=2}^{\infty}(\eta-1)[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)|a_\eta|}{1-\sum\limits_{\eta=2}^{\infty}[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)|a_\eta|}.$$

This last expression is bounded above by $(1-\upsilon)$ by

$$\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(1+\varrho)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)|a_\eta| \leq 1-\upsilon,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for $u(z)$ of the form (1.6) to be in the class $TS(\tau,\upsilon,\varrho)$, $-1 \leq \upsilon < 1$, $0 \leq \tau \leq 1$, $\varrho \geq 0$ is that*

$$\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(1+\varrho)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)|a_\eta| \leq 1-\upsilon. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $u \in TS(\tau,\upsilon,\varrho)$ and z is real then

$$\frac{1-\sum\limits_{\eta=2}^{\infty}\eta[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)a_\eta z^{\eta-1}}-\upsilon$$

$$\geq \varrho\left|\frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)|a_\eta|}{1-\sum\limits_{\eta=2}^{\infty}[1+\tau(\eta-1)]\Phi_n(\mu,b,\alpha)|a_\eta|}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(1+\varrho)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)|a_\eta| \leq 1-\upsilon. \qquad \square$$

Theorem 2.3. *Let $u(z)$ defined by (1.6) and $g(z) = z - \sum_{\eta=2}^{\infty} b_\eta z^\eta$ be in the class $TS(\tau, \upsilon, \varrho)$. Then the function $h(z)$ defined by*

$$h(z) = (1-\zeta)u(z) + \zeta g(z) = z - \sum_{\eta=2}^{\infty} c_\eta z^\eta,$$

where $c_\eta = (1-\zeta)a_\eta + \zeta b_\eta$, $0 \le \zeta < 1$ is also in the class $TS(\tau, \upsilon, \varrho)$.

Proof. Let the function

$$u_j = z - \sum_{\eta=2}^{\infty} a_{\eta,j} z^\eta, \quad a_{\eta,j} \ge 0, \quad j = 1, 2, \tag{2.3}$$

be in the class $TS(\tau, \upsilon, \varrho)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \zeta u_1(z) + (1-\zeta)u_2(z), \quad 0 \le \zeta \le 1,$$

is in the class $TS(\tau, \upsilon, \varrho)$. Since

$$g(z) = z - \sum_{\eta=2}^{\infty} [\zeta a_{\eta,1} + (1-\zeta)a_{\eta,2}] z^\eta,$$

an easy computation with the aid of Theorem 2.2 gives,

$$\sum_{\eta=2}^{\infty} [1+\tau(\eta-1)][\eta(\varrho+1) - (\upsilon+\varrho)]\Phi_n(\mu, b, \alpha)\zeta a_{\eta,1}$$

$$+ \sum_{\eta=2}^{\infty} [1+\tau(\eta-1)][\eta(\varrho+1) - (\upsilon+\varrho)]\Phi_n(\mu, b, \alpha)(1-\zeta) a_{\eta,2}$$

$$\le \zeta(1-\upsilon) + (1-\zeta)(1-\upsilon)$$

$$\le 1-\upsilon,$$

which implies that $g \in TS(\tau, \upsilon, \varrho)$.

Hence $TS(\tau, \upsilon, \varrho)$ is convex. □

3. Extreme points. The proof of Theorem 3.1, follows on lines similar to the proof of the theorem on extreme points given in Silverman [28].

Theorem 3.1. *Let $u_1(z) = z$ and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)} z^\eta, \tag{3.1}$$

for $\eta = 2, 3, \ldots$. Then $u(z) \in TS(\tau,\upsilon,\varrho)$ if and only if $u(z)$ can be expressed in the form $u(z) = \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z)$, where $\zeta_\eta \geq 0$ and $\sum_{\eta=1}^{\infty} \zeta_\eta = 1$.

Next we prove the following closure theorem.

4. Closure theorem.

Theorem 4.1. *Let the function $u_j(z)$, $j = 1, 2, \ldots, l$ defined by (2.3) be in the classes $TS(\tau,\upsilon_j,\varrho,\mu,s)$, $j = 1, 2, \ldots, l$, respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{l}\sum_{\eta=2}^{\infty}\left(\sum_{j=1}^{l} a_{\eta,j}\right) z^\eta$$

is in the class $TS(\tau,\upsilon,\varrho)$, where $\upsilon = \min_{1\leq j\leq l}\{\upsilon_j\}$, where $-1 \leq \upsilon_j \leq 1$.

Proof. Since $u_j(z) \in TS(\tau,\upsilon_j,\varrho,\mu,s)$, $j = 1, 2, \cdots, l$ by applying Theorem 2.2 to (2.3), we observe that

$$\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)\left(\frac{1}{l}\sum_{j=1}^{l} a_{\eta,j}\right)$$

$$=\frac{1}{l}\sum_{j=1}^{l} a_{\eta,j}\left(\sum_{\eta=2}^{\infty}[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)a_{\eta,j}\right)$$

$$\leq\frac{1}{l}\sum_{j=1}^{l}(1-\upsilon_j)$$

$$\leq 1-\upsilon$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\tau, \upsilon, \varrho)$ and so the proof is complete. □

Theorem 4.2. *Let $u \in TS(\tau, \upsilon, \varrho)$. Then*

(1). *u is starlike of order δ, $0 \le \delta < 1$, in the disc $|z| < r_1$*

i.e., $\Re\left\{\dfrac{zu'(z)}{u(z)}\right\} > \delta$, $|z| < r_1$, *where*

$$r_1 = \inf_{\eta \ge 2}\left\{\left(\frac{1-\delta}{\eta-\delta}\right)\frac{[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{1-\upsilon}\right\}^{\frac{1}{\eta-1}}.$$

(2). *u is convex of order δ, $0 \le \delta < 1$, in the disc $|z| < r_1$*

i.e., $\Re\left\{1+\dfrac{zu''(z)}{u'(z)}\right\} > \delta$, $|z| < r_2$, *where*

$$r_2 = \inf_{\eta \ge 2}\left\{\left(\frac{1-\delta}{\eta-\delta}\right)\frac{[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{1-\upsilon}\right\}^{\frac{1}{\eta}}.$$

Each of these results are sharp for the extremal function $u(z)$ given by (3.1).

Proof. (1). Given $u \in A$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| < 1 - \delta. \tag{4.1}$$

For the left hand side (4.1), we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \le \frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)a_\eta|z|^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}a_\eta|z|^{\eta-1}}.$$

The last expression is less that $1-\delta$ if

$$\sum_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}a_\eta|z|^{\eta-1} < 1.$$

Using the fact, that $u \in TS(\tau, \upsilon, \varrho)$ if and only if

$$\sum_{\eta=2}^{\infty} \frac{[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{1-\upsilon} a_\eta < 1.$$

We can say (4.1) is true if

$$\frac{\eta-\delta}{1-\delta}|z|^{\eta-1} < \frac{[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{1-\upsilon}$$

Or equivalently,

$$|z|^{\eta-1} < \frac{(1-\delta)[1+\tau(\eta-1)][\eta(\varrho+1)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{(\eta-\delta)(1-\upsilon)}$$

which yields the starlikeness of the family.

(2). Using the fact that u is convex if and only if zu' is starlike, we can prove (2), on lines similar to the proof of (1). □

5. Partial sums. Following the earlier works by Silverman [19] and Silvia [20] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $S(\tau, \upsilon, \varrho)$ and obtain sharp lower bounds for the ratios of real part of $u(z)$ to $u_q(z)$ and $u'(z)$ to $u'_q(z)$.

Theorem 5.1. *Let $u(z) \in S(\tau, \upsilon, \varrho)$ be given by (1.1) and define the partial sums $u_1(z)$ and $u_q(z)$ by*

(5.1) $$u_1(z) = z \textit{ and } u_q(z) = z + \sum_{\eta=2}^{q} a_\eta z^\eta, \quad (q \in \mathbb{N} \setminus \{1\}).$$

Suppose that $\sum_{\eta=2}^{\infty} d_\eta |a_\eta| \leq 1$,

(5.2) $$\text{where } d_\eta = \frac{[1+\tau(\eta-1)][\eta(1+\varrho)-(\upsilon+\varrho)]\Phi_n(\mu,b,\alpha)}{1-\upsilon}$$

Then $u \in S(\tau, \upsilon, \varrho)$.

(5.3) $$\text{Further more, } \Re\left[\frac{u(z)}{u_q(z)}\right] > 1 - \frac{1}{d_{q+1}}, \; z \in E, \; q \in \mathbb{N}$$

$$\text{and } \Re\left[\frac{u_q(z)}{u(z)}\right] > \frac{d_{q+1}}{1+d_{q+1}}. \tag{5.4}$$

P r o o f. For the coefficients d_η given by (5.2) it is not difficult to verify that

$$d_{\eta+1} > d_\eta > 1. \tag{5.5}$$

$$\text{Therefore we have } \sum_{\eta=2}^{q}|a_\eta| + d_{q+1}\sum_{\eta=q+1}^{\infty}|a_\eta| \le \sum_{\eta=2}^{\infty} d_\eta|a_\eta| \le 1 \tag{5.6}$$

by using the hypothesis (5.2). By setting

$$g_1(z) = d_{q+1}\left[\frac{u(z)}{u_q(z)} - \left(1 - \frac{1}{d_{q+1}}\right)\right]$$

$$= 1 + \frac{d_{q+1}\sum\limits_{\eta=q+1}^{\infty} a_\eta z^{\eta-1}}{1+\sum\limits_{\eta=2}^{q} a_\eta z^{\eta-1}} \tag{5.7}$$

and applying (5.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{d_{q+1}\sum\limits_{\eta=q+1}^{\infty}|a_\eta|}{2-2\sum\limits_{\eta=2}^{q}|a_\eta| - d_{q+1}\sum\limits_{\eta=q+1}^{\infty}|a_\eta|} \le 1 \tag{5.8}$$

which readily yields the assertion (5.3) of Theorem 5.1. In order to see that

(5.9) $\quad u(z) = z + \dfrac{z^{q+1}}{d_{q+1}}$ gives sharp result, we observe that for $z = re^{\frac{i\pi}{q}}$ that

$$\frac{u(z)}{u_q(z)} = 1 + \frac{z^q}{d_{q+1}} \;\to\; 1 - \frac{1}{d_{q+1}} \text{ as } z \to 1^-.$$

Similarly, if we take

$$g_2(z) = (1+d_{q+1})\left(\frac{u_q(z)}{u(z)} - \frac{d_{q+1}}{1+d_{q+1}}\right)$$

$$= 1 - \frac{(1 + d_{\eta+1}) \sum_{\eta=q+1}^{\infty} a_\eta z^{\eta-1}}{1 + \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1}} \tag{5.10}$$

and making use of (5.6), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{q+1}) \sum_{\eta=q+1}^{\infty} |a_\eta|}{2 - 2 \sum_{\eta=2}^{q} |a_\eta| - (1 - d_{q+1}) \sum_{\eta=q+1}^{\infty} |a_\eta|}$$

which leads is immediately to the assertion (5.4) of Theorem 5.1.

The bound in (5.4) is sharp for each $q \in \mathbb{N}$ with the external function $u(z)$ given by (5.9). Thus the proof of the Theorem 5.1 is complete. □

Theorem 5.2. *If* $u(z)$ *of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re \left[\frac{u'(z)}{u'_q(z)} \right] \geq 1 - \frac{q+1}{d_{q+1}}. \tag{5.11}$$

Proof. By setting

$$g(z) = d_{q+1} \left[\frac{u'(z)}{u'_q(z)} \right] - \left(1 - \frac{q+1}{d_{q+1}} \right)$$

$$= \frac{1 + \frac{d_{q+1}}{q+1} \sum_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1} + \sum_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}$$

$$= 1 + \frac{\frac{d_{q+1}}{q+1} \sum_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}$$

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\frac{d_{q+1}}{q+1} \sum_{\eta=q+1}^{\infty} \eta |a_\eta|}{2 - 2 \sum_{\eta=2}^{q} \eta |a_\eta| - \frac{d_{q+1}}{q+1} \sum_{\eta=q+1}^{\infty} \eta |a_\eta|}. \tag{5.12}$$

$$\text{Now } \left|\frac{g(z)-1}{g(z)+1}\right| \leq 1 \text{ if } \sum_{\eta=2}^{q} \eta|a_\eta| + \frac{d_{q+1}}{q+1} \sum_{\eta=q+1}^{\infty} \eta|a_\eta| \leq 1. \tag{5.13}$$

Since the left hand side of(5.13)is bounded above by $\sum_{\eta=2}^{q} d_\eta|a_\eta|$ if

$$\sum_{\eta=2}^{q}(d_\eta-\eta)|a_\eta| + \sum_{\eta=q+1}^{\infty} d_\eta - \frac{d_{q+1}}{q+1}\eta|a_\eta| \geq 0. \tag{5.14}$$

and the proof is complete.

The result is sharp for the extremal function $u(z) = z + \dfrac{z^{q+1}}{d_{q+1}}$. □

Theorem 5.3. *If* $u(z)$ *of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re\left[\frac{u_q'(z)}{u'(z)}\right] \geq \frac{d_{q+1}}{q+1+d_{q+1}}. \tag{5.15}$$

Proof. By setting

$$g(z) = [q+1+d_{q+1}]\left[\frac{u_q'(z)}{u'(z)} - \frac{d_{q+1}}{q+1+d_{q+1}}\right]$$

$$= 1 - \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1}}{1+\sum\limits_{\eta=2}^{q} \eta a_\eta z^{\eta-1}}$$

and making use of (5.14), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|}{2-2\sum\limits_{\eta=2}^{q} \eta|a_\eta| - \left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|} \leq 1$$

which leads us immediately to the assertion of the Theorem 5.3. □

6. Neighbouhood for the class $S^{\xi}(\tau, \upsilon, \varrho)$. In this section, we determine the neighbourhoods for the class $S^{\xi}(\tau, \upsilon, \varrho)$ which we define as follows:

Definition 6.1. *A function $u \in A$ is said to be in the class $S^{\xi}(\tau, \upsilon, \varrho)$ if there exist a function $g \in S(\tau, \upsilon, \varrho)$ such that*

$$\left|\frac{u(z)}{g(z)} - 1\right| < 1 - \upsilon, \ \ (z \in U, 0 \leq \upsilon < 1). \tag{6.1}$$

For any function $u(z) \in A, z \in U$ and $\delta \geq 0$, we define

$$N_{\eta,\delta}(u) = \left\{ g \in \Sigma : g(z) = z + \sum_{\eta=2}^{\infty} b_\eta z^\eta \text{ and } \sum_{\eta=2}^{\infty} \eta |a_\eta - b_\eta| \leq \delta \right\} \tag{6.2}$$

which is the (η, δ)-neighbourhood of $u(z)$.

The concept of neighbourhoods was first introduced by Goodman [8] and generalized by Ruscheweyh [17].

Theorem 6.2. *If $g \in S(\tau, \upsilon, \varrho)$ and*

$$\xi = 1 - \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)]} \tag{6.3}$$

then $N_{\eta,\delta}(g) \subset S^{\xi}(\tau, \upsilon, \varrho)$.

Proof. Suppose $u \in N_{\eta,\delta}(g)$. We then find from (6.2) that

$$\sum_{\eta=2}^{\infty} n |a_\eta - b_\eta| \leq \delta \tag{6.4}$$

which yields the coefficient inequality

$$\sum_{\eta=2}^{\infty} |a_\eta - b_\eta| \leq \frac{\delta}{2} \ (\eta \in \mathbb{N}). \tag{6.5}$$

Next, since $g \in S(\tau, \upsilon, \varrho)$, we have

$$\sum_{\eta=2}^{\infty} b_\eta \leq \frac{(1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)}{1-\upsilon}. \tag{6.6}$$

So that

$$\left|\frac{u(z)}{g(z)} - 1\right| < \frac{\sum\limits_{\eta=2}^{\infty} |a_\eta - b_\eta|}{1 - \sum\limits_{\eta=2}^{\infty} b_\eta}$$

$$= \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)]}$$

$$= 1 - \xi$$

provided ξ is given by (6.3). Thus the proof of the is completed. □

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