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CERTAIN SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY *q*-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we define certain subclass of harmonic univalent function in the unit disc $U = \{z \in C : |z| < 1\}$ by using q- differential operator. Also we obtain coefficient inequalities, growth and distortion theorems for this subclass.

1. Introduction. Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. In fact, by introducing new subclasses, Silverman [11], Silverman and Silvia [12], Jahangiri [3], Sangle and Yadav [9], Dixit and Porwal [2], Singh and Porwal [13] and Ravindar et al. [14] etc. presented a systematic and unified study of harmonic univalent functions.

The concepts of q-calculus has many applications in subfields of science, some of them are q-difference equations and geometric function theory. Motivated

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by the research work done by Jahangiri [3, 4], Joshi and Sangle [6, 5], Purohit et al. [7], we define some subclasses of harmonic mappings using the Salagean q-differential operator.

Also, we determine extreme points and coefficient estimates of $S_H^q(m, \alpha, u)$ and $\overline{S}_H^q(m, \alpha, u)$.

Let A be family of analytic functions in unit disk U and A^0 be the class of all normalized analytic functions. For $0 < q < 1$ and for positive integer u, the q-integer number is denoted by $[u]_q$ and also it is written as

$$[u]_q = \frac{1-q^u}{1-q} = \sum_{k\geq 0}^{u-1} q^k. \tag{1.1}$$

By making use of differential calculus, we can check that $\lim_{q\to 1^-} [u]_q = u$. For $h \in A$, the q-difference operator [7] is specified as

$$\partial_q h(z) = \frac{h(z) - h(qz)}{(1-q)\, z} \tag{1.2}$$

where $\lim_{q\to 1^-} \partial_q h(z) = h'(z)$.
Let the functions $h \in A$ be of the form

$$h(z) = z + \sum_{u\geq 2}^{\infty} a_u\, z^u. \tag{1.3}$$

J. M. Jahangiri [4] defined the Salagean q-differential operator for the above functions h as

$$D_q^0 h(z) = h(z)$$

$$D_q^1 h(z) = z\partial_q h(z) = \frac{h(z) - h(qz)}{(1-q)\, z}, \ldots$$

$$D_q^m h(z) = z\partial_q D_q^{m-1} h(z) = h(z) * \left(z + \sum_{u\geq 2}^{\infty} [u]_q^m\ z^u \right) = z + \sum_{u\geq 2}^{\infty} [u]_q^m\ a_u\, z^u, \tag{1.4}$$

where m is a positive integer. The operator D_q^m is called Salagean q-differential operator.

The complex-valued harmonic functions can be written as $f = h + \overline{g}$ in where h and g have the following power series expansions

$$h(z) = z + \sum_{u\geq 2}^{\infty} a_u\, z^u, \quad g(z) = \sum_{u\geq 1}^{\infty} b_u\, z^u, |b_1| < 1. \tag{1.5}$$

Clunie and Sheil-Small [1] defined the function of form $f = h+\overline{g}$ that are locally univalent, sense-preserving and harmonic in U. A sufficient condition for the harmonic functions f to be univalent in U is that $\left|h'(z)\right| \geq \left|g'(z)\right|$ in U.

J. M. Jahangiri [4] defined the Salagean q-differential operator for the harmonic functions f by

$$D_q^m f(z) = D_q^m h(z) + (-1)^m \overline{D_q^m g(z)} \tag{1.6}$$

where D_q^m is defined by (1.4).

Now, for $0 \leq \alpha < 1, m \in N_0$ and $z \in U$, suppose that $S_H^q(m,\alpha,u)$ denote the family of harmonic univalent function f of the form $f = h + \overline{g}$ such that

$$Re\left\{\frac{D_q^m h(z) + D_q^m g(z)}{z}\right\} > \alpha \tag{1.7}$$

where $D_q^m f(z)$ is defined by J. M. Jahangiri [4].

Further, let the subclass $\overline{S}_H^q(m,\alpha,u)$ consisting harmonic functions $f = h+\overline{g}$ in $\overline{S}_H^q(m,\alpha,u)$ so that h and g are of the form

$$h(z) = z - \sum_{u\geq 2}^{\infty} |a_u| z^u \quad \text{and} \quad g(z) = \sum_{u\geq 1}^{\infty} |b_u| z^u. \tag{1.8}$$

2. Main results.

Theorem 2.1. *Let the function $f = h+\overline{g}$ be such that h and g are given by* (1.5). *Furthermore*

$$\sum_{u\geq 2}^{\infty} [u]_q^m \ |a_u| \ + \sum_{u\geq 1}^{\infty} [u]_q^m \ |b_u| \ \leq (1-\alpha)\,, \tag{2.1}$$

where $0 \leq \alpha < 1$ and $m \in N_0$. Then f is harmonic univalent, sense-preserving in U and $f \in S_H^q(m,\alpha,u)$.

Proof. If $z_1 \neq z_2$ then,

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1 - \left|\frac{\sum\limits_{u\geq 1}^{\infty} b_u({z_1}^u - {z_2}^u)}{z_1 - z_2 + \sum\limits_{k\geq 2}^{\infty} a_u({z_1}^u - {z_2}^u)}\right|\end{aligned}$$

$$
\begin{aligned}
&> 1-\frac{\sum\limits_{u\geq 1}^{\infty} u\,|b_u|}{1-\sum\limits_{u\geq 2}^{\infty} u\,|a_u|}\\
&\geq 1-\frac{\sum\limits_{u\geq 1}^{\infty} \frac{[u]_q^m}{1-\alpha}\ |b_u|}{1-\sum\limits_{u\geq 2}^{\infty} \frac{[u]_q^m}{1-\alpha}\ |a_u|}\\
&\geq 0.
\end{aligned}
$$

Hence f is univalent in U. f is sense-preserving in U. This is because

$$
\begin{aligned}
\left|h'(z)\right| &\geq 1-\sum_{u\geq 2}^{\infty} u\ |a_u||z|^{u-1}\\
&> 1-\sum_{u\geq 2}^{\infty} u\ |a_u|\\
&\geq 1-\sum_{u\geq 2}^{\infty} \frac{[u]_q^m}{1-\alpha}\ |a_u|\\
&\geq \sum_{u\geq 1}^{\infty} \frac{[u]_q^m}{1-\alpha}\ |b_u|\\
&\geq \sum_{u\geq 1}^{\infty} u\ \ |b_u||z|^{u-1}\\
&\geq \left|g'(z)\right|.
\end{aligned}
$$

Now, we show that $f\in S_H^q(m,\alpha,u)$.Using the fact that $\mathrm{Re}\,(w)>\alpha$ if and only if $|1-\alpha+w|>|1+\alpha-w|$. it suffices to show that

$$
\left|(1-\alpha)+\frac{D_q^m h(z)+D_q^m g(z)}{z}\right|-\left|(1+\alpha)-\frac{D_q^m h(z)+D_q^m g(z)}{z}\right|>0 \tag{2.2}
$$

Substituting for $D_q^m h(z)$ and $D_q^m g(z)$ in (2.2), we obtain

$$
\begin{aligned}
=&\left|(2-\alpha)+\sum_{u\geq 2}^{\infty} [u]_q^m\ a_u\, z^{u-1}+\sum_{u\geq 1}^{\infty} [u]_q^m\ b_u\, z^{u-1}\right|\\
&-\left|\alpha-\sum_{u\geq 2}^{\infty} [u]_q^m\ a_u\, z^{u-1}-\sum_{u\geq 1}^{\infty} [u]_q^m\ b_u\, z^{u-1}\right|
\end{aligned}
$$

$$\geq 2(1-\alpha)\left\{1-\sum_{u\geq 2}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|a_u|\,|z|^{u-1}-\sum_{u\geq 1}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|b_u|\,|z|^{u-1}\right\}$$

$$> 2(1-\alpha)\left\{1-\sum_{u\geq 2}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|a_u|-\sum_{u\geq 1}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|b_u|\right\}$$

The harmonic mappings

$$f(z)=z+\sum_{u\geq 2}^{\infty}\frac{1-\alpha}{[u]_q^m}\,x_u\,z^u+\sum_{u\geq 1}^{\infty}\frac{1-\alpha}{[u]_q^m}\,\overline{y_u\,z^u},$$

where $\sum_{u\geq 2}^{\infty}|x_u|+\sum_{u\geq 1}^{\infty}|y_u|=1$, show that coefficient bound given by (2.1) is sharp. □

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f=h+\overline{g}$ where h and g are of the form (1.8).

Theorem 2.2. *Let $f=h+\overline{g}$ be given by (1.8). Then $f\in\overline{S}_H^q(m,\alpha,u)$ if and only if*

$$\sum_{u\geq 2}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|a_u|+\sum_{u\geq 1}^{\infty}\frac{[u]_q^m}{1-\alpha}\;|b_u|\leq 1 \tag{2.3}$$

where $0\leq\alpha<1$ and $m\in N_0$.

P r o o f. The if part follows from Theorem 2.1. For the only if part, we show that $f\in\overline{S}_H^q(m,\alpha,u)$ if the condition (2.3) holds. We notice that the condition

$$\mathrm{Re}\left\{\frac{D_q^m\,h(z)+D_q^m\,g(z)}{z}\right\}>\alpha$$

is equivalent to

$$\mathrm{Re}\left\{1-\sum_{u\geq 2}^{\infty}[u]_q^m\;|a_u|\,|z|^{u-1}-\sum_{u\geq 1}^{\infty}[u]_q^m\;|b_u|\,|z|^{u-1}\right\}>\alpha.$$

The above required condition must hold for all values of z in U. Taking the values of z on the positive real axis, where $0\leq|z|=r<1$, we must have

$$1-\sum_{u\geq 2}^{\infty}[u]_q^m\;|a_u|\;-\sum_{u\geq 1}^{\infty}[u]_q^m\;|b_u|\;\geq\alpha$$

which is precisely the assertion (2.3). □

Next, we determine the extreme points of closed convex hulls of class $\overline{S}_H^q(m,\alpha,u)$.

Theorem 2.3. *Let f be given by* (1.8). *Then $\overline{S}_H^q(m,\alpha,u)$ if and only if*

$$f(z)=\sum_{u\geq 1}^{\infty}(x_u\,h_u(z)+y_u\,g_u(z)),$$

where $h_1(z)=z$,

$$h_k(z)=z-\frac{1-\alpha}{[u]_q^m}\,z^u\,,(u=\ 2,3,4,\dots)\ ,g_k(z)=z-\frac{1-\alpha}{[u]_q^m}\,\overline{z}^u\,,(u=\ 1,2,3,4,\dots)\,,$$

$x_u\geq 0\,,y_u\geq 0\,,\sum_{u=1}^{\infty}x_u{+}y_u=1$. *In particular the extreme points of $\overline{S}_H^q(m,\alpha)$ are $\{h_u\}$ and $\{g_u\}$.*

The following theorem gives the bounds for functions in $\overline{S}_H^q(m,\alpha,u)$ which yields a covering result for this class.

Theorem 2.4. *Let $f\in\overline{S}_H^q(m,\alpha,u)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_1|)\,r+\frac{1}{2^n}\,(1-|b_1|-\alpha)\,r^2,\quad |z|=r<1$$

and

$$|f(z)|\geq(1-|b_1|)\,r-\frac{1}{2^n}\,(1-|b_1|-\alpha)\,r^2,\quad |z|=r<1.$$

Proof. Let $f\in\overline{S}_H^q(m,\alpha,u)$.Taking the absolute value of $f(z)$, we have

$$\begin{aligned}|f(z)|&\leq(1+|b_1|)r+\sum_{u\geq 2}^{\infty}(|a_u|+|b_u|)r^u\\&\leq(1+|b_1|)\,r+\sum_{u\geq 2}^{\infty}(|a_u|+|b_u|)r^2\\&\leq(1+|b_1|)\,r+\frac{1}{[2]_q^m}\sum_{u\geq 2}^{\infty}[u]_q^m\,(|a_u|+|b_u|)r^2\\&\leq(1+|b_1|)\,r+\frac{1}{[2]_q^m}\,(1-\alpha-|b_1|)\,r^2\end{aligned}$$

and

$$|f(z)|\geq(1-|b_1|)r-\sum_{u\geq 2}^{\infty}(|a_u|+|b_u|)r^u$$

$$\geq (1-|b_1|)\,r - \sum_{u\geq 2}^{\infty}(|a_u|+|b_u|)r^2$$

$$\geq (1-|b_1|)\,r - \frac{1}{[2]_q^m}\sum_{u\geq 2}^{\infty}[u]_q^m\,(|a_u|+|b_u|)r^2$$

$$\geq (1-|b_1|)\,r - \frac{1}{[2]_q^m}\,(1-\alpha-|b_1|)\,r^2$$

The functions $z+|b_1|\,\overline{z}+\dfrac{1}{[2]_q^m}\,(1-\alpha-|b_1|)\,\overline{z}^2$ and $z-|b_1|\,z-\dfrac{1}{[2]_q^m}\,(1-\alpha-|b_1|)\,z^2$ for $|b_1|\leq(1-\alpha)$. □

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