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TOPICS ON REAL AND COMPLEX CONVEXITY

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Abstract. We investigate the study of convex, strictly plurisubharmonic and the special class consisted of convex and strictly plurisubharmonic functions in convex domains of $\mathbb{C}^n$, $n \geq 1$.

Let $h : \mathbb{C}^n \to \mathbb{C}$ be pluriharmonic. We prove that $\{b \in \mathbb{C} \ /|h + b|$ is a convex function on $\mathbb{C}^n\} = \emptyset$, or $\{\alpha\}$, or $\mathbb{C}$, where $\alpha \in \mathbb{C}$.

Now let $\varphi_1, \varphi_2, \varphi_3 : D \to \mathbb{C}$ be three holomorphic functions, D is a domain of $\mathbb{C}^n$. Put $u(z,w) = |w - \overline{\varphi_1}(z)||w - \overline{\varphi_2}(z)||w - \overline{\varphi_3}(z)|$, for $(z,w) \in D \times \mathbb{C}$. We prove that u is psh on $D \times \mathbb{C}$ if and only if $(\varphi_1 + \varphi_2 + \varphi_3)$ and $(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3)$ are constant on D, or $(\varphi_1 + \varphi_2 + \varphi_3)$ is non constant and $\varphi_1 = \varphi_2 = \varphi_3$ on D.

1. Introduction. Let $n, N \in \mathbb{N}\backslash\{0\}$. An original question of complex analysis is to characterize all the pluriharmonic functions $K_1, \dots, K_N : \mathbb{C}^n \to \mathbb{C}$ such that $(|K_1|^2 + \cdots + |K_N|^2)$ is convex on $\mathbb{C}^n$, $N \geq 1$.

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In Section 2, we study a special family of plurisubharmonic (psh) functions which is invariant by a class of functions. We conclude this section by Theorem 2.7. Note that we begin this section by a technical remark on strictly psh functions.

We consider the application of the following technical result (see[4]). Let $f_1, \dots, f_N,\ g_1, \dots, g_N : D \to \mathbb{C}$ be $2N$ holomorphic functions, D is a domain of $\mathbb{C}^n$ and $(a_1, b_1, \dots, a_N, b_N) \in \mathbb{C}^{2N}$. Let $\varphi_1 = \sum_{j=1}^{N} |g_j - \overline{f_j}|^2$, $\varphi_2 = \sum_{j=1}^{N} (|f_j + a_j|^2 + |g_j + b_j|^2)$. Then φ_1 and φ_2 have the same hermitian Levi form on D. Moreover, φ_1 is strictly psh on D if and only if φ_2 is strictly psh on D.

In Section 3, we discuss technical properties between pluriharmonic and strictly psh functions.

In Section 4, we consider the following problem. Let $A_1, \dots, A_N \in \mathbb{C}\backslash\{0\}$, $B_1, \dots, B_N \in \mathbb{C}$ and $g_1, \dots, g_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions, $N \geq 2$. Define $u(z, w) = |(A_1 w + B_1) - \overline{g_1}(z)| \cdots |(A_N w + B_N) - \overline{g_N}(z)|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Find conditions $N, g_1, \dots, g_N$ should satisfy such that u is plurisubharmonic (psh) on $\mathbb{C}^n \times \mathbb{C}$.

Some technical results between strictly convex functions and the hermitian product are proved. Moreover, at the end we obtain a number of technical theorems using absolute values, holomorphic functions and analytic polynomials.

The aim of Section 5 is to extend some results concerning psh and strictly psh functions discussed in section 4.

In section 6 we answer to the following Question. Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Put $\psi(z, w) = |g_1(w + \overline{z}) + \overline{g_2}(w + \overline{z})|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Find exactly all the analytic expressions of g_1 and g_2 such that v is psh (or strictly psh) on $\mathbb{C}^n \times \mathbb{C}^n$. Moreover, the rest of the Section 6 is the study of the following questions. Now Let $h_1, h_2 : \mathbb{C}^n \to \mathbb{C}$ be two pluriharmonic functions. Define $\psi_2(z, w) = |h_1(w + \overline{z}) + h_2(w + \overline{z})|$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Characterize h_1, h_2 by their analytic expressions such that ψ_2 is psh on $\mathbb{C}^n$. Consider $f, g : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Define $\psi_3(z, w) = |w^2 + \overline{f}(z) w + \overline{g}(z)|$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Assume that $|f|$ is convex on $\mathbb{C}^n$. Find f and g by their expressions such that ψ_3 is psh on $\mathbb{C}^n \times \mathbb{C}$.

Deduce all the expressions of the holomorphic functions $f, g, h, k : \mathbb{C}^n \to \mathbb{C}$ such that $|f|$, $|h|$ are convex functions and ψ_3, ψ_4, $(\psi_3 + \psi_4)$ are psh on $\mathbb{C}^n \times \mathbb{C}$. Where $\psi_4(z, w) = |w^2 + \overline{h}(z) w + \overline{k}(z)|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Put $\psi_5(z, w) = |A_1 w - h_1(z)|^2 + |A_2 w - h_2(z)|^2$, for $h_1, h_2 : \mathbb{C}^n \to \mathbb{C}$ be two pluriharmonic functions, $A_1, A_2 \in \mathbb{C}\backslash\{0\}$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Characterize all the representations of h_1, h_2 such that ψ_5 is convex on $\mathbb{C}^n \times \mathbb{C}$.

Although we deduce all the representations of the four pluriharmonic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $h_1, h_2 : \mathbb{C}^n \to \mathbb{C}$ such that ψ_6 is convex on $\mathbb{C}^n \times \mathbb{C}$. $\psi_6(z,w) = |\varphi_1(w) - h_1(z)|^2 + |\varphi_2(w) - h_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$, $n, m \geq 1$.

Consequently, now it is possible to characterize n, m and all the functions ψ_6, ψ_7 such that ψ_6 and ψ_7 are convex functions on $\mathbb{C}^n \times \mathbb{C}$; and $v = (\psi_6 + \psi_7)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. Here $\psi_7(z,w) = |\varphi_3(w) - h_3(z)|^2 + |\varphi_4(w) - h_4(z)|^2$, where $\varphi_3, \varphi_4 : \mathbb{C}^m \to \mathbb{C}$ and $h_3, h_4 : \mathbb{C}^n \to \mathbb{C}$ be four pluriharmonic (prh) functions. Let $k_1, k_2 : \mathbb{C}^n \to \mathbb{R}$ be two pluriharmonic functions (of real values). Note that, in the sequel, an important problem of complex analysis is to characterize all the holomorphic functions $F_1, F_2 : \mathbb{C}^n \to \mathbb{C}$ such that $(|F_1|^2 + |F_2|^2)$ is convex on $\mathbb{C}^n$. Now it is possible to find all the representations of k_1 and k_2 such that $(k_1^2 + k_2^2)$ is convex on $\mathbb{C}^n$.

Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be two functions. Define $\psi_8(z,w) = |w^2 + f(z)w + g(z)|$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We prove that ψ_8 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, $g = \dfrac{f^2}{4}$, f is harmonic on $\mathbb{C}$ and $|\dfrac{\partial f}{\partial \bar{z}}(z)| > 0$, for each $z \in \mathbb{C}$. Note that in this situation, we can study the problem

$$\begin{cases} \psi_8 \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C} \\ \psi_9 \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C} \\ v_1 = (\psi_8 + \psi_9) \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \end{cases}$$

where $\psi_9(z,w) = |w^2 + f_1(z)w + g_1|$, $f_1, g_1 : \mathbb{C}^n \to \mathbb{C}$ be two functions.

In Section 7, we consider the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ (where $(c, \gamma \in \mathbb{C}$ and $k : \mathbb{C} \to \mathbb{C}$ be a holomorphic nonconstant function). This differential equation plays a technical tool in several problems of real and complex convexity and their generalizations. Indeed, in the sequel, we can prove that $\gamma \in \left\{ \dfrac{s-1}{s}, 1 \ / \ s \in \mathbb{N}\backslash\{0\} \right\}$ independently of the constant c.

Section 8 discusses technical properties between real and complex convexity.

The aim of Section 9 is to establish theorem 9.1 and some related topics. Some good references for the study of convex functions and their variations in complex convex domains are [8, 5, 4, 2, 11]. For the study of holomorphic functions we cite the references [9, 10, 11, 14]. For the study of the properties and the extension problems of holomorphic, plurisubharmonic (or quasiplurisubharmonic), pluriharmonic functions and the continuation of analytic objects in several complex variables we cite the references [12, 13, 14, 15, 16, 6, 7] and others.

Let U be a domain of $\mathbb{R}^d, (d \geq 2)$ and $f : U \to \mathbb{C}$ be a function. $|f|$ is the modulus of f, $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are respectively the real and the imaginary parts of f. supp(f) is the support of the function f. m_d is the Lebesgue measure on $\mathbb{R}^d$. Let $g : D \to \mathbb{C}$ be an analytic function, D is a domain of $\mathbb{C}$. We denote by $g^{(0)} = g, g^{(1)} = g'$ is the holomorphic derivative of g on D. $g^{(2)} = g'', g^{(3)} = g'''$. In general $g^{(m)} = \dfrac{\partial^m g}{\partial z^m}$ is the derivative of g of order m, for all $m \in \mathbb{N}$.

$$C^k(U) = \{\varphi : U \to \mathbb{C} \ /\varphi \text{ is of class } C^k \text{ in } U\}, k \in \mathbb{N} \cup \{\infty\}\backslash\{0\}.$$

$$C_c^\infty(U) = \{\varphi \in C^\infty(U)/\varphi \text{ has a compact support on } U\}$$

and

$$C(U) = \{\varphi : U \to \mathbb{C}/\varphi \text{ is continuous on } U\}.$$

Let $h : U \to \mathbb{C}^n$, $h = (h_1, \ldots, h_n)$. We write $\|h\|^2 = (|h_1|^2 + \cdots + |h_n|^2)$. $\mathrm{sh}(U)$ is the set of all subharmonic functions on U. Let D be a domain of $\mathbb{C}^n, (n \geq 1)$. $\mathrm{psh}(D)$ and $\mathrm{prh}(D)$ are respectively the class of plurisubharmonic and pluriharmonic functions on D. For all $a \in \mathbb{C}, |a|$ is the modulus of a. $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are respectively the real and imaginary parts of a. The habitual hermitian product over the vector space $\mathbb{C}^n$ is denoted by $\langle \cdot / \cdot \rangle$. If $b \in \mathbb{C}^n$, $\|b\|$ is the Euclidean norm of b, $B(b, r) = \{z \in \mathbb{C}^n / \|z - b\| < r\}$ and $\partial B(b, r) = \{z \in \mathbb{C}^n / |z - b| = r\}$, for $r > 0$. For $n = 1$, $B(b, r) = D(b, r)$ and $\partial B(b, r) = \partial D(b, r)$. $M_n(\mathbb{C})$ is the set of all matrix with coefficients in $\mathbb{C}$ and of type (n, n). If p is a holomorphic polynomial on $\mathbb{C}$, $\deg(p)$ is the degree of p.

2. A family of strictly plurisubharmonic functions and the invariance by a special class of functions. We begin this section by the following remark.

There exists $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions, $h_1, h_2 : \mathbb{C} \to \mathbb{C}$ be two harmonic functions such that if we put $u(z, w) = |w - g_1(z) - h_1(z)|^2 + |w - g_2(z) - h_2(z)|^2$ and $v(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$ for $(z, w) \in \mathbb{C}^2$. Then u is strictly psh on $\mathbb{C}^2$, but v is not strictly psh at every point of $\mathbb{C}^2$.

In fact we can choose $g_1(z) = g_2(z) = z^2$, $h_1(z) = z^2 + z$ and $h_2(z) = z^2$, for $z \in \mathbb{C}$. Although there exists $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ be holomorphic functions and $k_1, k_2 : \mathbb{C} \to \mathbb{C}$ be harmonic functions and if we define $u_1(z, w) = |w - \overline{f_1}(z) - k_1(z)|^2 + |w - \overline{f_2}(z) - k_2(z)|^2$ and $v_1(z, w) = |w - \overline{f_1}(z)|^2 + |w - \overline{f_2}(z)|^2$, for $(z, w) \in \mathbb{C}^2$. Then u_1 is strictly psh on $\mathbb{C}^2$, but v_1 is not strictly psh on $\mathbb{C}^2$. For example let $f_1(z) = z^2$, $f_2(z) = 2z^2$, $k_1(z) = k_2(z) = \mathrm{Re}(z)$, $z \in \mathbb{C}$.

This remark yields and investigate the following technical four theorems in complex analysis.

Theorem 2.1. *Let* $\varphi_1, \ldots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$ *be* N *holomorphic functions,* $n, N \geq 1$. *Put* $u(z,w) = |w - \overline{\varphi_1}(z)|^2 + \cdots + |w - \overline{\varphi_N}(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. *Suppose that the function* u *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$.

For $h_1, \ldots, h_N : \mathbb{C}^n \to \mathbb{R}$ *be* N *pluriharmonic functions, we define*

$$\begin{aligned} v(z,w) &= |w - \overline{\varphi_1}(z) - h_1(z)|^2 + \cdots + |w - \overline{\varphi_N}(z) - h_N(z)|^2 \\ &\quad + |w - \varphi_1(z)|^2 + \cdots + |w - \varphi_N(z)|^2, \\ v_1(z,w) &= |w - \overline{\varphi_1}(z) - h_1(z)|^2 + \cdots + |w - \overline{\varphi_N}(z) - h_N(z)|^2 \\ &\quad + |w - \varphi_1(z) - h_1(z)|^2 + \cdots + |w - \varphi_N(z) - h_N(z)|^2, \\ v_2(z,w) &= (h_1(z))^2 + |w - \overline{\varphi_1}(z) - h_1(z)|^2 + \cdots + |w - \overline{\varphi_N}(z) - h_N(z)|^2, \\ v_3(z,w) &= |w - \overline{\varphi_1}(z)|^2 + |w - \overline{\varphi_1}(z) - h_1(z)|^2 + \cdots + |w - \overline{\varphi_N}(z) - h_N(z)|^2, \\ v_4(z,w) &= |w - \varphi_1(z)|^2 + |w - \overline{\varphi_1}(z) - h_1(z)|^2 + \cdots + |w - \overline{\varphi_N}(z) - h_N(z)|^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. *Then the functions* v, v_1, v_2, v_3 *and* v_4 *are strictly psh on* $\mathbb{C}^n \times \mathbb{C}$.

Proof. We have v, v_1, v_2, v_3 and v_4 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Denote by $w = z_{n+1} \in \mathbb{C}$. The Levi hermitian form of v is

$$L(v)(z,w)(\alpha,\beta) = \sum_{j,k=1}^{n+1} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z,w)\alpha_j \overline{\alpha_k},$$

$z = (z_1, \ldots, z_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $\beta = \alpha_{n+1} \in \mathbb{C}$. Write $h_j = f_j + \overline{f_j}$, $1 \leq j \leq n$, where $f_j : \mathbb{C}^n \to \mathbb{C}$ is holomorphic. By [4, Lemma 2.6], the functions v and F have the same hermitian Levi form on $\mathbb{C}^n \times \mathbb{C}$, where

$$\begin{aligned} F(z,w) = |w - f_1(z)|^2 + |f_1(z) + \varphi_1(z)|^2 + \cdots + |w - f_N(z)|^2 \\ + |f_N(z) + \varphi_N(z)|^2 + |w - \varphi_1(z)|^2 + \cdots + |w - \varphi_N(z)|^2. \end{aligned}$$

F is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.

$$\begin{aligned} L(F)(z,w)(\alpha,\beta) &= \left|\beta - \sum_{j=1}^n \frac{\partial f_1}{\partial z_j}(z)\alpha_j\right|^2 + \left|\sum_{j=1}^n \frac{\partial f_1}{\partial z_j}(z)\alpha_j + \sum_{j=1}^n \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j\right|^2 + \cdots \\ &\quad + \left|\beta - \sum_{j=1}^n \frac{\partial f_N}{\partial z_j}(z)\alpha_j\right|^2 + \left|\sum_{j=1}^n \frac{\partial f_N}{\partial z_j}(z)\alpha_j + \sum_{j=1}^n \frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j\right|^2 \\ &\quad + \left|\beta - \sum_{j=1}^n \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j\right|^2 + \cdots + \left|\beta - \sum_{j=1}^n \frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j\right|^2. \end{aligned}$$

Then $L(F)(z,w)(\alpha,\beta)=0$ implies that

$$\text{(I)}\qquad \left|\beta-\sum_{j=1}^{n}\frac{\partial f_1}{\partial z_j}(z)\alpha_j\right|^2+\left|\sum_{j=1}^{n}\frac{\partial f_1}{\partial z_j}(z)\alpha_j+\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j\right|^2+\cdots$$

$$+\left|\beta-\sum_{j=1}^{n}\frac{\partial f_N}{\partial z_j}(z)\alpha_j\right|^2+\left|\sum_{j=1}^{n}\frac{\partial f_N}{\partial z_j}(z)\alpha_j+\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j\right|^2=0$$

and

$$\text{(II)}\qquad \left|\beta-\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j\right|^2+\cdots+\left|\beta-\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j\right|^2=0.$$

It follows that (I) implies $\beta=\sum_{j=1}^{n}\frac{\partial f_1}{\partial z_j}(z)\alpha_j=-\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j$.

(II) implies that $\beta=\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j$. Then $\beta=0$ and $\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j=0$.

$\vdots$

The last step implies that $0=\beta=\sum_{j=1}^{n}\frac{\partial f_N}{\partial z_j}(z)\alpha_j=-\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j$ and $\beta=\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j$. Therefore $\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j=0$. Consequently, we have $\beta=0$ and $\sum_{j=1}^{n}\frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j=0,\ldots,\sum_{j=1}^{n}\frac{\partial \varphi_N}{\partial z_j}(z)\alpha_j=0$.

Since now u is strictly psh on $\mathbb{C}^n\times\mathbb{C}$, then by [3], we have $\alpha_1=\cdots=\alpha_n=0$, for each $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. It follows that v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. The rest of this proof follows from the above proof. □

We have

Theorem 2.2. *Let $\varphi_1,\ldots,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be N analytic functions, n, $N\geq 1$. For $(z,w)\in\mathbb{C}^n\times\mathbb{C}$, denote by $u_1(z,w)=|w-\varphi_1(z)|^2+\cdots+|w-\varphi_N(z)|^2$. Suppose that u_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. Then we have*

(a) For all pluriharmonic functions $h_1,\ldots,h_N:\mathbb{C}^n\to\mathbb{R}$, the function v_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$, where $v_1(z,w)=|w-\varphi_1(z)-h_1(z)|^2+\cdots+|w-\varphi_N(z)-h_N(z)|^2$ for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.

(b) Let $T:\mathbb{C}^n\to\mathbb{C}^n$ be a $\mathbb{C}-$ linear bijective transformation.

Put $u_2(z,w) = |w-\varphi_1(\overline{T(z)})|^2+\cdots+|w-\varphi_N(\overline{T(z)})|^2$, *for* $(z,w) \in \mathbb{C}^n\times\mathbb{C}$. *Then* u_2 *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$ *and* $N \geq (n+1)$.

Proof. (a) u_1 and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Since $h_1, \ldots, h_N$ have real values, we can write $h_1 = g_1 + \overline{g_1}, \ldots, h_N = g_N + \overline{g_N}$, $g_1, \ldots, g_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions. $v_1(z,w) = |w-\varphi_1(z)-g_1(z)-\overline{g_1}(z)|^2 + \cdots + |w - \varphi_N(z) - g_N(z) - \overline{g_N}(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

By [4, Lemma 2.6], v_1 and ψ have the same hermitian Levi form on $\mathbb{C}^n\times\mathbb{C}$. Where $\psi(z,w) = |w - \varphi_1(z) - g_1(z)|^2 + |g_1(z)|^2 + \cdots + |w - \varphi_N(z) - g_N(z)|^2 + |g_N(z)|^2$. ψ is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Let $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Denote by $w = z_{n+1}$. Let $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and denote by $\beta = \alpha_{n+1}$. The Levi hermitian form of ψ is

$$\begin{aligned} L(\psi)(z,w)(\alpha,\beta) &= \sum_{j,k=1}^{n+1} \frac{\partial^2\psi}{\partial z_j \partial \overline{z_k}}(z,w)\alpha_j\overline{\alpha_k} \\ &= \left|\beta - \sum_{j=1}^{n}\frac{\partial\varphi_1}{\partial z_j}(z)\alpha_j - \sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j\right|^2 + \left|\sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j\right|^2 + \cdots \\ &+ \left|\beta - \sum_{j=1}^{n}\frac{\partial\varphi_N}{\partial z_j}(z)\alpha_j - \sum_{j=1}^{n}\frac{\partial g_N}{\partial z_j}(z)\alpha_j\right|^2 + \left|\sum_{j=1}^{n}\frac{\partial g_N}{\partial z_j}(z)\alpha_j\right|^2. \end{aligned}$$

We have $L(\psi)(z,w)(\alpha,\beta) = 0$ implies that

$$\beta - \sum_{j=1}^{n}\frac{\partial\varphi_1}{\partial z_j}(z)\alpha_j - \sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j = 0 \quad\text{and}\quad \sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j = 0.$$

Therefore, $$\sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j = 0 \quad\text{and}\quad \beta - \sum_{j=1}^{n}\frac{\partial\varphi_1}{\partial z_j}(z)\alpha_j = 0.$$

$$\vdots$$

$$\sum_{j=1}^{n}\frac{\partial g_N}{\partial z_j}(z)\alpha_j = 0 \quad\text{and}\quad \beta - \sum_{j=1}^{n}\frac{\partial\varphi_N}{\partial z_j}(z)\alpha_j = 0.$$

Since now u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then $\beta = 0$ and $\alpha_1 = \cdots = \alpha_n = 0$. □

Remark 2.1. In Theorem 2.1, if $h_j : \mathbb{C}^n \to \mathbb{C}$, $(1 \leq j \leq N)$, the result is false in general. There exists several cases where u is strictly psh but k is not strictly psh on $\mathbb{C}^n\times\mathbb{C}$, where $k(z,w) = (|w-\overline{\varphi_1}(z)-h_1(z)|^2+\cdots+|w-\overline{\varphi_N}(z)-h_N(z)|^2)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Moreover, if $h_j : \mathbb{C}^n \to i\mathbb{R}$, for each $(1 \leq j \leq N)$, the result is true.

We have the following technical converse two theorems.

Theorem 2.3. *Let $h_1, k_1, \dots, h_N, k_N : \mathbb{C}^n \to \mathbb{R}$ be $(2N)$ pluriharmonic functions, $n, N \geq 1$ and $(A_1, \dots, A_N) \in \mathbb{C}^N$. Assume that $(h_1 + ik_1), \dots, (h_N + ik_N)$ are holomorphic functions on $\mathbb{C}^n$. Define $H_1(z, w) = |A_1 w - h_1(z)|^2 + \cdots + |A_N w - h_N(z)|^2$, $K_1(z, w) = |A_1 w - k_1(z)|^2 + \cdots + |A_N w - k_N(z)|^2$, $H(z, w) = |A_1 w - h_1(z) - k_1(z)|^2 + \cdots + |A_N w - h_N(z) - k_N(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Put $K = (H_1 + K_1)$. We have*

(I) *Suppose that K is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then H, H_1 and K_1 are strictly psh on $\mathbb{C}^n \times \mathbb{C}$.*

(II) *Suppose that H is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then H_1 and K_1 are strictly psh on $\mathbb{C}^n \times \mathbb{C}$.*

(III) *In fact we have H_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if K_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.*

Theorem 2.4. *Let $h_1, \dots, h_N : \mathbb{C}^n \to \mathbb{R}$ be N prh functions and $(A_1, \dots, A_N) \in \mathbb{C}^N$. Put $v(z, w) = |A_1 w - h_1(z)|^2 + \cdots + |A_N w - h_N(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then for all holomorphic functions $\varphi_1, \dots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$, the function v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, where $v_1(z, w) = |A_1 w - h_1(z) - \varphi_1(z)|^2 + \cdots + |A_N w - h_N(z) - \varphi_N(z)|^2$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.*

P r o o f. We have v and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Now if $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ such that the hermitian Levi form of v_1 satisfies $L(v_1)(z, w)(\alpha, \beta) = 0$, where $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$. We prove by ([4], lemma 2.6), that $L(v)(z, w)(\alpha, \beta) = 0$ and then $\alpha = 0$, $\beta = 0$. Observe that if we denote by

$$v_2(z, w) = |A_1 w - h_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - h_N(z) - \overline{\varphi_N}(z)|^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Then there exists several cases where v_2 is not strictly psh on $\mathbb{C}^n \times \mathbb{C}$. □

Note that the complex structure plays a classical role on the above four theorems. We can use the previous theorems several times in the exercises and complex analysis problems.

Moreover, the four above theorems are not true for convex and strictly convex functions in convex domains. But we have

Theorem 2.5. *Let $(A_1, \dots, A_N) \in \mathbb{C}^N$ and $g_1, \dots, g_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions, $n, N \geq 1$. Put $u(z, w) = |A_1 w - \overline{g_1}(z)|^2 + \cdots + |A_N w - \overline{g_N}(z)|^2$ and $v(z, w) = |A_1 w - g_1(z)|^2 + \cdots + |A_N w - g_N(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.*

(I) *Assume that u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then for all holomorphic functions $\varphi_1, \dots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$, the function u_1 is srictly psh on $\mathbb{C}^n \times \mathbb{C}$. $u_1(z, w) =$*

$|A_1 w - \overline{g_1}(z) - \varphi_1(z)|^2 + \cdots + |A_N w - \overline{g_N}(z) - \varphi_N(z)|^2$, *for* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

(II) *Assume that* v *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$. *Then for all* $\varphi_1, \ldots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$ *analytic functions, the function* v_1 *is srictly psh on* $\mathbb{C}^n \times \mathbb{C}$. $v_1(z,w) = |A_1 w - g_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - g_N(z) - \overline{\varphi_N}(z)|^2$, *for* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Moreover, we have

Theorem 2.6. *Let* $N, s \in \mathbb{N}\backslash\{0\}$. *Consider* $(A_1, \ldots, A_N) \in \mathbb{C}^N$, $(B_1, \ldots, B_s) \in \mathbb{C}^s$ *and let* $f_1, \ldots, f_N, g_1, \ldots, g_s : \mathbb{C}^n \to \mathbb{C}$ *be holomorphic functions. Put* $u(z,w) = |A_1 w - f_1(z)|^2 + \cdots + |A_N w - f_N(z)|^2 + |B_1 w - \overline{g_1}(z)|^2 + \cdots + |B_s w - \overline{g_s}(z)|^2$, *for* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. *Assume that* u *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$. *Then for all* $(c_1, \ldots, c_s) \in \mathbb{C}^s$ *and for each holomorphic functions* $\varphi_1, \ldots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$, *the function* v *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}$.

$$v(z,w) = |A_1 w - f_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - f_N(z) - \overline{\varphi_N}(z)|^2 + |B_1 w - \overline{g_1}(z) - c_1|^2 + \cdots + |B_s w - \overline{g_s}(z) - c_s|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Example. Let $n, N \in \mathbb{N}\backslash\{0\}$ and $(A_1, \ldots, A_N) \in \mathbb{C}^N$. Let $g_1, \varphi_1, \ldots, g_N, \varphi_N : D \to \mathbb{C}$ be $2N$ holomorphic functions, D is a domain of $\mathbb{C}^n$. Define

$$u(z,w) = |A_1 w - g_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - g_N(z) - \overline{\varphi_N}(z)|^2 \text{ and}$$

$$u_1(z,w) = |A_1 w - g_1(z)|^2 + \cdots + |A_N w - g_N(z)|^2, \quad \text{for} \quad (z,w) \in D \times \mathbb{C}.$$

Assume that u is strictly psh on $D \times \mathbb{C}$. In general we can not conclude that u_1 is strictly psh on $D \times \mathbb{C}$. Indeed, let $N = 1$, $v(z,w) = |w - z - \overline{z}|^2$, $v_1(z,w) = |w - z|^2$, for $(z,w) \in \mathbb{C}^2$. $f(z) = z$, $k(z) = 2z$, $z \in \mathbb{C}$. f and k are analytic functions on $\mathbb{C}$. v and v_1 are functions of class C^∞ on $\mathbb{C} \times \mathbb{C}$. v is strictly psh on $\mathbb{C}^2$, but v_1 is not strictly psh at each point of $\mathbb{C}^2$. For $(z,w) \in \mathbb{C}^2$, put

$$\begin{aligned}
K(z,w) &= |A_1 w - g_1(z)|^2 + \cdots + |A_N w - g_N(z)|^2 + |A_1 w - \overline{g_1}(z)|^2 + \cdots \\
&\quad + |A_N w - \overline{g_N}(z)|^2, \\
K_1(z,w) &= |A_1 w - g_1(z) - \varphi_1(z)|^2 + \cdots + |A_N w - g_N(z) - \varphi_N(z)|^2 \\
&\quad + |A_1 w - \overline{g_1}(z) - \varphi_1(z)|^2 + \cdots + |A_N w - \overline{g_N}(z) - \varphi_N(z)|^2, \\
K_2(z,w) &= |A_1 w - g_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - g_N(z) - \overline{\varphi_N}(z)|^2 \\
&\quad + |A_1 w - \overline{g_1}(z) - \varphi_1(z)|^2 + \cdots + |A_N w - \overline{g_N}(z) - \varphi_N(z)|^2, \\
K_3(z,w) &= |A_1 w - g_1(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - g_N(z) - \overline{\varphi_N}(z)|^2 \\
&\quad + |A_1 w - \overline{g_1}(z) - \overline{\varphi_1}(z)|^2 + \cdots + |A_N w - \overline{g_N}(z) - \overline{\varphi_N}(z)|^2.
\end{aligned}$$

If K is strictly psh on $D\times\mathbb{C}$, then K_1, K_2 and K_3 are strictly psh on $D\times\mathbb{C}$. Moreover, we can study the problem

$$(P)\qquad \begin{cases} |w-\overline{\varphi_1}|^2+|w-\overline{\varphi_2}|^2 \text{ is strictly psh on } \mathbb{C}^2\\ |w-\overline{\varphi_1}||w-\overline{\varphi_2}| \text{ is psh on } \mathbb{C}^2\end{cases}$$

where $\varphi_1,\varphi_2:\mathbb{C}\to\mathbb{C}$ be two analytic functions.

In general we can study for N functions the above system, for $N\geq 2$.

Question. Suppose that (P) is true. Then there exists an infinite number of harmonic functions $h:\mathbb{C}\to\mathbb{R}$ such that

$$\begin{cases} |w-\overline{\varphi_1}-h|^2+|w-\overline{\varphi_2}-h|^2 \text{ is strictly psh on } \mathbb{C}^2, \text{ and}\\ |w-\overline{\varphi_1}-h||w-\overline{\varphi_2}-h| \text{ is psh on } \mathbb{C}^2.\end{cases}$$

We can in fact investigate this problem for N-functions on $\mathbb{C}^n$, $n\geq 1$.

Note that we have the following remark.

Let $\varphi_1,\varphi_2:\mathbb{C}\to\mathbb{C}$ be two analytic functions. Put $u(z,w)=|w-\overline{\varphi_1}(z)|^2+|w-\overline{\varphi_2}(z)|^2$, $(z,w)\in\mathbb{C}^2$. Given $h:\mathbb{C}\to\mathbb{R}$ be an harmonic function. Put $v(z,w)=|w-\overline{\varphi_1}(z)-h(z)|^2+|w-\overline{\varphi_2}(z)-h(z)|^2$. Suppose that u is strictly psh on $\mathbb{C}^2$. Then v is strictly psh on $\mathbb{C}^2$. The converse is also true.

In the sequel, Combining the above theorems, we deduce the following technical result.

Theorem 2.7. *Let $\varphi_1,\dots,\varphi_N,g_1,\dots,g_N:\mathbb{C}^n\to\mathbb{C}$ be $2N$ holomorphic functions, $n,N\geq 1$. Put*

$$\begin{aligned}
u(z,w)&=|w-\varphi_1(z)|^2+\cdots+|w-\varphi_N(z)|^2+|w-\overline{\varphi_1}(z)|^2+\cdots+|w-\overline{\varphi_N}(z)|^2,\\
v(z,w)&=|w-\overline{\varphi_1}(z)|^2+\cdots+|w-\overline{\varphi_N}(z)|^2,\\
u_1(z,w)&=|w-\overline{\varphi_1}(z)|^2+\cdots+|w-\overline{\varphi_N}(z)|',\\
v_1(z,w)&=|w-\overline{\varphi_1}(z)-g_1(z)|^2+\cdots+|w-\overline{\varphi_N}(z)-g_N(z)|^2,\\
v_2(z,w)&=|w-\overline{\varphi_1}(z)-\overline{g_1}(z)|^2+\cdots+|w-\overline{\varphi_N}(z)-\overline{g_N}(z)|^2,\ (z,w)\in\mathbb{C}^n\times\mathbb{C}.
\end{aligned}$$

(a) Suppose that u is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. Then v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ and $N\geq(n+1)$. The converse is also true.

(b) Assume that u_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. Then v_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$, but in general v_2 is not strictly psh on $\mathbb{C}^n\times\mathbb{C}$.

(c) Let $h:\mathbb{C}^n\to\mathbb{C}$ be prh. Then v_1 is not strictly convex on every not empty Euclidean open ball subset of $\mathbb{C}^n\times\mathbb{C}$, if $n\geq N+1$.

Moreover, there exists several cases for $n=N+1$, where F is strictly psh on $\mathbb{C}^n\times\mathbb{C}$, where $F(z,w)=|w-\varphi_1(z)|^2+\cdots+|w-\varphi_N(z)|^2+|w-h(z)|^2$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.

Remark 2.2. Let $\varphi_1(z) = z$, $\varphi_2(z) = z^2$, $z \in \mathbb{C}$. $u_1(z,w) = |w-\varphi_1(z)|^2 + |w-\varphi_2(z)|^2$, $u(z,w) = |w-\overline{\varphi_1}(z)|^2 + |w-\overline{\varphi_2}(z)|^2$, $(z,w) \in \mathbb{C}^2$. u is strictly psh on $\mathbb{C}^2$. But u_1 is not strictly psh on $D\left(\frac{1}{2},1\right) \times \mathbb{C}$. That is antiholomorphic functions are a good class (instead of holomorphic functions) for the study of the class of strictly plurisubharmonic functions and their properties.

We have

Corollary 2.1. *Let $\varphi_1,\dots,\varphi_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions, $n, N \in \mathbb{N}\backslash\{0\}$. Put*

$$\begin{aligned} u(z,w) &= |w-\varphi_1(z)|^2 + \cdots + |w-\varphi_N(z)|^2,\\ v(z,w) &= |w-\overline{\varphi_1}(z)|^2 + \cdots + |w-\overline{\varphi_N}(z)|^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and $N \geq (n+1)$. The converse is unfortunately false.

3. Pluriharmonicity and strictly psh functions.

We have

Theorem 3.1. *Let $f_1,\dots,f_N : \mathbb{C}^n \to \mathbb{C}$ be N pluriharmonic (prh) functions, $n, N \in \mathbb{N}\backslash\{0\}$. Let $f_j = g_j + \overline{k}_j$, where $g_j, k_j : \mathbb{C}^n \to \mathbb{C}$ are two analytic functions for each $j \in \{1,\dots,N\}$. Put*

$$\begin{aligned} u(z,w) &= |w-f_1(z)|^2 + \cdots + |w-f_N(z)|^2,\\ u_1(z,w) &= |w-g_1(z)-k_1(z)|^2 + \cdots + |w-g_N(z)-k_N(z)|^2,\\ v(z,w) &= |w-f_1(z)-\overline{f_1}(z)|^2 + \cdots + |w-f_N(z)-\overline{f_N}(z)|^2,\\ \varphi(z,w) &= |w-\overline{f_1}(z)|^2 + \cdots + |w-\overline{f_N}(z)|^2, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}. \end{aligned}$$

(a) Suppose that u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Indeed, the converse is false in general.

(b) Assume that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Then $(u+\varphi)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Moreover, if $(u+\varphi)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not conclude that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Indeed, if $f_1,\dots,f_N$ are holomorphic functions, we have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $(u+\varphi)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

Proof. Since u, u_1, v and φ are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$, the proof is obvious by using the Levi form of each function. □

Example. Let $f(z) = 2z + (\overline{z^2})$, for $z \in \mathbb{C}$. f is harmonic on $\mathbb{C}$. Put $u(z,w) = |w-f(z)|^2$, $v(z,w) = |w-f(z)-\overline{f}(z)|^2$ and $\varphi(z,w) = |w-\overline{f}(z)|^2$, for $(z,w) \in \mathbb{C}^2$. u, v and φ are functions of class C^∞ on $\mathbb{C}^2$. Let $(z_0,w_0) = (-1,0)$. We have v is not strictly psh at $(0,-1)$, but $(u+\varphi)$ is strictly psh on $\mathbb{C}^2$.

We have

Proposition 3.1. *Let $g, k : \mathbb{C} \to \mathbb{C}$ be two analytic functions. Define*

$$\begin{aligned} u_1(z,w) &= |w - g(z) - \overline{k}(z)|^2, \\ u_2(z,w) &= |w - \overline{g}(z) - k(z)|^2 \quad \textit{and} \\ u_3(z,w) &= |w - \overline{g}(z) - \overline{k}(z)|^2, \ \textit{for } (z,w) \in \mathbb{C}^2. \end{aligned}$$

Suppose that u_1 is strictly psh on $\mathbb{C}^2$. Then u_3 is strictly psh on $\mathbb{C}^2$.
Suppose that u_2 is strictly psh on $\mathbb{C}^2$. Then u_3 is strictly psh on $\mathbb{C}^2$.
Suppose that u_1 is strictly psh on $\mathbb{C}^2$. We can not conclude that u_2 is strictly psh on $\mathbb{C}^2$ and conversely.
If u_3 is strictly psh on $\mathbb{C}^2$, we can not in general conclude that u_1 or u_2 is strictly psh on $\mathbb{C}^2$.

Moreover, let $h_1, \dots, h_N : \mathbb{C}^n \to \mathbb{C}$ be N pluriharmonic functions, N, $n \geq 1$. Put

$$\begin{aligned} \varphi(z,w) &= |w - h_1(z)|^2 + \cdots + |w - h_N(z)|^2 \quad \text{and} \\ \psi(z,w) &= |w - \overline{h}_1(z)|^2 + \cdots + |w - \overline{h}_N(z)|^2, \ \text{for } (z,w) \in \mathbb{C}^n \times \mathbb{C}. \end{aligned}$$

If φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not deduce that ψ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

Note that the above proposition have many applications in problems and exercises.

Claim 3.1. *Let $h : \mathbb{C} \to \mathbb{C}$ be a harmonic function. Put $u(z,w) = |w - h(z)|^2 + |w - \overline{h}(z)|^2$, for $(z,w) \in \mathbb{C}^2$. Suppose that u is strictly psh on $\mathbb{C}^2$. Then we can not conclude that $|w - h|^2$ or $|w - \overline{h}|^2$ is strictly psh on $\mathbb{C}^2$.*

Example. Let $u_1(z,w) = |w - \frac{z^2}{2} - \overline{z}^2 - \overline{z}|^2 + |w - z^2 - z - \frac{\overline{z}^2}{2}|^2$, $(z,w) \in \mathbb{C}^2$. The function u_1 is strictly psh on $\mathbb{C}^2$. Put $h_1(z) = \frac{z^2}{2} + \overline{z}^2 + \overline{z}$; then h_1 is harmonic on $\mathbb{C}$. But $|w - h_1|^2$ and $|w - \overline{h_1}|^2$ are not strictly psh on $\mathbb{C}^2$. Recall that for analytic functions we have the following. If $v(z,w) = |w - g(z)|^2 + |w - \overline{g}(z)|^2$ is strictly psh on $\mathbb{C}^2$, then $|w - \overline{g}|^2$ is strictly psh on $\mathbb{C}^2$, where $g : \mathbb{C} \to \mathbb{C}$ is holomorphic.

Moreover, strictly plurisubharmonic functions plays a fundamental role in the theory of holomorphic or antiholomorphic partial differential equations and some interesting results in a slightly different direction are obtained in [3] and [4].

The following theorem is a technical result which is a necessary tool in function theory and related topics.

Theorem 3.2. *Let $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $g = h + ik$, $h = \mathrm{Re}(g)$ and $k = \mathrm{Im}(g)$. Put $u_1(z,w) = |w - \overline{g}(z)|^2$, $u_2(z,w) = |w - h(z)|^2$, $u_3(z,w) = |w - k(z)|^2$, $u_4(z,w) = |w - h(z) - k(z)|^2$ and $u_5(z,w) = |w - g(\overline{z})|^2$, for $(z,w) \in \mathbb{C}^2$.*

The following statements are equivalent

(I) u_1 *is strictly psh on* $\mathbb{C}^2$;

(II) u_2 *is strictly psh on* $\mathbb{C}^2$;

(III) u_3 *is strictly psh on* $\mathbb{C}^2$;

(IV) u_4 *is strictly psh on* $\mathbb{C}^2$;

(V) u_5 *is strictly psh on* $\mathbb{C}^2$.

Proof. Obvious. Moreover, we can see [3]. □

Observe that we have the following simple technical remark.

Let $g, h : D \to \mathbb{C}$ be two functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$. Suppose that g is holomorphic and h is pluriharmonic on D. Put $u(z,w) = |w - g(z)|^2$ and $v(z,w) = |w - h(z)|^2$, for $(z,w) \in D \times \mathbb{C}$. Then u is not strictly psh at any point of $D \times \mathbb{C}$, for all $n \geq 1$. Moreover, if $n = 1$, there exists several cases where v is strictly psh on $D \times \mathbb{C}$.

Actually Section 2 is a generalization of Theorem 3.2.

Remark 3.1. Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Put $u(z,w) = |w - f(z)|^4 + |w - g(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. u is not strictly psh on $\mathbb{C}^n \times \mathbb{C}$, for each $n \geq 1$.

Recall that there exists $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions such that v is strictly psh on $\mathbb{C}^2$, where $v(\xi,\zeta) = |\zeta - \varphi(\xi)|^2 + |\zeta - \psi(\xi)|^2$ for $(\xi,\zeta) \in \mathbb{C}^2$.

Moreover, for all holomorphic functions $\varphi_1, \psi_1 : \mathbb{C} \to \mathbb{C}$, the function v_1 is not strictly psh at the point $(z_0, w_0) \in \mathbb{C}^2$ (for each $z_0 \in \mathbb{C}$ with $\varphi_1'(z_0) = 0$ and $w_0 = \overline{\psi_1}(z_0)$), where $v_1(z,w) = |w - \overline{\varphi_1}(z)|^4 + |w - \overline{\psi_1}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. But if $v_2(z,w) = |w - \overline{K}(z)|^2$, $K : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $|K'| > 0$ on $\mathbb{C}$. Then v_2 is strictly psh on $\mathbb{C}^2$. On the other hand, there exists two holomorphic functions $\varphi_2, \psi_2 : \mathbb{C} \to \mathbb{C}$, such that $v_3(z,w) = |w - \overline{\varphi_2}(z)|^2 + |w - \overline{\psi_2}(z)|^2$ and $v_4(z,w) = |w - \overline{\varphi_2}(z)|^4 + |w - \overline{\psi_2}(z)|^4$. v_3 is strictly psh on $\mathbb{C}^2$ but v_4 is not.

Example. $\varphi_2(z) = \psi_2(z) = z$, for $z \in \mathbb{C}$.

Indeed, there exists three holomorphic functions $f_1, g_1, k_1 : \mathbb{C}^n \to \mathbb{C}$ such that u_1 is strictly psh on a neighborhood of $(z_0, w_0) = (0, -i)$, but u_2 is not strictly psh at (z_0, w_0). Here $u_1(z,w) = |w - f_1(z)|^4 + |w - g_1(z)|^4 + |w - k_1(z)|^4$, $u_2(z,w) = |w - \overline{f_1}(z)|^4 + |w - \overline{g_1}(z)|^4 + |w - \overline{k_1}(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. In the sequel, let $f_1(z) = z + i$, $g_1(z) = 2z + i$ and $k_1(z) = -z + i$, for $z \in \mathbb{C}$.

Put $u_3(z,w) = |w - f_1(z)|^4$ and $u_4(z,w) = |w - \overline{f_1}(z)|^4$. Then u_3 and u_4 are functions of class C^∞ on $\mathbb{C}^2$. The hermitian Levi form of u_3 is

$L(u_3)(z,w)(\alpha,\beta) = |w - f_1(z)|^2|\beta - f_1'(z)\alpha|^2$, and also $L(u_4)(z,w)(\alpha,\beta) = 2|-f_1'(z)(w-\overline{f_1}(z))\alpha+\beta(\overline{w}-f_1(z))|^2+2|f_1'(z)|^2|w-\overline{f_1}(z)|^2+2|w-\overline{f_1}(z)|^2|\beta|^2$. Now let $\theta_1,\dots,\theta_N : \mathbb{C}^n \to \mathbb{C}$ be N pluriharmonic functions, $n \geq 1$ and $N \geq 2$.

Put $u_5(z,w) = |w-\theta_1(z)|^2+\cdots+|w-\theta_N(z)|^2$, for $(z,w) \in \mathbb{C}^n\times\mathbb{C}$. Observe that the exponent 2 is a technical tool for the study of the strict plurisubharmonicity of the function u_5.

Example. For $s \in \mathbb{N}\backslash\{0,1\}$, $N \in \mathbb{N}\backslash\{0\}$, put $f_j(z) = jz + i$, for $1 \leq j \leq N$. f_j is a holomorphic function on $\mathbb{C}$. Let $(z_0,w_0) = (0,-i)$. Put $u(z,w) = (|w - f_1(z)|^{2s} + \cdots + |w - f_N(z)|^{2s})$ and $v(z,w) = u(z,\overline{w})$ for $(z,w) \in \mathbb{C}\times\mathbb{C}$. Then u is strictly psh on a neighborhood of $(0,-i)$, but v is not strictly psh at $(0,-i)$.

This example yields and investigate the problem of the study of strongly psh functions on $\mathbb{C}^n\times\mathbb{C}$ defined like u and v.

Remark 3.2. Let $f(z) = 1 + e^z$, $g(z) = 1 - e^z$, $z \in \mathbb{C}$. f and g are holomorphic functions on $\mathbb{C}$. Put $u(z,w) = |w - \overline{f}(z)|^4 + |w - \overline{g}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. We have

(I) u is strictly psh on $\mathbb{C}^2$, because $|f - g| > 0$, $|f'| > 0$, $|g'| > 0$ on $\mathbb{C}$.

(II) u_1 is not strictly psh on $\mathbb{C}^2$, for all holomorphic functions $f_1, g_1 : \mathbb{C} \to \mathbb{C}$, where $u_1(z,w) = |w - f_1(z)|^4 + |w - g_1(z)|^4$. Indeed,

(III) Put $\varphi_1(z) = e^z$, $\varphi_2(z) = 2e^z$ and $\varphi_3(z) = 3e^z$ and $v(z,w) = |w-\varphi_1(z)|^4 + |w - \varphi_2(z)|^4 + |w - \varphi_3(z)|^4$, for $(z,w) \in \mathbb{C}^2$. $\varphi_1,\varphi_2,\varphi_3$ are holomorphic functions on $\mathbb{C}$ and v is strictly psh in all the domain $\mathbb{C}^2$.

We have

Lemma 3.1. *Let $f_1,\dots,f_N : D \to \mathbb{C}$ be holomorphic functions, D is a domain of $\mathbb{C}^n$, $n,N \geq 1$. Put*

$$\begin{aligned}
u(z,w) &= |w - f_1(z)|^4 + \cdots + |w - f_N(z)|^4,\\
v(z,w) &= |w - \overline{f_1}(z)|^4 + \cdots + |w - \overline{f_N}(z)|^4,\\
u_1(z,w) &= |w - f_1(z)|^2 + \cdots + |w - f_N(z)|^2 \quad \textit{and}\\
v_1(z,w) &= |w - \overline{f_1}(z)|^2 + \cdots + |w - \overline{f_N}(z)|^2, \quad \textit{for } (z,w) \in D\times\mathbb{C}.
\end{aligned}$$

We have

(I) *Assume that u is strictly psh on $D\times\mathbb{C}$. Then u_1 is strictly psh on $D\times\mathbb{C}$.*

(II) *Suppose that v is strictly psh on $D\times\mathbb{C}$. Then v_1 is strictly psh on $D\times\mathbb{C}$.*

(III) *Suppose that u_1 is strictly psh on $D \times \mathbb{C}$. Then v_1 is strictly psh on $D \times \mathbb{C}$. But u is strictly psh on $D \times \mathbb{C}$, does not imply that v is strictly psh on $D \times \mathbb{C}$. (Also v_1 is strictly psh does not implies that u_1 is strictly psh).*

Proof. Obvious. □

In the sequel, thanks to the above properties, our framework in another paper, we are interested in the study of the structure (convex and strictly psh) of the above special classes of functions.

Concerning the product of pluriharmonic functions, we have

Lemma 3.2. *Let $\varphi_1, \ldots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions, n, $N \geq 1$. Put $u(z,w) = |w - \overline{\varphi}_1(z)| \cdots |w - \overline{\varphi}_N(z)|$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, $N = 2$, $\varphi_2 = \varphi_1$ and $|\dfrac{\partial \varphi_1}{\partial z}| > 0$ on $\mathbb{C}$.*

Proof. Recall that if we put $\psi(w) = |w - a_1| \cdots |w - a_s|$, where $s \in \mathbb{N}\backslash\{0\}$, $a_1, \ldots, a_s \in \mathbb{C}$ and $w \in \mathbb{C}$. Then ψ is strictly sh on $\mathbb{C}$ if and only if $s = 2$ and $a_1 = a_2$. Therefore $N = 2$ and $\varphi_1 = \varphi_2$ on $\mathbb{C}^n$. Consequently, $u(z,w) = |w - \overline{\varphi_1}(z)|^2$, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Now the hermitian Levi form of u is

$$L(u)(z,w)(\alpha,\beta) = |\beta|^2 + |\sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j|^2 > 0$$

for each $z \in \mathbb{C}^n$ and $(\alpha,\beta) = ((\alpha_1, \ldots, \alpha_n), \beta) \in \mathbb{C}^n \times \mathbb{C}\backslash\{0\}$. It follows that $n = 1$ and $\dfrac{\partial \varphi_1}{\partial z}(z) \neq 0$, for every $z \in \mathbb{C}$. □

Although, using technical results of the below section 4, we can study the problem of the characterization of all holomorphic functions f_1, φ_1, f_2, φ_2, g_1, ψ_1, g_2, $\psi_2 : \mathbb{C}^n \to \mathbb{C}$, such that u_1 and u_2 are psh and $u = (u_1^2 + u_2^2)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

$$\begin{aligned} u_1(z,w) &= |w - f_1(z) - \overline{\varphi_1}(z)||w - f_2(z) - \overline{\varphi_2}(z)|, \\ u_2(z,w) &= |w - g_1(z) - \overline{\psi_1}(z)||w - g_2(z) - \overline{\psi_2}(z)|, \quad \text{for } (z,w) \in \mathbb{C}^n \times \mathbb{C}. \end{aligned}$$

For the study of this problem, note that the function u_1 is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if we have the following two cases (see section 4).

Case 1. $(\varphi_1 + \varphi_2)$ is constant on $\mathbb{C}^n$. In this case we have $(f_1 + \overline{\varphi_1} - f_2 - \overline{\varphi_2})^2$ is pluriharmonic (prh) on $\mathbb{C}^n$. Therefore $(\varphi_1 - \varphi_2)$ or $(f_1 - f_2)$ is constant on $\mathbb{C}^n$.

Case 2. $(\varphi_1 + \varphi_2)$ is nonconstant on $\mathbb{C}^n$. Then $(f_1 + \overline{\varphi_1} - f_2 - \overline{\varphi_2}) = 0$ on $\mathbb{C}^n$. We obtain then $f_1 - f_2 = \overline{\varphi_2} - \overline{\varphi_1} = c$, where $c \in \mathbb{C}$. In general, an open problem is the following.

Let $n \geq 1$, $N \geq 2$ and D be a domain of $\mathbb{C}^n$. Let $f_j, g_j, \varphi_j, \psi_j : D \to \mathbb{C}$ be holomorphic functions, for each $1 \leq j \leq N$. Put

$$v_j(z,w) = |w - f_j(z) - \overline{\varphi_j}(z)||w - g_j(z) - \overline{\psi_j}(z)|, \text{ for } (z,w) \in D \times \mathbb{C}.$$

Define $v = \sum_{j=1}^{N} v_j^2$. Find conditions on $n, N, f_1, g_1, \varphi_1, \psi_1, \ldots, f_N, g_N, \varphi_N, \psi_N$ such that

(a) For each $j \in \{1, \ldots, N\}$, v_j is psh on $D \times \mathbb{C}$, and
(b) v is strictly psh on $D \times \mathbb{C}$.

4. On the product of plurisubharmonic functions and related topics. Recall that it is well known that the product of several psh functions is in general not psh. In this section, we consider essentially the question of the product of several absolute values of pluriharmonic functions and we study the question of their plurisubharmonicity, or convexity (also, we are interested and we consider a precise estimate of the strict convexity of a special class which is well defined).

We need the following additional lemma

Lemma 4.1. *Let $a, b, c \in \mathbb{C}$. Put $K(\alpha, \beta) = a\alpha\overline{\alpha} + b\beta\overline{\beta} + 2Re[c\overline{\alpha}\beta]$, $(\alpha, \beta) \in \mathbb{C}^2$. We have*

(I) $(K(\alpha,\beta) > 0, \forall(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\})$ if and only if $(|c|^2 < ab$ and $a > 0$, $b > 0)$.

(II) $(K(\alpha,\beta) \geq 0, \forall(\alpha,\beta) \in \mathbb{C}^2)$ if and only if $(|c|^2 \leq ab$ and $a \geq 0$, $b \geq 0)$.

Proof. Obviously follows from Abidi [2], (we can see also [3]). □

The next proposition gives the exact characterization of the first study in the sequel.

Proposition 4.1. *Let $\varphi_1, \varphi_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions, $n \geq 1$. Put $u(z,w) = |w - \overline{\varphi_1}(z)||w - \overline{\varphi_2}(z)|$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.*

The following are equivalent

(I) u is psh on $\mathbb{C}^n \times \mathbb{C}$;

(II) $\varphi_1 = \varphi_2$ on $\mathbb{C}^n$, or $(\varphi_1 + \varphi_2)$ is constant on $\mathbb{C}^n$.

Proof. We choose the following analysis proof.

Without loss of generality, we assume that $n = 1$ in this proof.

(I) implies (II). Let $v(z,w) = (u(z,w))^2$, for $(z,w) \in \mathbb{C}^2$. Then v is a function of class C^∞ and psh on $\mathbb{C}^2$. $v(z,w) = |w^2 - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)|^2$. Now we can observe that if the function $(\varphi_1 + \varphi_2)$ is constant on $\mathbb{C}$, then $u(z,w) = |h(z,w)|$, where h is a prh function on $\mathbb{C}^2$. Consequently, u is psh on $\mathbb{C}^2$. Now assume that $(\varphi_1 + \varphi_2)$ is nonconstant on $\mathbb{C}$. Observe that we have $\dfrac{\partial^2 v}{\partial \overline{z} \partial z}(z,w) \geq 0$ and $\dfrac{\partial^2 v}{\partial \overline{w} \partial w}(z,w) \geq 0$, for $(z,w) \in \mathbb{C}^2$. Since the hermitian Levi form of v, denoted by $L(v)(z,w)(\alpha,\beta)$, satisfy

$$L(v)(z,w)(\alpha,\beta) = \frac{\partial^2 v}{\partial \overline{z} \partial z}(z,w)|\alpha|^2 + \frac{\partial^2 v}{\partial \overline{w} \partial w}(z,w)|\beta|^2 + 2\mathrm{Re}\left(\frac{\partial^2 v}{\partial \overline{w} \partial z}(z,w)\alpha\overline{\beta}\right) \geq 0,$$

for each $(z,w), (\alpha,\beta) \in \mathbb{C}^2$.

By Lemma 4.1, we have then $|\dfrac{\partial^2 v}{\partial \overline{w} \partial z}(z,w)|^2 \leq \dfrac{\partial^2 v}{\partial \overline{z} \partial z}(z,w)\dfrac{\partial^2 v}{\partial \overline{w} \partial w}(z,w)$, for each $(z,w) \in \mathbb{C}^2$. Therefore,

$$\begin{aligned} &| - \overline{(\varphi_1'(z) + \varphi_2'(z))}[w^2 - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)]|^2 \\ &\qquad \leq | - (\overline{\varphi_1'(z) + \varphi_2'(z)})w + \overline{(\varphi_1\varphi_2)'(z)}|^2 |2w - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))|^2, \end{aligned}$$

for each $(z,w) \in \mathbb{C}^2$.

Now, fix $z \in \mathbb{C}$ such that $\dfrac{\partial(\varphi_1 + \varphi_2)}{\partial z}(z) \neq 0$. Since the following two holomorphic nonconstant polynomials (of the complex variable w),

$$-\overline{(\varphi_1'(z) + \varphi_2'(z))}[w^2 - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)] \quad \text{and}$$

$$(-(\overline{\varphi_1'(z) + \varphi_2'(z)})w + \overline{(\varphi_1\varphi_2)'}(z))(2w - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z)))$$

satisfies the last above inequality for each $w \in \mathbb{C}$, then there exits $c \in \mathbb{C}\backslash\{0\}$ such that

$$\begin{aligned} &-\overline{(\varphi_1'(z) + \varphi_2'(z))}[w^2 - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)] = \\ &c[-(\overline{\varphi_1'(z) + \varphi_2'(z)})w + \overline{(\varphi_1\varphi_2)'}(z)][2w - (\overline{\varphi_1}(z) + \overline{\varphi_2}(z))], \end{aligned}$$

for each $w \in \mathbb{C}$. Then $c = \dfrac{1}{2}$. By identification, we have then

$$\frac{1}{2}(\varphi_1'(z) + \varphi_2'(z))(\varphi_1(z) + \varphi_2(z)) = (\varphi_1\varphi_2)'(z), \quad \text{and}$$

$$(\varphi_1'(z) + \varphi_2'(z))\varphi_1(z)\varphi_2(z) = \frac{1}{2}(\varphi_1(z) + \varphi_2(z))(\varphi_1\varphi_2)'(z).$$

Therefore, $\frac{1}{4}(\varphi_1(z)+\varphi_2(z))^2 = \varphi_1(z)\varphi_2(z)$, because $(\varphi_1'(z)+\varphi_2'(z)) \neq 0$. Consequently, $\varphi_1(z) = \varphi_2(z)$, for each $z \in \mathbb{C}$ such that $(\varphi_1'(z)+\varphi_2'(z)) \neq 0$. Since φ_1 and φ_2 are holomorphic functions on $\mathbb{C}$ and the set $E = \{\xi \in \mathbb{C}/(\varphi_1'(\xi)+\varphi_2'(\xi)) = 0\}$ is polar on $\mathbb{C}$, then $\mathbb{C}\backslash E$ is dense in $\mathbb{C}$. It follows that $\varphi_1 = \varphi_2$ on $\mathbb{C}$.

(II) implies (I). Obvious. The proof is now complete. □

Now the following theorem have a great importance in complex function theory. In the sequel, It plays a key role in several future problems in complex analysis.

Theorem 4.1. *Let $\varphi_1, \varphi_2, \varphi_3 : \mathbb{C}^n \to \mathbb{C}$ be analytic functions, $n \geq 1$. Put $v(z,w) = |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent*

(I) *v is psh on $\mathbb{C}^n \times \mathbb{C}$;*

(II) *$(\varphi_1+\varphi_2+\varphi_3)$ and $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)$ are constant functions on $\mathbb{C}^n$, or $\varphi_1 = \varphi_2 = \varphi_3$ on $\mathbb{C}^n$.*

Proof. Throughout in all of the proof we assume that $n = 1$, because the case $n \geq 2$ follows from the previous case by using the fibration problem. Define $u = v^2$. Then u is a function of class C^∞ on $\mathbb{C}^2$.

(I) implies (II). Note that the hermitian Levi form of u satisfies

$$L(u)(z,w)(\alpha,\beta) \geq 0, \quad \text{for all } (z,w),(\alpha,\beta) \in \mathbb{C}^2.$$

Therefore,

$$L(u)(z,w)(\alpha,\beta) = \frac{\partial^2 u}{\partial\overline{z}\partial z}(z,w)|\alpha|^2 + \frac{\partial^2 u}{\partial\overline{w}\partial w}(z,w)|\beta|^2 + 2\mathrm{Re}(\frac{\partial^2 u}{\partial\overline{w}\partial z}(z,w)\alpha\overline{\beta}) \geq 0,$$

for each (z,w), $(\alpha,\beta) \in \mathbb{C}^2$.

By Lemma 4.1, for all $w \in \mathbb{C}$, we have the inequality (E)

$$\begin{aligned} &|-2(\varphi_1'+\varphi_2'+\varphi_3')\overline{w} + (\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'|^2|\overline{w}^3 - (\varphi_1+\varphi_2+\varphi_3)\overline{w}^2 \\ &\qquad +(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)\overline{w} - \varphi_1\varphi_2\varphi_3|^2 \\ &\leq |-(\varphi_1'+\varphi_2'+\varphi_3')\overline{w^2} + (\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'\overline{w} - (\varphi_1\varphi_2\varphi_3)'|^2 \times \\ &\qquad |3\overline{w^2} - 2(\varphi_1+\varphi_2+\varphi_3)\overline{w} + \varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3|^2, \text{ on } \mathbb{C}. \end{aligned}$$

Case 1. $(\varphi_1+\varphi_2+\varphi_3)' = 0$ on $\mathbb{C}$.

State 1. $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)' = 0$ on $\mathbb{C}$. Then $v = |h|$, where h is a prh function on $\mathbb{C}^2$. It follows that v is psh on $\mathbb{C}^2$.

State 2. $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)' \neq 0$ on $\mathbb{C}$. Now let $z \in \mathbb{C}$ such that $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'(z) \neq 0$. From the inequality (E), we can see that there exists $c \in \mathbb{C}$ such that for each $w \in \mathbb{C}$, we have (P)

$$\begin{aligned}&(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'[\overline{w}^3-(\varphi_1+\varphi_2+\varphi_3)\overline{w}^2\\ &\qquad+(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)\overline{w}-(\varphi_1\varphi_2\varphi_3)]\\ &=c((\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'\overline{w}-(\varphi_1\varphi_2\varphi_3)')(3\overline{w^2}-2(\varphi_1+\varphi_2+\varphi_3)\overline{w}\\ &\qquad+\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3), \text{ on } \mathbb{C}.\end{aligned}$$

Then $c=\dfrac{1}{3}$. Now let $q(w)=(w-\overline{\varphi_1}(z))(w-\overline{\varphi_2}(z))(w-\overline{\varphi_3}(z))$, for $w \in \mathbb{C}$. q is a holomorphic polynomial on $\mathbb{C}$ of degree three. Note that $\overline{\varphi_1}(z)$, $\overline{\varphi_2}(z)$, $\overline{\varphi_3}(z)$ are only the three zeros of q on $\mathbb{C}$. From the equality (P) and since the holomorphic polynomial q_2 is of degree one on $\mathbb{C}$, where

$$q_2(w)=[\overline{(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'}(z)w-\overline{(\varphi_1\varphi_2\varphi_3)'}(z)];$$

then $\overline{\varphi_1}(z)$, or $\overline{\varphi_2}(z)$, or $\overline{\varphi_3}(z)$ is a zero of the polynomial $q_1(w)=q'(w)$. Suppose that $\overline{\varphi_1}(z)$ is a zero of q_1. Therefore $\overline{\varphi_1}(z)$ is a zero of q of order ≥ 2. It follows that $\overline{\varphi_1}(z)=\overline{\varphi_2}(z)$ or $\overline{\varphi_1}(z)=\overline{\varphi_3}(z)$. Assume that $\overline{\varphi_1}(z)=\overline{\varphi_2}(z)$ and $\overline{\varphi_1}(z) \neq \overline{\varphi_3}(z)$. Therefore, $\overline{\varphi_3}(z)$ is a zero of q of order one. Consequently, $\overline{\varphi_3}(z)$ is not a zero of q_1. By the equality (P), we deduce that $\overline{\varphi_3}(z)$ is the only zero of the holomorphic polynomial q_2 and $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z)\}$ is the set of all zeros of the holomorphic polynomial q_1q_2 on $\mathbb{C}$. Then $\overline{\varphi_1}(z)$ is a zero of q_1 of order 2. Therefore $\overline{\varphi_1}(z)$ is a zero of q of order 3. Consequently, $\overline{\varphi_3}(z)=\overline{\varphi_1}(z)$. A contradiction.

Therefore, $\overline{\varphi_1}(z)=\overline{\varphi_2}(z)=\overline{\varphi_3}(z)$. It follows that $\overline{\varphi_1}=\overline{\varphi_2}=\overline{\varphi_3}$ on $\mathbb{C}$. Now since $(\varphi_1+\varphi_2+\varphi_3)'=0$, on $\mathbb{C}$. Then $\varphi_1'=0$ on $\mathbb{C}$. Consequently, $(\varphi_1)'\varphi_1=0$ on $\mathbb{C}$. Now since $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)' \neq 0$ on $\mathbb{C}$. We get a contradiction.

Finally, we conclude that the state 2 is impossible.

Case 2. $(\varphi_1+\varphi_2+\varphi_3)' \neq 0$, on $\mathbb{C}$. Fix $z \in \mathbb{C}$ such that $(\varphi_1+\varphi_2+\varphi_3)'(z) \neq 0$. From the inequality (E) and using properties of holomorphic functions in one variable, we have

$$\begin{aligned}&[-2(\varphi_1'(z)+\varphi_2'(z)+\varphi_3'(z))\overline{w}+(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'(z)][\overline{w^3}-(\varphi_1(z)+\varphi_2(z)\\ &\quad+\varphi_3(z))\overline{w^2}+(\varphi_1(z)\varphi_2(z)+\varphi_1(z)\varphi_3(z)+\varphi_2(z)\varphi_3(z))\overline{w}-\varphi_1(z)\varphi_2(z)\varphi_3(z)]\\ &=c[-(\varphi_1'(z)+\varphi_2'(z)+\varphi_3'(z))\overline{w^2}+(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)'(z)\overline{w}-(\varphi_1\varphi_2\varphi_3)'(z)]\\ &\quad\times[3\overline{w^2}-2(\varphi_1(z)+\varphi_2(z)+\varphi_3(z))\overline{w}+\varphi_1(z)\varphi_2(z)+\varphi_1(z)\varphi_3(z)+\varphi_2(z)\varphi_3(z)],\end{aligned}$$

for each $w \in \mathbb{C}$, where $c \in \mathbb{C}\backslash\{0\}$, ($c$ is independent of w).

By identification we can prove that $c = \frac{2}{3}$. Now define

$$q_1(w) = (w - \overline{\varphi_1}(z))(w - \overline{\varphi_2}(z))(w - \overline{\varphi_3}(z)), \quad \text{for} \quad w \in \mathbb{C}.$$

q_1 is a holomorphic polynomial on $\mathbb{C}$ of degree three. Note that $\overline{\varphi_1}(z)$, $\overline{\varphi_2}(z)$, $\overline{\varphi_3}(z)$ are only the three zeros of q_1 on $\mathbb{C}$. Let $q_2 = q_1'$ and $q_4 = q_3'$, where

$$q_3(w)=[-(\overline{\varphi_1'(z) + \varphi_2'(z) + \varphi_3'(z)})w^2+\overline{(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3)'}(z)w-\overline{(\varphi_1\varphi_2\varphi_3)'}(z)].$$

We have $q_1q_4 = \frac{2}{3}q_2q_3$ on $\mathbb{C}$. Indeed, q_2, q_3 and q_4 are holomorphic polynomials on $\mathbb{C}$ of degree respectively $2, 2$ and 1.

State 1. The set $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z)\}$ have a cardinal equal 3. Then $\overline{\varphi_1}(z), \overline{\varphi_2}(z)$ and $\overline{\varphi_3}(z)$ are zeros of q_1 of order 1. Therefore, $\overline{\varphi_1}(z), \overline{\varphi_2}(z)$ and $\overline{\varphi_3}(z)$ are not zeros of q_2. Since $q_1q_4, = \frac{2}{3}q_2q_3$, then $\overline{\varphi_1}(z), \overline{\varphi_2}(z)$ and $\overline{\varphi_3}(z)$ are three zeros of q_3. But q_3 is a holomorphic polynomial of degree 2. A contradiction. Consequently, this state is impossible.

State 2. $\varphi_1(z) = \varphi_2(z)$ and $\varphi_1(z) \neq \varphi_3(z)$. Then $\overline{\varphi_1}(z)$ is a zero of $q_1' = q_2$ of order 1. $\overline{\varphi_3}(z)$ is not a zero of q_2. Since $q_1q_4 = \frac{2}{3}q_2q_3$, we conclude that the set of all zeros of q_1 is equal to the set of all zeros of q_3 on $\mathbb{C}$. Since the degree of q_3 is 2, then $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z)\}$ is the set of all zeros of q_3 on $\mathbb{C}$. Now observe that $\overline{\varphi_1}(z)$ and $w_0(z) = \frac{1}{3}(\overline{\varphi_1}(z) + 2\overline{\varphi_3}(z))$ are the two zeros of the polynomial q_2. Now we have $\overline{\varphi_1}(z) \neq w_0(z)$. Because if there is equality, then $\overline{\varphi_1}(z)$ is a zero of q_1 of order 3 and we conclude a contradiction. Observe also that $\overline{\varphi_3}(z) \neq w_0(z)$. Because if $w_0(z) = \overline{\varphi_3}(z)$, then $\overline{\varphi_3}(z)$ is a zero of $q_2 = q_1'$, which contradict the hypothesis $\overline{\varphi_3}(z)$ is not a zero of q_2. Now since $q_1q_4 = \frac{2}{3}q_2q_3$ on $\mathbb{C}$, then $w_0(z)$ is a zero of q_4. Therefore $w_0(z) = \dfrac{\overline{(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3)'}(z)}{2(\overline{\varphi_1'}(z) + \overline{\varphi_2'}(z) + \overline{\varphi_3'}(z))}$. Now we have $\overline{\varphi_1}(z) + \overline{\varphi_3}(z) = \dfrac{\overline{(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3)'}(z)}{(\overline{\varphi_1'}(z) + \overline{\varphi_2'}(z) + \overline{\varphi_3'}(z))}$, because $\overline{\varphi_1}(z)$ and $\overline{\varphi_3}(z)$ are the two zeros of q_3. Observe now that $\overline{\varphi_1}(z) + \overline{\varphi_3}(z) = 2w_0(z)$. Since $w_0(z) = \frac{1}{3}(\overline{\varphi_1}(z) + 2\overline{\varphi_3}(z))$, then $\overline{\varphi_1}(z) + \overline{\varphi_3}(z) = \frac{2}{3}(\overline{\varphi_1}(z) + 2\overline{\varphi_3}(z))$. Consequently, $\overline{\varphi_1}(z) = \overline{\varphi_3}(z)$. A contradiction.

Therefore this state is impossible.

In this case, we conclude that $\varphi_1(z) = \varphi_2(z) = \varphi_3(z)$. Now since the set $\{\xi \in \mathbb{C} \ / \ (\varphi_1 + \varphi_2 + \varphi_3)'(\xi) \neq 0 \ \}$ is a domain dense on $\mathbb{C}$, we deduce that $\varphi_1 = \varphi_2 = \varphi_3$ on $\mathbb{C}$.

(II) implies (I). Obvious. □

Let now $n, m \geq 1$. It is clear to see that the problem is open when we consider the product of N absolute values of prh functions for each $N \geq 2$ in the following form. Let $n, m \geq 1$. Given $g_j : \mathbb{C}^n \to \mathbb{C}$, $h_j : \mathbb{C}^m \to \mathbb{C}$, g_j is holomorphic nonconstant and h_j is prh, for $1 \leq j \leq N$. We can study the structure of u defined by $u(z,w) = |g_1(z) - h_1(w)| \cdots |g_N(z) - h_N(w)|$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$. Moreover, let $\varphi_1, \ldots, \varphi_s : D \to \mathbb{C}$ be holomorphic functions, D is a domain of $\mathbb{C}^n$ and $s \in \mathbb{N}\backslash\{0,1\}$. A technical question in complex analysis and geometry is the study of the plurisubharmonicity of u_1 defined by $u_1(z,w) = \prod_{1\leq j\leq s} |w - \overline{\varphi_j}(z)|$, for $(z,w) \in D \times \mathbb{C}$. Note that in the sequel, when $s \geq 4$, the proof is independent in my proof for the product of three or two modulus of pluriharmonic functions described as above.

For instance, in the sequel, basing on the above situation, we can study the complex structure (plurisubharmonicity) of the function v, defined by $v(z,w) = |w^k + \overline{f_{k-1}}(z)w^{k-1} + \cdots + \overline{f_1}(z)w + \overline{f_0}(z)|$, for $(z,w) \in D \times \mathbb{C}$. Here D is a domain of $\mathbb{C}^n$, $f_0, f_1, \ldots, f_{k-1}$ are holomorphic (respectively prh) functions on D and $k \geq 2$.

Now we have,

Remark 4.1. Let $g : \mathbb{C} \to \mathbb{C}$ be analytic and nonconstant. Then there exists 3 holomorphic nonconstant functions $g_1, g_2, g_3 : \mathbb{C} \to \mathbb{C}$, such that $g_1 g_2 g_3 = g^3$, $(g_1 \neq g_2, g_1 \neq g_3, g_2 \neq g_3)$, $(g_1 + g_2 + g_3)' = 0$, $(g_1 g_2 + g_1 g_3 + g_2 g_3)' = 0$ and u is psh on $\mathbb{C}^2$. Where $u(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)|$, for $(z,w) \in \mathbb{C}^2$.

The construction. Put $g_1(z) = ag(z), g_2(z) = bg(z), g_3(z) = cg(z)$, where $a, b, c \in \mathbb{C}$ to construct satisfying the above conditions. We choose a, b, $c \in \mathbb{C}$, such that $a + b + c = 0$, $ab + ac + bc = 0$. Take $a = 1$. Then $b + c = -1$, $bc = -(b+c)$. We solve the equation $X^2 - (b+c)X + bc = X^2 + X + 1 = 0$, on $\mathbb{C}$.

For example, $b = \dfrac{-1 - i\sqrt{3}}{2}$, $c = \dfrac{-1 + i\sqrt{3}}{2}$. In this case $g_1(z) = g(z)$, $g_2(z) = (\dfrac{-1 - i\sqrt{3}}{2})g(z)$, $g_3(z) = (\dfrac{-1 + i\sqrt{3}}{2})g(z)$, for $z \in \mathbb{C}$. We have then $u(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)|$. Therefore u is psh on $\mathbb{C}^2$.

Example. Define the three holomorphic functions on $\mathbb{C}$ by, $g_1(z) = z + 1$, $g_2(z) = \left(\dfrac{-1 - i\sqrt{3}}{2}\right) z + 1$ and $g_3(z) = \left(\dfrac{-1 + i\sqrt{3}}{2}\right) z + 1$, $z \in \mathbb{C}$. We have $g_1(z)g_2(z)g_3(z) = (z^3 + 1)$ and then $(g_1 g_2 g_3)' \neq 0$, on $\mathbb{C}$. $(g_1 \neq g_2, g_1 \neq g_3, g_2 \neq g_3)$. $(g_1 + g_2 + g_3) = 3$ and then $(g_1 + g_2 + g_3)' = 0$ on $\mathbb{C}$.

$(g_1g_2+g_1g_3+g_2g_3) = 3$ on $\mathbb{C}$ and therefore the function $(g_1g_2+g_1g_3+g_2g_3)' = 0$. But $v(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)| = |w^3-3w^2+3w-(\overline{z^3+1})|$, for $(z,w) \in \mathbb{C}^2$. Therefore, v is psh on $\mathbb{C}^2$.

Remark 4.2. Let $\varphi_1(z) = \varphi_2(z) = z$, $\varphi_3(z) = -z$, for $z \in \mathbb{C}$. φ_1, φ_2 and φ_3 are holomorphic functions on $\mathbb{C}$. Put

$$\begin{aligned} u(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|, \\ u_1(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)|, \\ u_2(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_3}(z)| \text{ and} \\ u_3(z,w) &= |w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|, \quad \text{for } (z,w) \in \mathbb{C}^2. \end{aligned}$$

Then u_1, u_2 and u_3 are psh functions on $\mathbb{C}^2$. But u is not psh on $\mathbb{C}^2$.

Remark 4.3. For 4 functions, Theorem 4.1 is not true.

Example. $g_1(z) = z = g_2(z)$, $g_3(z) = g_4(z) = -z$, for $z \in \mathbb{C}$. g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$. Let

$$\begin{aligned} v(z,w) &= |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)| \\ &= |w-\overline{z}|^2|w+\overline{z}|^2 = |w^4-2\overline{z}^2w^2+\overline{z}^4| = |w^2-\overline{z^2}|^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^2$. v is psh on $\mathbb{C}^2$, but $g_1 \neq g_3$, $g_2 \neq g_4$. Indeed, we have

$$\begin{aligned} (g_1+g_2+g_3+g_4)' &= 0, \\ (g_1g_2+g_1g_3+g_1g_4+g_2g_3+g_2g_4+g_3g_4)' &\neq 0, \\ (g_1g_2g_3+g_1g_2g_4+g_1g_3g_4+g_2g_3g_4)' &= 0, \\ (g_1g_2g_3g_4)' &\neq 0, \quad \text{on } \mathbb{C}. \end{aligned}$$

But v is psh on $\mathbb{C}^2$. In particular we have

Example. Let $\varphi_1(z) = z$, $\varphi_2(z) = -z$, $\varphi_3(z) = 1$ and $\varphi_4(z) = -1$, for $z \in \mathbb{C}$. φ_1, φ_2, φ_3 and φ_4 are holomorphic functions on $\mathbb{C}$. Define

$$\begin{aligned} u(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)||w-\overline{\varphi_4}(z)|, \\ u_1(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|, \\ u_2(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_4}(z)|, \\ u_3(z,w) &= |w-\overline{\varphi_1}(z)||w-\overline{\varphi_3}(z)||w-\overline{\varphi_4}(z)| \text{ and} \\ u_4(z,w) &= |w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)||w-\overline{\varphi_4}(z)|, \quad \text{for } (z,w) \in \mathbb{C}^2. \end{aligned}$$

We have u_1, u_2, u_3 and u_4 are not psh functions on $\mathbb{C}^2$. But u is psh on $\mathbb{C}^2$. Consequently, for 4 functions there exists another characterization.

Let us fix $N \in \mathbb{N}$, $N \geq 4$. Thanks to the above situations, for N holomorphic functions, we have another conditions to describe separately and which are different from the above situations studied in different cases (Proposition 4.1, Theorem 4.1 and the three above remarks).

Theorem 4.2. *Let D be a domain of $\mathbb{C}^n$, $n \geq 1$. Consider $g_1, \ldots, g_N : D \to \mathbb{C}^n$ be N analytic functions, $N \geq 1$. Let*

$$u(z,w) = \sum_{j=1}^{N} (\|w - g_j(z)\|^2 + \|w - \overline{g_j}(z)\|^2), \quad v(z,w) = \sum_{j=1}^{N} (\|w - \overline{g_j}(z)\|^2),$$

$(z,w) \in D \times \mathbb{C}^n$. Then u is strictly psh on $D \times \mathbb{C}^n$ if and only if v is strictly psh on $D \times \mathbb{C}^n$.

Corollary 4.1. *Let $g_1, \ldots, g_N : D \to \mathbb{C}$ be N analytic functions, D is a domain of $\mathbb{C}^n$, $n, N \geq 1$. Given $A_j, B_j \in \mathbb{R}_+ \backslash \{0\}$ and $a_j, b_j \in \mathbb{C}$, $1 \leq j \leq N$. Define*

$$u(z,w) = \sum_{j=1}^{N} (A_j |w - g_j(z) + a_j|^2 + B_j |w - \overline{g}_j(z) + b_j|^2),$$
$$v(z,w) = \sum_{j=1}^{N} (|w - \overline{g}_j(z)|^2), \quad \text{for } (z,w) \in D \times \mathbb{C}.$$

We have

(I) *u is strictly psh on $D \times \mathbb{C}$ if and only if v is strictly psh on $D \times \mathbb{C}$.*

(II) *Assume that $N \leq n$. Then u and v are not strictly convex on each not empty Euclidean open ball subset of $D \times \mathbb{C}$.*

Analogously, let $D = B(a,R)$ be an open ball of $\mathbb{C}^n$, $(a \in \mathbb{C}^n, R > 0)$. Let $\varphi : D \to \mathbb{C}$ be a function. Put $\psi(z,w) = |w - \varphi(z)|^2 + |w - \overline{\varphi}(z)|^2$, $\psi_1(z,w) = |w - \varphi(z)|^2$, $\psi_2(z,w) = |w - \overline{\varphi}(z)|^2$, for $(z,w) \in D \times \mathbb{C}$. Then ψ is convex on $D \times \mathbb{C}$ if and only if ψ_1 (respectively ψ_2) is convex on $D \times \mathbb{C}$. While we have

Theorem 4.3. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Put*

$$u(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2 + |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2,$$
$$u_1(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2,$$
$$u_2(z,w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2, \quad \text{for } (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

We have the assertions

(I) *u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n \in \{1,2\}$ and u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and there exists several cases where u_1 is not strictly psh. But*

(II) *Suppose that u is strictly convex on $\mathbb{C}^n \times \mathbb{C}$. Then we can not conclude that u_2 (respectively, u_1) is strictly convex on $\mathbb{C}^n \times \mathbb{C}$.*

(III) *Assume that u is convex on $\mathbb{C}^n \times \mathbb{C}$. Then we can not deduce that u_2 (respectively, u_1) is convex on $\mathbb{C}^n \times \mathbb{C}$, for each $n \geq 1$.*

Proof. (III) Without loss of generality we suppose that $n = 1$.

We can write

$$\begin{aligned} 2u_1(z,w) &= |2w - g_1(z) - g_2(z)|^2 + |g_1(z) - g_2(z)|^2, \\ 2u_2(z,w) &= |2w - \overline{g_1}(z) - \overline{g_2}(z)|^2 + |g_1(z) - g_2(z)|^2, \end{aligned}$$

for $(z,w) \in \mathbb{C}^2$. In this situation we have

$$\begin{aligned} 4u(z,w) = |4w - g_1(z) - g_2(z) - \overline{g_1}(z) - \overline{g_2}(z)|^2 + |g_1(z) + g_2(z) \\ - \overline{g_1}(z) - \overline{g_2}(z)|^2 + 4|g_1(z) - g_2(z)|^2. \end{aligned}$$

Since u is convex on $\mathbb{C}^2$, observe now that the functions $(g_1 + g_2)$, $(\overline{g_1} + \overline{g_2})$, $(g_1 + g_2 + \overline{g_1} + \overline{g_2})$ and $(g_1 + g_2 - \overline{g_1} - \overline{g_2})$ are affine on $\mathbb{C}^2$. $4u$ (or u) is the sum of a C^∞ convex function φ_1 on $\mathbb{C}^2$

$$(\varphi_1(z,w) = |4w - g_1(z) - g_2(z) - \overline{g_1}(z) - \overline{g_2}(z)|^2)$$

and a C^∞ convex function φ_2 on $\mathbb{C}$

$$(\varphi_2(z) = |g_1(z) + g_2(z) - \overline{g_1}(z) - \overline{g_2}(z)|^2 + 4|g_1(z) - g_2(z)|^2, z \in \mathbb{C}).$$

Now we consider the following example.

Let $g_1(z) = z + \frac{1}{2}(z^2 + 1)$ and $g_2(z) = z - \frac{1}{2}(z^2 + 1)$, for $z \in \mathbb{C}$. Using the above notation for u, u_1 and u_2. We have u is convex on $\mathbb{C}^2$. In fact, because φ_2 is strictly convex on $\mathbb{C}$ and the function $\varphi_1(z, .)$ is strictly convex on $\mathbb{C}$, for each $z \in \mathbb{C}$, it follows that u is strictly convex on $\mathbb{C}^2$. But u_1 is not convex in all $D\left(0, \frac{\sqrt{3}}{3}\right) \times \mathbb{C}$, because for example ψ is not convex in all $D\left(0, \frac{\sqrt{3}}{3}\right)$ where $\psi(z) = u_1(z,z)$, for $z \in \mathbb{C}$. Observe that u_2 is also not convex on $\mathbb{C}^2$. □

Remark 4.4. For $N \in \mathbb{N}$, $N \geq 3$, we consider $\varphi_1(z) = iz + \frac{1}{2}(z^2 - 1)$, $\varphi_2(z) = iz - \frac{1}{2}(z^2 - 1)$, $\varphi_3(z) = \cdots = \varphi_N(z) = 0$, for $z \in \mathbb{C}$.

Put $v_1(z,w)=\sum_{j=1}^{N}|w-\varphi_j(z)|^2$, $v_2(z,w)=\sum_{j=1}^{N}|w-\overline{\varphi_j}(z)|^2$, for $(z,w)\in\mathbb{C}^2$.

Let $v=(v_1+v_2)$. Then v is convex on $\mathbb{C}^2$, but v_1 and v_2 are not convex on $\mathbb{C}^2$. Now, we obtain there some interesting and sharp results in the framework of theorem 4.3, but in a slightly different direction. We list them as questions.

Question 1. Let $\varphi:\mathbb{C}\to\mathbb{C}$ be a holomorphic nonconstant function and $N\geq 1$. Given $f_1,g_1,\varphi_1,\psi_1,\ldots,f_N,g_N,\varphi_N,\psi_N:\mathbb{C}^n\to\mathbb{C}$ be $4N$ holomorphic functions. Define

$$\begin{aligned}
F_j(z,w) &= |\varphi(w)-f_j(z)|^2+|\varphi(w)-g_j(z)|^2,\\
K_j(z,w) &= |\varphi(w)-\overline{f_j}(z)|^2+|\varphi(w)-\overline{g_j}(z)|^2,\\
S_j(z,w) &= |\varphi(w)-\varphi_j(z)|^2+|\varphi(w)-\psi_j(z)|^2,\\
T_j(z,w) &= |\varphi(w)-\overline{\varphi_j}(z)|^2+|\varphi(w)-\overline{\psi_j}(z)|^2,
\end{aligned}$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$ and $1\leq j\leq N$.

Put $u_j=(F_j+K_j)$, $v_j=(S_j+T_j)$, $u=\sum_{j=1}^{N}u_j$ and $v=\sum_{j=1}^{N}v_j$.

Find all the holomorphic functions φ, f_j,g_j,φ_j,ψ_j, $(1\leq j\leq N)$, such that we have the following three conditions.

(a) F_j, K_j, S_j and T_j are not convex functions on $\mathbb{C}^n\times\mathbb{C}$, for each $j\in\{1,\ldots,N\}$.

(b) u_j and v_j are convex functions on $\mathbb{C}^n\times\mathbb{C}$, for every $j\in\{1,\ldots,N\}$.

(c) u and v are not strictly psh on $\mathbb{C}^n\times\mathbb{C}$, but $\theta=(u+v)$ is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. In the sequel, we can replace all the holomorphic functions by pluriharmonic (prh) functions.

Question 2. Let $n,d\in\mathbb{N}\backslash\{0\}$. Find all the holomorphic functions $\varphi_1,\varphi_2:\mathbb{C}^d\to\mathbb{C}$, $f_1,f_2,g_1,g_2:\mathbb{C}^n\to\mathbb{C}$ such that if we put

$$\begin{aligned}
u_1(z,w) &= |\varphi_1(w)-f_1(z)|^2+|\varphi_2(w)-f_2(z)|^2,\\
u_2(z,w) &= |\varphi_1(w)-\overline{f_1}(z)|^2+|\varphi_2(w)-\overline{f_2}(z)|^2,\\
v_1(z,w) &= |\varphi_1(w)-g_1(z)|^2+|\varphi_2(w)-g_2(z)|^2,\\
v_2(z,w) &= |\varphi_1(w)-\overline{g_1}(z)|^2+|\varphi_2(w)-\overline{g_2}(z)|^2,
\end{aligned}$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}^d$, $u=(u_1+u_2)$ and $v=(v_1+v_2)$.

We can study the problem u_1, u_2, v_1 and v_2 are not convex functions, but u and v are convex functions with $(u+v)$ is strictly psh on $\mathbb{C}^n\times\mathbb{C}^d$.

Note that we can consider the same problem if $\varphi_1,\varphi_2,f_1,f_2,g_1,g_2$ are pluriharmonic functions.

In the sequel, by theorem 4.3 we observe that we have

Proposition 4.2. *For each $c \in \mathbb{C}\backslash\{0\}$, there exists two holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that, $(g_1 + g_2)$ is an affine function, u is convex on $\mathbb{C}^n \times \mathbb{C}$, but u_1 and u_2 are not convex on $\mathbb{C}^n \times \mathbb{C}$. Where*

$$\begin{aligned} u(z,w) &= |w-g_1(z)|^2 + |w-g_2(z)|^2 + |w-\overline{g_1}(z)|^2 + |w-\overline{g_2}(z)|^2, \\ u_1(z,w) &= |w-g_1(z)|^2 + |w-g_2(z)|^2 \quad \textit{and} \\ u_2(z,w) &= |w-\overline{g_1}(z)|^2 + |w-\overline{g_2}(z)|^2, \quad \textit{for } (z,w) \in \mathbb{C}^n \times \mathbb{C}. \end{aligned}$$

But we have

$$\left| c^2 \left(\sum_{j=1}^{n} \alpha_j \right)^2 + 4(\overline{g_1}(z) - \overline{g_2}(z)) \sum_{j,k=1}^{n} \frac{\partial^2 (g_1 - g_2)(z)}{\partial z_j \partial z_k} \alpha_j \alpha_k \right|$$

$$\leq |c|^2 \left| \sum_{j=1}^{n} \alpha_j \right|^2 + 4 \left| \sum_{j=1}^{n} \frac{\partial (g_1 - g_2)(z)}{\partial z_j} \alpha_j \right|^2$$

for each $z = (z_1, \ldots, z_n), \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

P r o o f. Obvious by the proof of Theorem 4.3. □

Observe that we have

Corollary 4.2. *For each $c \in \mathbb{C}\backslash\{0\}$, there exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $|g|^2$ is not convex on $\mathbb{C}$, but we have*

$$|c^2 + g''(z)\overline{g}(z)| \leq |c|^2 + |g'(z)|^2$$

for every $z \in \mathbb{C}$.

In pluripotential theory, we have

Theorem 4.4. *For all $n \geq 1$, there does not exists a function $u : \mathbb{C}^n \to [-\infty, +\infty[$ such that K is strictly psh (or convex and strictly psh) on a not empty Euclidean open ball subset of $\mathbb{C}^n \times \mathbb{C}^n$, where $K(z,w) = u(w-z)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$.*

P r o o f. Let $u : \mathbb{C}^n \to [-\infty, +\infty[$. Put $K(z,w) = u(w-z)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$.

Case 1. u is a function of class C^2 on $\mathbb{C}^n$, (then $u : \mathbb{C}^n \to \mathbb{R}$). Therefore K is a function of class C^2 on $\mathbb{C}^n \times \mathbb{C}^n$. Let $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$ and

$\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}) \in \mathbb{C}^{2n}$. The Levi hermitian form of u is

$$L(u)(z)(\alpha_{n+1} - \alpha_1, \ldots, \alpha_{2n} - \alpha_n) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(z)(\alpha_{n+j} - \alpha_j)(\overline{\alpha_{n+k} - \alpha_k})$$

$$= \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(z)\alpha_j \overline{\alpha_k} + \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(z)\alpha_{n+j}\overline{\alpha_{n+k}} - 2\mathrm{Re}\left[\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(z)\alpha_j \overline{\alpha_{n+k}}\right].$$

Put $z = (z_1, \ldots, z_n), w = (z_{n+1}, \ldots, z_{2n})$. The hermitian Levi form of K is

$$\begin{aligned} L(K)(z,w)(\alpha) &= \sum_{j,k=1}^{2n} \frac{\partial^2 K}{\partial \xi_j \partial \overline{\xi_k}}(z,w)\alpha_j \overline{\alpha_k} = \sum_{j,k=1}^{n} \frac{\partial^2 K}{\partial \xi_j \partial \overline{\xi_k}}(z,w)\alpha_j \overline{\alpha_k} \\ &\quad + \sum_{j,k=n+1}^{2n} \frac{\partial^2 K}{\partial \xi_j \partial \overline{\xi_k}}(z,w)\alpha_j \overline{\alpha_k} + \sum_{j=1}^{n}\sum_{k=n+1}^{2n} \frac{\partial^2 K}{\partial \xi_j \partial \overline{\xi_k}}(z,w)\alpha_j \overline{\alpha_k} \\ &\quad + \sum_{j=n+1}^{2n}\sum_{k=1}^{n} \frac{\partial^2 K}{\partial \xi_j \partial \overline{\xi_k}}(z,w)\alpha_j \overline{\alpha_k} \\ &= \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_j \overline{\alpha_k} + \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_{j+n} \overline{\alpha_{k+n}} \\ &\quad - \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_j \overline{\alpha_{k+n}} - \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_{n+j} \overline{\alpha_k} \\ &= \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_j \overline{\alpha_k} + \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_{j+n} \overline{\alpha_{k+n}} \\ &\quad - 2\mathrm{Re}\left[\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial \xi_j \partial \overline{\xi_k}}(w-z)\alpha_j \overline{\alpha_{k+n}}\right], \end{aligned}$$

for each $\alpha \neq 0$. But if $\alpha_{n+1} = \alpha_1, \ldots, \alpha_{2n} = \alpha_n$, we have

$$\begin{aligned} &L(K)(z,w)(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}) \\ &\qquad = L(u)(w-z)(\alpha_{n+1} - \alpha_1, \ldots, \alpha_{2n} - \alpha_n) = 0, \end{aligned}$$

independently of $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. Consequently, K is not strictly psh at each point of $\mathbb{C}^n \times \mathbb{C}^n$.

Case 2. $u : \mathbb{C}^n \to [-\infty, +\infty[$.

State 1. Without loss of generality, assume that K is strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$ for the simplicity. For $z_0, w_0 \in \mathbb{C}^n$, we have the functions $K(z_0, .) \neq -\infty$ and $K(., w_0) \neq -\infty$ on $\mathbb{C}^n$. For example suppose that $K(z_0, .) = -\infty$ on $\mathbb{C}^n$, where $z_0 \in \mathbb{C}^n$. Let $\rho : \mathbb{C}^n \to \mathbb{R}_+$ be a function of class C^∞, ρ is a radial function, $\mathrm{supp}(\rho) \subset B(0,1)$ and $\int_{\mathbb{C}^n} \rho(\xi) dm_{2n}(\xi) = 1$. Put $\rho_\delta(\xi) = \frac{1}{\delta^{2n}} \rho\left(\frac{\xi}{\delta}\right)$, for $\xi \in \mathbb{C}^n$ and $\delta \in \mathbb{R}_+ \backslash \{0\}$. Note that ρ_δ is a function of class C^∞, ρ_δ is a radial function, $\mathrm{supp}(\rho_\delta) \subset B(0,\delta)$ and $\int_{\mathbb{C}^n} \rho_\delta(\xi) dm_{2n}(\xi) = 1$. Then

$$\begin{aligned}
-\infty &= (K(z_0, .) * \rho_\delta)(w) = \int K(z_0, w - \xi) \rho_\delta(\xi) dm_{2n}(\xi) \\
&= \int u(w - \xi - z_0) \rho_\delta(\xi) dm_{2n}(\xi) = \int u(w - \zeta - \xi - (z_0 - \zeta)) \rho_\delta(\xi) dm_{2n}(\xi),
\end{aligned}$$

for all $\zeta \in \mathbb{C}^n$. Then $K(z_0 - \zeta, w - \zeta) = -\infty$, $\forall w \in \mathbb{C}^n$. Consequently, the function $K(z_0 - \zeta, .) = -\infty$ on $\mathbb{C}^n$, $\forall \zeta \in \mathbb{C}^n$. Then $K = -\infty$ on $\mathbb{C}^n \times \mathbb{C}^n$. A contradiction.

We deduce now that for all $z \in \mathbb{C}^n$, the function $K(z, .) \neq -\infty$ on $\mathbb{C}^n$. By the same method we conclude that for all $w \in \mathbb{C}^n$, the function $K(., w) \neq -\infty$ on $\mathbb{C}^n$. Consequently, the functions $K(z, .)$ and $K(., w)$ are psh on $\mathbb{C}^n$.

State 2. Let $\delta > 0$. Consider now

$$\begin{aligned}
(K(z, .) * \rho_\delta)(w) &= \int K(z, w - \xi) \rho_\delta(\xi) dm_{2n}(\xi) = \int u(w - \xi - z) \rho_\delta(\xi) dm_{2n}(\xi) \\
&= u * \rho_\delta(w - z) = \varphi_\delta(w - z) = K_\delta(z, w),
\end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. $\varphi_\delta = u * \rho_\delta$ is then a C^∞ function on $\mathbb{C}^n$. $K_\delta(z, w) = (K(z, .) * \rho_\delta)(w)$. Since K is strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$, then K_δ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$.

By the case 1, we have a contradiction. Therefore K is not strictly psh on each Euclidean not empty open ball subset of $\mathbb{C}^n \times \mathbb{C}^n$. □

Corollary 4.3. *For all $u : \mathbb{C}^n \to \mathbb{R}$, define $v(z, w) = u(w - \overline{z})$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Then the function v is not strictly convex on each not empty Euclidean open ball subset of $\mathbb{C}^n \times \mathbb{C}^n$.*

Note that there exists several cases where v is strictly psh (or convex and strictly psh) on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. Assume that v is strictly convex on a convex domain $G \subset \mathbb{C}^n \times \mathbb{C}^n$, $G \neq \emptyset$. Consider $T : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$, defined by $T(z, w) =$

$(\overline{z}, w)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Then T is an $\mathbb{R}-$ linear bijective transformation. It follows that $v_1 = voT$ is strictly convex on $T^{-1}(G)$. But $v_1(z, w) = u(w - z)$. Therefore v_1 is strictly psh on $T^{-1}(G)$ and $T^{-1}(G) \neq \emptyset$. A contradiction by Theorem 4.4. □

There exists a fundamental relation between pluriharmonicity and the strict plurisubharmonicity, however we deduce the next useful theorem.

Theorem 4.5. *Let $h_1, \dots, h_N : \mathbb{C}^n \to \mathbb{R}$ be N pluriharmonic functions (of real valued), where $n, N, s \in \mathbb{N}\backslash\{0\}$. Suppose that $N \leq n - 1$. Then $u = (h_1^{2s} + \cdots + h_N^{2s})$ is not strictly psh at each point of $\mathbb{C}^n$. Consequently, u is not strictly psh on every not empty Euclidean open ball subset of $\mathbb{C}^n$.*

Proof. Obvious by using the Levi hermitian form of the C^∞ function u. □

The next theorem gives a characterization to obtain the real strict convexity, using the classical hermitian product and recall some properties of the geometry, which we shall use later. This properties reveals both the originality and richness of this inequalities in complex function theory.

Theorem 4.6. *Let $n, N \geq 1$.*

(I) *Put $u(z) = |\langle z/a\rangle + \langle b/z\rangle + c|^2$, for $z \in \mathbb{C}^n$, where $a, b \in \mathbb{C}^n$, $c \in \mathbb{C}$. Then, u is strictly convex on $\mathbb{C}^n$ if and only if $n = 1$ and $|2ab| < |a|^2 + |b|^2$.*

(II) *Let $a_1, \dots, a_N \in \mathbb{C}^n$, $c_1, \dots, c_N \in \mathbb{C}$. Put*

$$v(z) = |\langle z/a_1\rangle + \langle a_1/z\rangle + c_1|^2 + \cdots + |\langle z/a_N\rangle + \langle a_N/z\rangle + c_N|^2.$$

Assume that $N \leq 2n$. Then v is strictly convex on $\mathbb{C}^n$ if and only if $N = 2n$ and

$$\sum_{j,k=1}^{N} \langle \alpha/a_j\rangle^2 \langle a_k/\alpha\rangle^2 < \sum_{j,k=1}^{N} |\langle \alpha/a_j\rangle|^2 |\langle a_k/\alpha\rangle|^2,$$

for each $\alpha \in \mathbb{C}^n\backslash\{0\}$. We can also consider

$$\psi(z) = |\langle z/a_1\rangle - \langle a_1/z\rangle + c_1|^2 + \cdots + |\langle z/a_N\rangle - \langle a_N/z\rangle + c_N|^2.$$

But we have the following.

(III) *Let $a_1, \dots, a_n \in \mathbb{C}^n$, $c_1, \dots, c_n \in \mathbb{C}$. Then there exists $b_1, \dots, b_n \in \mathbb{C}^n$, such that v is strictly convex on $\mathbb{C}^n$. Where*

$$v(z) = |\langle z/a_1\rangle + \langle b_1/z\rangle + c_1|^2 + \cdots + |\langle z/a_n\rangle + \langle b_n/z\rangle + c_n|^2,$$

for $z \in \mathbb{C}^n$.

(IV) *Let* $a_1, \ldots, a_n \in \mathbb{C}^n$, $c_1, \ldots, c_n \in \mathbb{C}$. *Put*

$$v(z) = |\langle z/a_1\rangle + c_1|^2 + \cdots + |\langle z/a_n\rangle + c_n|^2,$$

for $z \in \mathbb{C}^n$. *Then* v *is strictly convex on* $\mathbb{C}^n$ *if and only if* $(a_1, \ldots, a_n)$ *is a basis of* $\mathbb{C}^n$.

Proof. (I) We have u is a function of class C^∞ on $\mathbb{C}^n$. Now assume that u is strictly convex on $\mathbb{C}^n$. Then

$$\left|\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\right| < \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k},$$

for each $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n\backslash\{0\}$. Therefore,

$$|2\langle\alpha/a\rangle\langle\alpha/b\rangle| < |\langle\alpha/a\rangle|^2 + |\langle\alpha/b\rangle|^2,$$

for every $\alpha \in \mathbb{C}^n\backslash\{0\}$. Thus $(|\langle\alpha/a\rangle|^2 - |\langle\alpha/b\rangle|^2)^2 > 0$, for each $\alpha \in \mathbb{C}^n\backslash\{0\}$.

Suppose that $n \geq 2$. Let $\alpha \in \mathbb{C}^n\backslash\{0\}$, such that $\langle\alpha/a-b\rangle = 0$, since $(a-b)$ is a one vector in the complex vector space $\mathbb{C}^n$ and $n \geq 2$. Then $\langle\alpha/a\rangle = \langle\alpha/b\rangle$ and consequently, $|\langle\alpha/a\rangle| = |\langle\alpha/b\rangle|$. A contradiction. It follows that $n = 1$.

We have $2|\alpha\overline{a}\alpha\overline{b}| < |\alpha\overline{a}|^2 + |\alpha\overline{b}|^2$, $\forall\alpha \in \mathbb{C}\backslash\{0\}$. Then $2|ab| < |a|^2 + |b|^2$. The converse is obvious.

(II) v is a function of class C^∞ on $\mathbb{C}^n$. Assume that v is strictly convex on $\mathbb{C}^n$. Therefore,

$$\left|\sum_{j=1}^{N}\langle\alpha/a_j\rangle^2\right| < \sum_{j=1}^{N}|\langle\alpha/a_j\rangle|^2$$

for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n\backslash\{0\}$. Thus

$$\sum_{j,k=1}^{N} \langle\alpha/a_j\rangle^2\langle a_k/\alpha\rangle^2 < \sum_{j,k=1}^{N} |\langle a_j/\alpha\rangle|^2|\langle\alpha/a_k\rangle|^2$$

for each $\alpha \in \mathbb{C}^n\backslash\{0\}$.

Assume that $N < 2n$. Consider now $\psi : \mathbb{C}^n \to \mathbb{R}^N$, defined by

$$\psi(\alpha) = (\langle\alpha/a_1\rangle + \langle a_1/\alpha\rangle, \ldots, \langle\alpha/a_N\rangle + \langle a_N/\alpha\rangle),$$

for $\alpha \in \mathbb{C}^n$. ψ is $\mathbb{R}-$ linear on $\mathbb{C}^n$, (we consider $\mathbb{C}^n$ an $\mathbb{R}$ vector space of real dimension $2n$). Since $N < 2n$, then ψ is not injective. Therefore, there exists $\alpha \in \mathbb{C}^n\backslash\{0\}$, such that $\psi(\alpha) = 0$. Then $\langle\alpha/a_j\rangle + \langle a_j/\alpha\rangle = 0$, for each

$j \in \{1, \ldots, N\}$. Thus $|\langle \alpha/a_j\rangle + \langle a_j/\alpha\rangle|^2 = 0$, for every $1 \le j \le N$. It follows that $2|\langle \alpha/a_j\rangle|^2 + 2\mathrm{Re}(\langle \alpha/a_j\rangle^2) = 0$ and then $|\langle \alpha/a_j\rangle|^2 = -\mathrm{Re}(\langle \alpha/a_j\rangle^2)$, for all $j \in \{1, \ldots, N\}$. Thus $\sum_{j=1}^{N} |\langle \alpha/a_j\rangle|^2 = -\mathrm{Re}\left(\sum_{j=1}^{N} \langle \alpha/a_j\rangle^2\right) \ge 0$. We obtain

$$\sum_{j=1}^{N} |\langle \alpha/a_j\rangle|^2 = \left| -\mathrm{Re}\left(\sum_{j=1}^{N} \langle \alpha/a_j\rangle^2\right)\right| \le \left|\sum_{j=1}^{N} \langle \alpha/a_j\rangle^2\right| < \sum_{j=1}^{N} |\langle \alpha/a_j\rangle|^2.$$

A contradiction. Consequently, $2n \le N$. Since $N \le 2n$, then $N = 2n$. □

We have

Claim 4.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^N$ be $\mathbb{R}-$ linear, $\psi = (\psi_1, \ldots, \psi_N)$, $n, N \ge 1$. We have the following assertions.*

(A) *$\|\psi\|^2$ is strictly convex on $\mathbb{R}^n$ if and only if $\|\psi\|^2$ is strictly convex at* 0, *(because $\frac{\partial \psi_k}{\partial x_j}(x)$ is independent of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, for each $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, N\}$).*

(B) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^N$, $\varphi = (\varphi_1, \ldots, \varphi_N)$, where $\varphi_k(b) = \sum_{j=1}^{n} \frac{\partial \psi_k}{\partial x_j}(0) b_j$, for $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $1 \le k \le N$.*

Then $\|\psi\|^2$ is strictly convex on $\mathbb{R}^n$ if and only if φ is injective (therefore $n \le N$).

P r o o f. We use the bilinear form associate to $\|\psi\|^2$ and the above proof. □

In the sequel, we conclude the following analysis corollary.

Corollary 4.4. *Let $n, N \in \mathbb{N}$, $N \le 2n-1$. Then for all $a_1, \ldots, a_N \in \mathbb{C}^n$, there exists $\alpha \in \mathbb{C}^n \backslash \{0\}$ such that*

$$\sum_{j,k=1}^{N} \langle \alpha/a_j\rangle^2 \langle a_k/\alpha\rangle^2 = \sum_{j,k=1}^{N} |\langle \alpha/a_j\rangle|^2 |\langle a_k/\alpha\rangle|^2.$$

Remark 4.5. Let

$u(z) = |\langle z/a_1\rangle + \langle a_1/z\rangle + c_1|^2 + |\langle z/a_2\rangle + \langle a_2/z\rangle + c_2|^2 + |\langle z/a_3\rangle + \langle a_3/z\rangle + c_3|^2$, $z \in \mathbb{C}^2$. Then u is not strictly convex at any point of $\mathbb{C}^2$. But there exists $A_1, B_1, A_2, B_2, (\lambda_1, \lambda_2) \in \mathbb{C}^2$, such that v is strictly convex on $\mathbb{C}^2$, where

$$v(z, w) = |\langle z/A_1\rangle + \langle B_1/z\rangle + \lambda_1|^2 + |\langle z/A_2\rangle + \langle B_2/z\rangle + \lambda_2|^2, \quad \text{for } z \in \mathbb{C}^2.$$

Remark 4.6. For the complex structure, let $f_1, \ldots, f_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions, $n, N \geq 1$. We have $(|f_1|^2 + \cdots + |f_N|^2)$ is strictly psh on $\mathbb{C}^n$ if and only if $(|f_1 + \overline{f_1}|^2 + \cdots + |f_N + \overline{f_N}|^2)$ is strictly psh on $\mathbb{C}^n$ (if and only if $(|f_1 - \overline{f_1}|^2 + \cdots + |f_N - \overline{f_N}|^2)$ is strictly psh on $\mathbb{C}^n$). But for the real convexity, this result is not true.

Example. $n = 2$. Let $f_1(z) = z_1$, $f_2(z) = z_2$, for $z = (z_1, z_2) \in \mathbb{C}^2$. Put $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $1 \leq j \leq 2$. $u = (|f_1|^2 + |f_2|^2)$ is strictly convex on $\mathbb{C}^2$. But if we put $v(z) = (|f_1(z) + \overline{f_1}(z)|^2 + |f_2(z) + \overline{f_2}(z)|^2) = 4x_1^2 + 4x_2^2$. Observe that v is not strictly convex at each point of $\mathbb{C}^2$.

Remark 4.7. Let $m, N \in \mathbb{N}\backslash\{0\}$, $a_1, \ldots, a_m, b_1, \ldots, b_N \in \mathbb{C}$.
Let $u(z, w) = \sum_{j=1}^{m} A_j |w - \overline{z} + a_j|^2 + \sum_{k=1}^{N} B_j |w - z + b_j|^2$, where $A_j \geq 0$, $B_k \geq 0$, $1 \leq j \leq m$, $1 \leq k \leq N$, such that $\sum_{j=1}^{m} A_j > 0$, $\sum_{k=1}^{N} B_j > 0$. u is strictly psh and convex on $\mathbb{C}^2$, but u is not strictly convex on each not empty Euclidean open ball of $\mathbb{C}^2$.

4.1. The absolute value and holomorphic or plurisubharmonic functions.

Proposition 4.3. *Let $\varphi : D \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$ and $\lambda \in \mathbb{R}$.*
Denote by $u(\xi, w) = |w - |\varphi(\xi)||$, $v(z) = |g(z) - \lambda|g(z)||^2$, for $(\xi, z, w) \in D \times \mathbb{C}^n \times \mathbb{C}$. We have

(I) *Suppose that u is psh on $D \times \mathbb{C}$. Then φ is constant on D.*

(II) *v is psh on $\mathbb{C}^n$, for each $\lambda \in \mathbb{R}$.*

Proof. (I) Since u is psh on $D \times \mathbb{C}$, then $|\varphi|$ is prh on D, by Abidi [1]. Assume that $\varphi \neq 0$. Therefore $\varphi(\xi) \neq 0$, for each $\xi \in D\backslash E$, where E is a pluripolar subset of D. Now since φ is holomorphic on D, it is easy to prove that φ is constant on D.
(II) Without loss of generality, we assume that $n = 1$. If $g = 0$ on $\mathbb{C}$, then g is constant and the proof is finished. Assume that $g \neq 0$ on $\mathbb{C}$. Let $z \in \mathbb{C}$ such that $g(z) \neq 0$. $v = (1 + \lambda^2)|g|^2 - \lambda|g|(g + \overline{g})$, on $\mathbb{C}$. We have then,

$$\frac{\partial^2 v}{\partial z \partial \overline{z}} = (1 + \lambda^2)|g'|^2 - \frac{3}{4}\lambda|g'|^2(\frac{g + \overline{g}}{|g|}) = \frac{|g'|^2}{|g|}[(1 + \lambda^2)|g| - \frac{3}{2}\lambda \mathrm{Re}(g)],$$

in a neighborhood of z. Now since $\dfrac{\lambda}{1+\lambda^2} < \dfrac{2}{3}$, then

$$|g| \geq |\mathrm{Re}(g)| \geq |\frac{3}{2}(\frac{\lambda}{1+\lambda^2})||\mathrm{Re}(g)| \geq +\frac{3}{2}(\frac{\lambda}{1+\lambda^2})\mathrm{Re}(g),$$

in a neighborhood of z. Thus $\dfrac{\partial^2 v}{\partial z \partial \overline{z}}(z) \geq 0$ in a small neighborhood of z. Since the function $[(1+\lambda^2)|g|^2 - \lambda|g|(g+\overline{g})]$ is continuous on $\mathbb{C}$ and the set $E = \{\xi \in \mathbb{C}/g(\xi) = 0\}$ is polar in $\mathbb{C}$, the proof is complete. □

Note that proposition 4.3 is only true for analytic functions. For example if we take $h(z) = x_1$, $u(z,w) = |w - |h(z)||$, for $(z,w) \in G \times \mathbb{C}$, $z = (x_1 + iy_1) \in G = D(1, \frac{1}{2})$, where $x_1 = \mathrm{Re}(z)$. Then h is harmonic on G. The function u is psh on $G \times \mathbb{C}$. But h is not constant on the domain G.

Example. Let $h_1(z) = \mathrm{Re}(e^{(z_1)})$, $h_2(z) = \mathrm{Re}(e^{(-z_1)})$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Put $z_1 = x_1 + iy_1$, $x_1, y_1 \in \mathbb{R}$. Then $h_1(z)h_2(z) = [\cos(y_1)]^2 \geq 0$, for each $z \in \mathbb{C}^n$. Note that h_1 and h_2 are real valued pluriharmonic functions on $\mathbb{C}^n$. But $h_2 \neq \lambda h_1$, for each $\lambda \in \mathbb{R}$. In general we have the following technical extension in the theory of functions. Moreover, it is important to consider the global domain $\mathbb{C}^n$.

Theorem 4.7. *Let $h : \mathbb{C}^n \to \mathbb{C}$ be harmonic. Put $u(z,w) = |w - |h(z)||$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that u is psh on $\mathbb{C}^n \times \mathbb{C}$. Then h is constant on $\mathbb{C}^n$.*

Comparing with the above situations, this theorem proves the importance of the classical global domain $\mathbb{C}^n$ (which plays a key role in several problems of complex analysis).

P r o o f. By [1], it follows that $|h|$ is pluriharmonic on $\mathbb{C}^n$. Since $|h| \geq 0$ on $\mathbb{C}^n$, then $|h| = c$, $c \in \mathbb{R}_+$. Then $\dfrac{\partial^2 |h|^2}{\partial z_j \partial \overline{z_j}} = 0$, on $\mathbb{C}^n$, for each $j \in \{1, \ldots, n\}$. Therefore, $\displaystyle\sum_{j=1}^{n} \frac{\partial^2 |h|^2}{\partial z_j \partial \overline{z_j}} = 0$. But we have

$$\sum_{j=1}^{n} \frac{\partial^2 |h|^2}{\partial z_j \partial \overline{z_j}} = \left(\sum_{j=1}^{n} \frac{\partial^2 h}{\partial z_j \partial \overline{z_j}} \right) \overline{h} + \left(\sum_{j=1}^{n} \frac{\partial^2 \overline{h}}{\partial z_j \partial \overline{z_j}} \right) h + \sum_{j=1}^{n} \left| \frac{\partial h}{\partial z_j} \right|^2 + \sum_{j=1}^{n} \left| \frac{\partial h}{\partial \overline{z_j}} \right|^2.$$

Now since h is harmonic on $\mathbb{C}^n$, it follows that

$$\sum_{j=1}^{n} \left| \frac{\partial h}{\partial z_j} \right|^2 = \sum_{j=1}^{n} \left| \frac{\partial h}{\partial \overline{z_j}} \right|^2 = 0, \quad \text{on} \quad \mathbb{C}^n.$$

Therefore, $\left|\frac{\partial h}{\partial z_j}\right|^2 = 0$ and $\left|\frac{\partial h}{\partial \overline{z_j}}\right|^2 = 0$, for each $j \in \{1, \dots, n\}$. It follows that $\frac{\partial h}{\partial z_j} = \frac{\partial h}{\partial \overline{z_j}} = 0$, for every $j \in \{1, \dots, n\}$. Consequently, h is constant on $\mathbb{C}^n$. □

Corollary 4.5. *Let $h_1, \dots, h_N : \mathbb{C}^n \to \mathbb{C}$ be N harmonic functions, $n, N \geq 1$. Put $v(z,w) = \sum_{j=1}^{N} |w - |h_j(z)||^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that v is psh on $\mathbb{C}^n \times \mathbb{C}$. Then $h_1, \dots, h_N$ are constant on $\mathbb{C}^n$.*

Proof. By [1], we have then $u = (|h_1| + \dots + |h_N|)$ is pluriharmonic on $\mathbb{C}^n$. Since $u \geq 0$ on $\mathbb{C}^n$, it follows that u is constant on $\mathbb{C}^n$. We have

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_j}} = 0.$$

Now since $|h_j|$ is continuous and $|h_j|$ is subharmonic on $\mathbb{C}^n$, for each $j \in \{1, \dots, n\}$, the condition $\sum_{j=1}^{n} \frac{\partial^2 (|h_1| + \dots + |h_N|)}{\partial z_j \partial \overline{z_j}} = 0$ on $\mathbb{C}^n$ implies that we have $\sum_{j=1}^{n} \frac{\partial^2 |h_k|}{\partial z_j \partial \overline{z_j}} = 0$, for each $k \in \{1, \dots, N\}$ and in all $\mathbb{C}^n$. Therefore $|h_1|, \dots, |h_N|$ are harmonic nonnegative functions on $\mathbb{C}^n$. Consequently, $|h_1|, \dots, |h_N|$ are constant on $\mathbb{C}^n$. By the above proof we conclude that h_k is constant, for every $k \in \{1, \dots, N\}$. □

But we have

Theorem 4.8. *Let $\varphi : G \to \mathbb{C}$ be analytic, G is a domain of $\mathbb{C}^n$. Define $u_1(z,w) = |w - \varphi(z)|^2 + |w - |\varphi(z)||^2$, $u_2(z,w) = u_1(z,w) + ||w| - |\varphi(z)||^2$, $u_3(z,w) = u_2(z,w) + ||w| - \varphi(z)|^2$, $(z,w) \in G \times \mathbb{C}$. We have u_1 is psh on $G \times \mathbb{C}$ if and only if φ is constant on G. u_2 is psh on $G \times \mathbb{C}$ if and only if $\varphi = 0$ on G. u_3 is psh on $G \times \mathbb{C}$ if and only if φ is constant, $\varphi = Re(\varphi) \leq 0$.*

Proof. Obvious by [1]. □

Remark 4.8. Theorem 4.8 is not true for pluriharmonic functions.

Example. Let $h(z) = x$, for $z \in \mathbb{C}$, $z = (x + iy)$, $x = \text{Re}(z)$, $y = \text{Im}(z)$. Put $v(z,w) = |w - h(z)|^2 + |w - |h(z)||^2$, for $(z,w) \in (\mathbb{C} \backslash E) \times \mathbb{C}$, $E = \{iy \,/\, y \in \mathbb{R}\}$. h is harmonic on $\mathbb{C} \backslash E$, v is psh on $(\mathbb{C} \backslash E) \times \mathbb{C}$, but h is not constant on $\mathbb{C} \backslash E$.

Corollary 4.6. *Let $g_1, \dots, g_N : \mathbb{C}^n \to \mathbb{C}$ be N holomorphic functions, $a, b \in \mathbb{C}^m$, $n, m \geq 1$, $N \geq 2$. Put*

$$\begin{aligned} u(z,w) &= |w - g_1(z)|^2 |w - g_2(z)|^2 \cdots |w - g_N(z)|^2 + |w - |g_1(z)||^2 |w \\ &\quad - |g_2(z)||^2 \cdots |w - |g_N(z)||^2, \\ u_1(z,w) &= |w - |g_1(z)|||w - |g_2(z)|| \cdots |w - |g_N(z)||, \\ v(z,\xi) &= |\langle \xi/a \rangle - g_1(z)|^2 + |\langle \xi/b \rangle - |g_1(z)||^2, \end{aligned}$$

for $(z, w, \xi) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^m$. Then have

(I) *u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $g_1, \dots, g_N$ are constant on $\mathbb{C}^n$.*

(II) *u_1 is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $g_1, \dots, g_N$ are constant functions.*

(III) *v is psh on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if g_1 is constant, or $b = 0$.*

Proof. Obvious by ([2], page 336), Theorem 4.8 and the proof of Corollary 4.5. □

Using the notation of Corollary 4.6, we can study the complex structure of the function

$$\begin{aligned} \psi(z,w) = (&|w - g_1(z)|^2 \cdots |w - g_N(z)|^2 \\ &+ |w - |g_1(z)||^2 \cdots |w - |g_N(z)||^2)(|w - \varphi_1(z)|^2 \cdots |w - \varphi_N(z)|^2 \\ &+ |w - |\varphi_1(z)||^2 \cdots |w - |\varphi_N(z)||^2), \end{aligned}$$

where $N \geq 1$, $\varphi_1, \dots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions.

An extension of the results. Now let $\psi_1, \dots, \psi_N, f_1, \dots, f_N : \mathbb{C}^n \to \mathbb{C}$ be $2N$ holomorphic functions, $N \geq 2$. Put

$$\begin{aligned} \psi(z,w) &= |w - |\psi_1(z)|\,|\,|w - |\psi_2(z)|\,| \cdots |w - |\psi_N(z)|\,|, \\ f(z,w) &= |w - \overline{f_1}(z)||w - \overline{f_2}(z)| \cdots |w - \overline{f_N}(z)|, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Observe that, using potential theory methods, we can prove that ψ is psh implies that $\psi_1, \dots, \psi_N$ are constant functions. But it is difficult to deduce the exact conditions satisfies by the functions $f_1, \dots, f_N$ such that f is psh on $\mathbb{C}^n \times \mathbb{C}$, if N has a decomposition in product of q prime numbers, $q \geq 2$.

4.2. Some properties of plurisubharmonic functions. In this subsection, we would like to extend Proposition 4.2 in the following situation.

Theorem 4.9. *Let $\varphi_0, \dots, \varphi_{k-1} : D \to \mathbb{C}$ be k analytic functions, D is a domain of $\mathbb{C}^n$ and $k \in \mathbb{N} \backslash \{0\}$. Define*

$$u(z,w) = |w^k + |\varphi_{k-1}(z)|w^{k-1} + \cdots + |\varphi_1(z)|w + |\varphi_0(z)||,$$

for each $(z,w) \in D \times \mathbb{C}$*. The following are equivalents*

(I) u *is psh on* $D \times \mathbb{C}$;

(II) $\varphi_{k-1}, \dots, \varphi_0$ *are constant functions on* D.

Proof. (I) implies (II). By [2], we have $|\varphi_{k-1}|$ is prh on D. Therefore φ_{k-1} is constant on D. We conclude in fact that, in this order, $\varphi_{k-2}, \dots, \varphi_0$ are constant on D.

(II) implies (I). Obvious. □

Remark 4.9. For an infinite sequence of analytic functions, the above theorem is false.

Example. Consider $u(z,w) = \left|\sum_{k=0}^{\infty}\right| e^z \left|\frac{w^k}{k!}\right| = |e^z||e^w| = e^{x_1+y_1}$, where $(z,w) \in \mathbb{C}^2$, $x_1 = \mathrm{Re(z)}$ and $y_1 = \mathrm{Re(w)}$. Then u is psh on $\mathbb{C}^2$. But the function φ_k is holomorphic nonconstant on $\mathbb{C}$ for all $k \in \mathbb{N}$, where $\varphi_k(z) = \frac{e^z}{k!}$ for $z \in \mathbb{C}$, for each $k \in \mathbb{N}$.

The following two lemmas are technical tools in the proof of the below theorem.

Lemma 4.2. *Let* $k \in \mathbb{N}\backslash\{0\}$ *and* $c_0, \dots, c_{k-1} \in \mathbb{R}_+$. *Let* $q(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_0$, *for* $x \in \mathbb{R}$. *Then there exists* $a \in \mathbb{R}_+$, *such that* $q(a) = 0$.

Proof. Note that q is a polynomial with real coefficients in $\mathbb{R}$. Therefore the function q is continuous on $\mathbb{R}$. We have $q(0) = -c_0 \leq 0$. Since $\lim_{x\to+\infty} q(x) = +\infty$. Then there exists $a \in \mathbb{R}_+$, such that $q(a) = 0$. □

Lemma 4.3. *Let* $k \in \mathbb{N}\backslash\{0\}$ *and* $c_0, \dots, c_{k-1} \in \mathbb{R}_+$. *Put* $u(w) = ||w|^k - c_{k-1}|w|^{k-1} - \cdots - c_0|$, *for each* $w \in \mathbb{C}$. *Then* u *is subharmonic (sh) on* $\mathbb{C}$ *if and only if* $c_{k-1} = \cdots = c_0 = 0$.

Proof. Assume that u is sh on $\mathbb{C}$. Observe that u is a radial function on $\mathbb{C}$. By lemma 4.2, there exists $a \in \mathbb{R}_+$, such that $u(a) = 0$. Assume that $a > 0$. Since u is subharmonic on $\mathbb{C}$, then $\sup_{|w|=a} u(w) = u(a) = 0$. Now by the maximum principle we have $\sup_{|w|=a} u(w) = \sup_{|w|\leq a} u(w) = 0$. Since $u \geq 0$ on $\mathbb{C}$, then $u = 0$, on the open disc $D(0,a)$. It follows that $\psi(x) = 0$, for each $x \in]-a,a[$, where ψ is the polynomial on $\mathbb{R}$, defined by, $\psi(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_0$, for $x \in \mathbb{R}$. Therefore the polynomial ψ satisfies $\psi = 0$, on $\mathbb{R}$. This is a contradiction. Therefore $a = 0$. $\psi(0) = 0$ implies that $c_0 = 0$. If $k = 1$, the proof is finished.

Now suppose that $k \geq 2$. It follows that $\psi(x) = x(x^{k-1} - c_{k-1}x^{k-2} - \cdots - c_1)$, for each $x \in \mathbb{R}$. Put $\psi_1(x) = x^{k-1} - c_{k-1}x^{k-2} - \cdots - c_1$, for every $x \in \mathbb{R}$. ψ_1 is a polynomial on $\mathbb{R}$ and we have $u(w) = u(|w|) = |w|\psi_1(|w|) = \psi(|w|)$. Now we consider the polynomial ψ_1 in the above state and we prove that $\psi_1(0) = 0$ and ψ_1 does not have a zero $b \in \mathbb{R}$, where $b > 0$. Therefore $c_1 = 0$. Consequently, $c_0 = c_1 = \cdots = c_{k-1} = 0$. □

Moreover, we have

Theorem 4.10. *Let $\varphi_0, \ldots, \varphi_{k-1} : D \to \mathbb{C}$ be k functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$ and $k \geq 1$. Let*

$$u(z,w) = ||w|^k - |\varphi_{k-1}(z)||w|^{k-1} - \cdots - |\varphi_1(z)||w| - |\varphi_0(z)||, \ (z,w) \in D \times \mathbb{C}.$$

Suppose that u is psh on $D \times \mathbb{C}$. Then $\varphi_{k-1} = \cdots = \varphi_0 = 0$ on D.

Proof. Obvious by the above two lemmas. □

Remark 4.10. For an infinite sequence of holomorphic functions, the above theorem is false.

Example. Let $v(z,w) = \left|\sum_{j\geq 0} -\frac{|e^z||w|^j}{j!}\right| = |e^z|e^{|w|} = e^{x+|w|}$, $(z,w) \in \mathbb{C}^2$, $x = \mathrm{Re(z)}$. The function v is psh on $\mathbb{C}^2$. But $\varphi_j \neq 0$, for each $j \in \mathbb{N}$, where $\varphi_j(z) = e^z$ for every $z \in \mathbb{C}$ and $j \in \mathbb{N}$.

5. Absolute values and strictly plurisubharmonic functions.

We have

Theorem 5.1. *Let $g_1, \ldots, g_N : D \to \mathbb{C}$ be N holomorphic functions, D is a domain of $\mathbb{C}^n$, $n, N \geq 1$. Put*

$$u(z,w) = |w - |g_1(z)|^2|^2 + \cdots + |w - |g_N(z)|^2|^2 + |w - g_1(z)|^2 + \cdots + |w - g_N(z)|^2,$$

for $(z,w) \in D \times \mathbb{C}$. Then u is not strictly psh in $D \times \mathbb{C}$.

Proof. Without loss of generality suppose that $n = 1$. Note that u is a function of class C^∞ on $\mathbb{C}^2$. Assume that u is strictly psh on $\mathbb{C}^2$. Let $(z,w) \in \mathbb{C}^2$. We have

$$\frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w) = 4|g_1'(z)|^2|g_1(z)|^2 + \cdots + 4|g_N'(z)|^2|g_N(z)|^2 - (|g_1'(z)|^2 + \cdots + |g_N'(z)|^2)(w + \overline{w}) + |g_1'(z)|^2 + \cdots + |g_N'(z)|^2.$$

$$\frac{\partial^2 u}{\partial z \partial \overline{w}}(z,w) = -g_1'(z)\overline{g_1}(z) - \cdots - g_N'(z)\overline{g_N}(z) - g_1'(z) - \cdots - g_N'(z)$$

$$\frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w) = 2N.$$

Since u is strictly psh on $\mathbb{C}^2$, then by Lemma 4.1,

$$|\frac{\partial^2 u}{\partial z \partial \overline{w}}(z,w)|^2 < \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)\frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w)$$

for each $(z,w) \in \mathbb{C}^2$. Therefore the function ψ is bounded below on $\mathbb{R}$, (for every z fixed in $\mathbb{C}$), where $\psi(x) = -2(|g_1'(z)|^2 + \cdots + |g_N'(z)|^2)x$, for $x \in \mathbb{R}$. It follows that $(|g_1'(z)|^2 + \cdots + |g_N'(z)|^2) = 0$, for any $z \in \mathbb{C}$.
Consequently, $g_1' = \cdots = g_N' = 0$ on $\mathbb{C}$. A contradiction.

In fact we have proved that if there exists $j \in \{1, \ldots, N\}$ such that g_j is nonconstant, then u is not psh on $\mathbb{C}^2$. □

In the sequel, we prove now

Theorem 5.2. *Let $f, g : D \to \mathbb{C}$, D is a domain of $\mathbb{C}^n$, $n \geq 1$. Put $u(z,w) = |w^2 + wf(z) + g(z)|$, for $(z,w) \in D \times \mathbb{C}$. The following conditions are equivalent*

(I) *u is strictly psh on $D \times \mathbb{C}$;*

(II) *$g = \frac{f^2}{4}$ on D, $n = 1$, f is harmonic on D and $\frac{\partial f}{\partial \overline{z}}(z) \neq 0$, for each $z \in D$.*

Proof. (I) implies (II). Fix $z \in D$. Then the function $u(z,.)$ is strictly sh on $\mathbb{C}$. Now let $v(w) = |w^2 + aw + b| = |(w + \frac{a}{2})^2 - \frac{a^2}{4} + b|$ for $w \in \mathbb{C}$, where $a, b \in \mathbb{C}$. Recall that v is strictly sh on $\mathbb{C}$ if and only if $b = \frac{a^2}{4}$. It follows that $g = \frac{f^2}{4}$ on D. Therefore, $u(z,w) = |w + \frac{f(z)}{2}|^2$. Since u is strictly psh on $D \times \mathbb{C}$, then u is psh on $D \times \mathbb{C}$.
By [1], f is prh on D. Let now $z_0 \in D$ and $R > 0$, such that $B(z_0, R) \subset D$. Write now $f = f_1 + \overline{f_2}$, on $B(z_0, R)$, where f_1 and f_2 are holomorphic functions on $B(z_0, R)$. Now since u is a function of class C^∞ and strictly psh on $B(z_0, R) \times \mathbb{C}$, the hermitian Levi form of u denoted by $L(u)(z,w)(\alpha, \beta)$ satisfies

$$L(u)(z,w)(\alpha,\beta) = |\beta + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + \left|\frac{1}{2}\sum_{j=1}^{n}\frac{\partial f_2}{\partial z_j}(z)\alpha_j\right|^2 > 0$$

for each $(z,w) \in B(z_0, R) \times \mathbb{C}$, and $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta) \in \mathbb{C}^n \times \mathbb{C} \backslash \{0\}$. Therefore $L(u)(z,w)(\alpha,\beta) = 0$, implies that $\sum_{j=1}^{n}\frac{\partial f_1}{\partial z_j}(z)\alpha_j = -2\beta$ and $\sum_{j=1}^{n}\frac{\partial f_2}{\partial z_j}(z)\alpha_j = 0$.

It follows that the condition $\sum_{j=1}^{n}\frac{\partial f_2}{\partial z_j}(z)\alpha_j = 0$ implies that $\alpha_1 = \cdots = \alpha_n = 0$. Therefore, the family $\left\{\frac{\partial f_2}{\partial z_1}(z), \ldots, \frac{\partial f_2}{\partial z_n}(z)\right\}$ is a free family in the complex vector space $\mathbb{C}$. Then $n = 1$ and $\frac{\partial f_2}{\partial z}(z) \neq 0$, for each $z \in D$. In particular, g is a function of class C^∞.

(II) implies (I). Obvious. □

Now using the below theorem 6.2 for the convexity of prh functions, we can prove

Theorem 5.3. *Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be two functions. Let*

$$u(z,w) = |w^2 + f(z)w + g(z)|, \quad \text{for} \ \ (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

Assume that u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and $|f|$ is convex on $\mathbb{C}^n$. Then we have $n = 1$, $g = \frac{f^2}{4}$, f is harmonic on $\mathbb{C}$, $f = f_1 + \overline{f_2}$, where f_1, f_2 are holomorphic functions on $\mathbb{C}$, such that f_1 and f_2 are affine functions and $f_2' \neq 0$, or f_1 is constant and $f_2(z) = e^{(Az+B)} - \overline{f_1}(0)$, for $z \in \mathbb{C}$, where $A, B \in \mathbb{C}$, $A \neq 0$.

The question posed now is to study the following problem. Characterize exactly all the $2N$ holomorphic functions $f_j, g_j : \mathbb{C}^n \to \mathbb{C}$ such that u_j is a psh function for each $1 \leq j \leq N$, $N \geq 2$; $v = \sum_{j=1}^{N} u_j$ is convex on $\mathbb{C}^n \times \mathbb{C}$. Where $u_j(z,w) = |A_j w^2 + \overline{f_j}(z)w + \overline{g_j}(z)|$, $A_j \in \mathbb{C}\backslash\{0\}$, $j = 1, \ldots, N$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. As application of Section 4, recall that if we put

$$u(z,w) = |w - \overline{\varphi_1}(z)|^{N_1} |w - \overline{\varphi_2}(z)|^{N_2}, \quad \text{for} \ \ (z,w) \in D \times \mathbb{C},$$

where $N_1, N_2 \in \mathbb{N}\backslash\{0\}$, D is a domain of $\mathbb{C}^n$, $\varphi_1, \varphi_2 : D \to \mathbb{C}$ be two holomorphic functions. u is psh on $D \times \mathbb{C}$ if and only if we have the following two possible cases.

Case 1. $N_1 = N_2$. Then $\varphi_1 = \varphi_2$, or $(\varphi_1 + \varphi_2)$ is constant on D.

Case 2. $N_1 \neq N_2$. Therefore, φ_1 and φ_2 are constant functions, or $\varphi_1 = \varphi_2$ on D. Now let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Put $v = (v_1 + v_2)$, where

$$v_1(z,w) = |w - \overline{\varphi_1}(z)|\,|w - \overline{\varphi_2}(z)|, \quad v_2(z,w) = |w - \overline{\varphi_3}(z)|\,|w - \overline{\varphi_4}(z)|,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Note that we can study the problem of the characterization of $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ such that v_1 and v_2 are psh functions and v is convex on $\mathbb{C}^n \times \mathbb{C}$.

We have

Theorem 5.4. *Let $k, s, t \in \mathbb{N}\backslash\{0,1\}$. Let $f_0, \ldots, f_{k-1}$, $g_0, \ldots, g_{s-1}$, $\varphi_0, \ldots, \varphi_{t-1} : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Define $u = u_1u_2u_3$, where*

$$\begin{aligned} u_1(z,w) &= \left|w^k + \overline{f_{k-1}}(z)w^{k-1} + \cdots + \overline{f_0}(z)\right|, \\ u_2(z,w) &= \left|w^s + \overline{g_{s-1}}(z)w^{s-1} + \cdots + \overline{g_0}(z)\right|, \\ u_3(z,w) &= w^t + \left|\overline{\varphi_{t-1}}(z)w^{t-1} + \cdots + \overline{\varphi_0}(z)\right|, \end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Assume that u_1, u_2 and u_3 are convex functions and u is psh on $\mathbb{C}^n \times \mathbb{C}$.

Then we have

$$u_1(z,w) = \left|w - \frac{\overline{f_{k-1}}(z)}{k}\right|^k, \; u_2(z,w) = \left|w - \frac{\overline{g_{s-1}}(z)}{s}\right|^s, \; u_3(z,w) = \left|w - \frac{\overline{\varphi_{t-1}}(z)}{t}\right|^t,$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if we have the following two cases.

Case 1. $t = s = k$. Then $(f_{k-1} + g_{k-1} + \varphi_{k-1})$ and $(f_{k-1}g_{k-1} + f_{k-1}\varphi_{k-1} + g_{k-1}\varphi_{k-1})$ are constant on $\mathbb{C}^n$, or $(f_{k-1} + g_{k-1} + \varphi_{k-1})$ is non constant and $f_{k-1} = g_{k-1} = \varphi_{k-1}$ on $\mathbb{C}^n$.

Case 2. $k \neq s$, or $k \neq t$, or $s \neq t$. Then f_{k-1}, g_{s-1} and φ_{t-1} are constant functions, or $f_{k-1} = g_{s-1} = \varphi_{t-1}$ on $\mathbb{C}^n$.

Proof. We can see section 4 and the paper [2]. □

6. The analysis of holomorphic and convex functions in one and several complex variables. There exist a fundamental relation between holomorphic functions, partial differential equations and the convexity property. Exactly we have the technical result.

Theorem 6.1. *Let $g : \mathbb{C} \to \mathbb{C}$ be analytic nonconstant and $c \in \mathbb{C}$. Put $v(z,w) = |g(w - \overline{z}) + c|$, for $(z,w) \in \mathbb{C}^2$. The following conditions are equivalent*

(a_1) *The function $|g + c|$ is convex on $\mathbb{C}$;*

(a_2) *There exists $\gamma \in \mathbb{C}$ such that $g''(g + c) = \gamma(g')^2$ on $\mathbb{C}$;*

(a_3) *There exists $\gamma \in S = \left\{0, \dfrac{s-1}{s}, 1/s \in \mathbb{N}\{0,1\}\right\}$ such that $g''(g + c) = \gamma(g')^2$ on D, where D is a not empty domain subset of $\mathbb{C}$;*

(a_4) *v is psh on $\mathbb{C}^2$;*

(a_5) *There exists an harmonic not analytic function $\varphi : \mathbb{C} \to \mathbb{C}$ such that u is psh on $\mathbb{C}^2$, where $u(z,w) = |g(w - \varphi(z)) + c|$, for $(z,w) \in \mathbb{C}^2$;*

(a_6) *There exists an harmonic nonconstant function* $\psi : \mathbb{C} \to \mathbb{R}$ *such that* u_1 *is psh on* $\mathbb{C}^2$, *where* $u_1(z,w) = |g(w - \psi(z)) + c|$, *for* $(z,w) \in \mathbb{C}^2$.

For a proof of this theorem, we can see [2]. Note that we can find exactly g by each of its analytic expression (also we can see [2]).

6.1. Some technical problems. We have

$$\begin{cases} h : \mathbb{C}^n \to \mathbb{C} \text{ be nonconstant} \\ u(z,w) = |w - h(z)| \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \\ v(z,w) = |h(w - \overline{z})| \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \end{cases}$$

this is the first technical problem.

The second problem is now

$$\begin{cases} h : \mathbb{C}^n \to \mathbb{C} \text{ be nonconstant} \\ u(z,w) = |w - h(z)| \text{ is strictly psh on } \mathbb{C}^n \times \mathbb{C}; \\ v(z,w) = |h(w - \overline{z})| \text{ is psh on } \mathbb{C}^n \times \mathbb{C}. \end{cases}$$

Now we begin this study by one of the following technical result between real and complex convexity.

Theorem 6.2. *Let* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ *be two analytic functions. Suppose that* $u = |g_1 + \overline{g_2}|$ *is convex on* $\mathbb{C}^n$.
Then we have an only one assertion of the following conditions

(I) g_1 *and* g_2 *are nonconstant functions. Then* g_1 *and* g_2 *are affine functions on* $\mathbb{C}^n$.

(II) g_2 *is constant on* $\mathbb{C}^n$. *Then* $|g_1 + \overline{g_2}(0)|$ *is convex on* $\mathbb{C}^n$. *Note that in this case* $(g_1 + \overline{g_2}(0))$ *is a holomorphic function and we can find exactly* g_1 *by its expression by* [2], *(see also* [3]*).*

(III) g_1 *is constant on* $\mathbb{C}^n$. *Then* $|g_2 + \overline{g_1}(0)|$ *is convex on* $\mathbb{C}^n$.

P r o o f. Let $T_1(z) = z$, $T_2(z) = \overline{z}$, for $z \in \mathbb{C}^n$. Then T_1 and T_2 are two $\mathbb{R}-$ linear transformations on $\mathbb{C}^n$. The family $\{T_1, T_2\}$ is in fact $\mathbb{C}-$ linearly free on $\mathbb{C}^n$, when we consider $\mathbb{C}^n$ a complex vector space of dimension n. Note that $g_1(z)$ depend only in the variable z. $\overline{g_2}(z)$ depend only in the variable $\overline{z}$. For each $\alpha, \beta \in \mathbb{C}$, the condition $|\alpha g_1(z) + \beta \overline{g_2}(z)| = 0$, (for every $z \in \mathbb{C}^n$), implies that $\alpha = \beta = 0$, if g_1 and g_2 are nonconstant functions. Then the condition u is convex on $\mathbb{C}^n$ implies that the new function, defined by $v(z,w) = |g_1(z) + \overline{g_2}(w)|$ for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$, satisfies v is convex on $\mathbb{C}^n \times \mathbb{C}^n$. Therefore we have the following cases.

Case 1. g_2 is constant on $\mathbb{C}^n$. Then $|g_1 + \overline{g_2}(0)|$ is convex on $\mathbb{C}^n$. Since $(g_1 + \overline{g_2}(0))$ is a holomorphic function on $\mathbb{C}^n$, then by [2], (see also [3]), we have the following two states. $g_1(z) = (< z/a_1 > +b_1)^m - \overline{g_2}(0)$, where $a_1 \in \mathbb{C}^n$, $b_1 \in \mathbb{C}$, $m \in \mathbb{N}$, for each $z \in \mathbb{C}^n$, or $g_1(z) = e^{(<z/c_1>+d_1)} - \overline{g_2}(0)$, where $c_1 \in \mathbb{C}^n$ and $d_1 \in \mathbb{C}$, for every $z \in \mathbb{C}^n$.

Case 2. g_1 is constant on $\mathbb{C}^n$. The conclusion is now by the case 1.

Case 3. g_1 and g_2 are nonconstant functions on $\mathbb{C}^n$. Let $\alpha, \beta \in \mathbb{C}$, $(\alpha \neq \beta)$, $\{\alpha, \beta\} \subset \overline{g_2}(\mathbb{C})$. Since v is convex on $\mathbb{C}^n \times \mathbb{C}^n$, then the functions $|g_1 + \alpha|$ and $|g_1 + \beta|$ are convex on $\mathbb{C}^n$. By [2], we conclude that g_1 is affine on $\mathbb{C}^n$. It follows that g_2 is affine on $\mathbb{C}^n$. The proof is now complete. □

Several applications of this theorem can be obtained in complex analysis, convex analysis, representation theory, harmonic analysis in several complex variables and holomorphic partial differential equations.

Remark 6.1. (I) Let $h_1, h_2 : \mathbb{C}^n \to \mathbb{C}$ be two pluriharmonic functions. Let $u(z) = |h_1(z) + h_2(\overline{z})|$ and $v(z, w) = |h_1(z) + h_2(w)|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Assume that u is convex on $\mathbb{C}^n$. Then we can not conclude that v is convex on $\mathbb{C}^n \times \mathbb{C}^n$, because $h_1(z)$ and $h_2(\overline{z})$ depend of the two variables z and $\overline{z}$.

Example. Let $h_1(z) = (z + 1)^2$, $h_2(z) = (\overline{z} + 1)^2$, for $z \in \mathbb{C}$. Put $u_1(z) = |h_1(z) + h_2(\overline{z})|$, $u_2(z, w) = |h_1(z) + h_2(w)|$, $(z, w) \in \mathbb{C}^2$. We have h_1 and h_2 are harmonic functions on $\mathbb{C}$. The function u_1 is convex on $\mathbb{C}$. But v is not convex on $\mathbb{C}^2$.

(II) Let $f_1(z) = 4x^2 + 2x$, $f_2(z) = -2x$, $z = (x + iy) \in \mathbb{C}$, $x = \mathrm{Re}(z)$, $w \in \mathbb{C}$. We have $u_1(z) = |f_1(z) + f_2(z)|^2 = 16x^4$. Then u_1 is convex on $\mathbb{C}$. But $u_2(z, w) = |f_1(z) + f_2(w)|$, satisfies u_2 is not convex on $\mathbb{C}^2$. In fact $f_1(z) = (f(z) + \overline{f}(z))^2 + (f(z) + \overline{f}(z))$, $f_2(z) = -2x = -\mathrm{Re}(f(z))$, where $f(z) = z$. Note that f is holomorphic on $\mathbb{C}$. In fact $f(z)$ is independent of $\overline{z}$. We have for example, $f_1(z)$ does not depend only in the variable z, f_2 does not depend only in the variable $\overline{z}$, but f_1 and f_2 are linearly independent on $\mathbb{C}$.

Now by the following proposition and the paper [4], we obtain an answer of the problem of the characterization of all real valued pluriharmonic functions $k_1, k_2 : \mathbb{C}^n \to \mathbb{R}$ such that ψ is psh on $\mathbb{C}^n \times \mathbb{C}^n$, where $\psi(z, w) = [(k_1(w - \overline{z}))^2 + (k_2(w - \overline{z}))^2]$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$.

Proposition 6.1. *Let $h_1, h_2 : \mathbb{C}^n \to \mathbb{R}$ be two pluriharmonic (prh) functions. Put $u = h_1^2 + h_2^2$. Assume that u is convex on $\mathbb{C}^n$. Then we can find exactly all the functions h_1 and h_2 by their analytic expressions.*

Proof. Observe that $u = |h_1 + ih_2|^2$ on $\mathbb{C}^n$. Now since h_1 and h_2 are prh functions on $\mathbb{C}^n$, then there exists two holomorphic functions g_1, g_2 on $\mathbb{C}^n$, such that $h_1 = (g_1 + \overline{g_1})$ and $h_2 = (g_2 + \overline{g_2})$. Therefore $u = |g_1 + \overline{g_1} + i(g_2 + \overline{g_2})|^2 = |g_1 + ig_2 + (\overline{g_1 - ig_2})|^2$ on $\mathbb{C}^n$. Now note that $(g_1 + ig_2)$ and $(g_1 - ig_2)$ are holomorphic functions on $\mathbb{C}^n$. By Theorem 6.2, we have the following three states.

State 1. $(g_1 + ig_2)$ is affine nonconstant on $\mathbb{C}^n$. In this case $(g_1 - ig_2)$ is an affine function. Consequently, we can find g_1 and g_2 by their expressions.

State 2. $(g_1 + ig_2)$ is not an affine function on $\mathbb{C}^n$. In this case $(g_1 - ig_2)$ is a constant function. Then $|g_1 + ig_2 + (\overline{g_1 - ig_2})(0)|^2$ is a convex function on $\mathbb{C}^n$. Therefore, we can find g_1 and g_2 by their expressions.

State 3. $(g_1 + ig_2)$ is constant on $\mathbb{C}^n$. In this case $|(g_1 - ig_2) + (\overline{g_1 + ig_2})(0)|^2$ is a convex function. Therefore, we can find g_1 and g_2 by their expressions. □

But in general if the number of real valued prh functions is $N \geq 3$, the problem remains open.

Corollary 6.1. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Assume that for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, we have the inequality,*

$$\left| \sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k} \alpha_j \alpha_k (g_2 + \overline{g_1}) + \sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k} \alpha_j \alpha_k (g_1 + \overline{g_2}) + 2 \sum_{j,k=1}^{n} \frac{\partial g_1}{\partial z_j} \frac{\partial g_2}{\partial z_k} \alpha_j \alpha_k \right| \leq \left| \sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j} \alpha_j \right|^2 + \left| \sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j} \alpha_j \right|^2$$

on $\mathbb{C}^n$.
Then we have the following three states.

State 1. g_2 is constant and

$$\begin{cases} g_1(z) = (< z/a_1 > + b_1)^{m_1} - \overline{g_2}(0), \text{ or} \\ g_1(z) = e^{(<z/c_1> + d_1)} - \overline{g_2}(0) \end{cases}$$

for each $z \in \mathbb{C}^n$, where $a_1, c_1 \in \mathbb{C}^n$, $b_1, d_1 \in \mathbb{C}, m_1 \in \mathbb{N}$.

State 2. g_1 is constant and

$$\begin{cases} g_2(z) = (< z/a_2 > + b_2)^{m_2} - \overline{g_1}(0), \text{ or} \\ g_2(z) = e^{(<z/c_2> + d_2)} - \overline{g_1}(0) \end{cases}$$

for each $z \in \mathbb{C}^n$, where $a_2, c_2 \in \mathbb{C}^n$, $b_2, d_2 \in \mathbb{C}, m_2 \in \mathbb{N}$.

State 3. g_1 and g_2 are affine functions on $\mathbb{C}^n$.

Theorem 6.3. *Let $h_1, h_2 : \mathbb{C}^n \to \mathbb{C}$ be two pluriharmonic functions. $h_1 = f_1 + \overline{g_1}$, $h_2 = f_2 + \overline{g_2}$, where f_1, g_1, f_2, g_2 are holomorphic functions on $\mathbb{C}^n$. Assume that $|h_1 + h_2|$ is convex on $\mathbb{C}^n$. Then we have only the following three cases.*

Case 1. $(f_1 + f_2)$ and $(g_1 + g_2)$ are affine functions on $\mathbb{C}^n$.

Case 2. $(g_1 + g_2)$ is constant on $\mathbb{C}^n$ and $|f_1 + f_2 + \overline{g_1}(0) + \overline{g_2}(0)|$ is a convex function on $\mathbb{C}^n$.

Case 3. $(f_1 + f_2)$ is constant on $\mathbb{C}^n$ and $|g_1 + g_2 + \overline{f_1}(0) + \overline{f_2}(0)|$ is a convex function on $\mathbb{C}^n$.

Proof. Obvious by Theorem 6.2. □

Observe that if $k : \mathbb{C}^n \to \mathbb{C}$ be prh, the condition $|k|$ is convex on $\mathbb{C}^n$, implies that k is affine, or k is holomorphic, or $\overline{k}$ is holomorphic on $\mathbb{C}^n$. Now we can prove that the complex structure is important in the richness of the above results.

Example. For $h : \mathbb{C}^n \to \mathbb{C}$ be harmonic such that $|h|$ is convex on $\mathbb{C}^n$, $n \geq 2$. We can not conclude that h is in the above form. Put $h_1(z) = (z_1 + \overline{z_2})^2$, $z = (z_1, z_2) \in \mathbb{C}^2$. h_1 is $2-$ harmonic and $|h_1|$ is convex on $\mathbb{C}^2$. But h_1 is not affine, h_1 is not holomorphic and $\overline{h_1}$ is not holomorphic in $\mathbb{C}^2$.
Indeed, let $k(z) = x_1 + i(|z_1|^2 - |z_2|^2)$, $z = (z_1, z_2) \in \mathbb{C}^2$, $x_1 = \mathrm{Re}(z_1)$. k is harmonic on $\mathbb{C}^2$. $|e^k|$ is convex on $\mathbb{C}^2$. But k is not prh on $\mathbb{C}^2$.

Recall also that we have the following.

Let $k_1 : \mathbb{C}^n \to \mathbb{C}$ be harmonic, $n \geq 2$. Let $T = (T_1 + iT_2) : \mathbb{C}^n \to \mathbb{C}^n$ be $\mathbb{R}$ linear and bijective. Assume that T_1 and T_2 have real values. If $T_1(k_1)$ and $T_2(k_1)$ are psh functions. Then h is prh on $\mathbb{C}^n$.

Remark 6.2. Let $h_1(z_1, \dots, z_n) = e^{((\overline{z_1})^2 + \cdots + (\overline{z_n})^2)}$, $h_2(z_1, \dots, z_n) = (\overline{z_1})^2$, $h_3(z_1, \dots, z_n) = (\overline{z_1} + \cdots + \overline{z_n})$, $(z_1, \dots, z_n) \in \mathbb{C}^n$, $n \geq 1$. h_1, h_2 and h_3 are pluriharmonic functions on $\mathbb{C}^n$. We have

$\{a \in \mathbb{C} \;/\; |h_1 + a|$ is a convex function on $\mathbb{C}^n\} = \emptyset$.
$\{a \in \mathbb{C} \;/\; |h_2 + a|$ is a convex function on $\mathbb{C}^n\} = \{0\}$.
$\{a \in \mathbb{C} \;/\; |h_3 + a|$ is a convex function on $\mathbb{C}^n\} = \mathbb{C}$.

The originality and richness of the above characterization and the complex structure gives the following technical theorem.

Theorem 6.4. *Let $k : \mathbb{C}^n \to \mathbb{C}$ be a pluriharmonic function. Then*

$$\{a \in \mathbb{C} \;/\; |k + a| \text{ is a convex function on } \mathbb{C}^n\} = \emptyset, \text{ or } \{\alpha\}, \text{ or } \mathbb{C},$$

where $\alpha \in \mathbb{C}$.

P r o o f. *Case 1.* $|k|$ is convex on $\mathbb{C}^n$. If k is affine. Then

$$\{a \in \mathbb{C} \; / \; |k + a| \text{ is a convex function on } \mathbb{C}^n\} = \mathbb{C}.$$

Now suppose that k is not affine.
Then k is holomorphic or $\overline{k}$ is holomorphic on $\mathbb{C}^n$, by Theorem 6.2. By [2], we deduce that,

$$\{a \in \mathbb{C} \; / \; |k + a| \text{ is a convex function on } \mathbb{C}^n\} = \emptyset \text{ or } \{\alpha\},$$

where $\alpha \in \mathbb{C}$.

Case 2. $|k|$ is not convex on $\mathbb{C}^n$. If there exists $b \in \mathbb{C}^n$, such that $|k + b|$ is convex on $\mathbb{C}^n$. We use the case 1 to obtain

$$\{a \in \mathbb{C} \; / \; |k + a| \text{ is a convex function on } \mathbb{C}^n\} = \{b\}.$$

If for all $\xi \in \mathbb{C}$, the function $|k + \xi|$ is not convex on $\mathbb{C}^n$. Then

$$\{a \in \mathbb{C} \; / \; |k + a| \text{ is a convex function on } \mathbb{C}^n\} = \emptyset.$$

The proof is now complete. □

Observe that this theorem extends some results of [2] concerning holomorphic functions.

Corollary 6.2. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be pluriharmonic and $a, b \in \mathbb{C}$, $(a \neq b)$. Assume that $|g + a|$ and $|g + b|$ are convex functions on $\mathbb{C}^n$. Then g is affine on $\mathbb{C}^n$.*

Remark 6.3. We give some simple examples of smooth functions with various behavior.

(I) Let $h(z) = z^2$, $z \in \mathbb{C}$. h is harmonic on $\mathbb{C}$. We have, for each $\epsilon > 0$, for every $\alpha \in D(0, \epsilon)$, the function $|h + \alpha|^2$ is convex on $\mathbb{C} \backslash \overline{D}(0, \epsilon)$, but h is not affine on $\mathbb{C}$. Therefore, for all domains U of $\mathbb{C}$, such that $\mathbb{C} \backslash U$ have not empty interior, there exists an harmonic and not affine function $k : \mathbb{C} \to \mathbb{C}$, satisfying $|k + \delta_j|^2$ is convex on $\mathbb{C} \backslash \overline{U}$, for each $j \in \{1, 2\}$, where $\delta_1, \delta_2 \in \mathbb{C}$, $(\delta_1 \neq \delta_2)$. This proves that in the above corollary, all the subset $\mathbb{C}^n$ is very important, for each $n \geq 1$.

(II) Let $h_1, h_2 : \mathbb{C}^n \to \mathbb{R}$ be prh. Assume that $(|h_1|^2 + |h_2|^2) = (h_1^2 + h_2^2)$ and $((h_1 + 1)^2 + (h_2)^2)$ are convex functions on $\mathbb{C}^n$. Then $h = (h_1 + ih_2)$ is affine on $\mathbb{C}^n$. Consequently, h_1 and h_2 are affine functions. But if $k_1, k_2 : \mathbb{C}^n \to \mathbb{C}$ be two prh functions satisfying $(|k_1|^2 + |k_2|^2)$ and $(|k_1 + 1|^2 + |k_2|^2)$ are convex functions on $\mathbb{C}^n$. We can not deduce that k_1 and k_2 are affine on $\mathbb{C}^n$.

Example. Let $k_1(z) = z^2 - 1$, $k_2(z) = 2z$, for $z \in \mathbb{C}$. k_1 and k_2 are harmonic functions on $\mathbb{C}$. We have $(|k_1|^2 + |k_2|^2)$ and $(|k_1 + 1|^2 + |k_2|^2)$ are convex functions on $\mathbb{C}$. Moreover, $(|k_1 + \alpha|^2 + |k_2 + \beta|^2)$ is convex on $\mathbb{C}$, for each $(\alpha, \beta) \in D(1,1) \times \mathbb{C}$. But k_1 and k_2 are not affine functions on $\mathbb{C}$.

Question. Let $h : \mathbb{C}^n \to \mathbb{C}$ be harmonic, $n \geq 2$ and $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$. Assume that $|h + \alpha|$ and $|h + \beta|$ are convex functions on $\mathbb{C}^n$. Is it true that h is affine on $\mathbb{C}^n$?

6.2. Applications in pluripotential theory. We have

Theorem 6.5. *Let $g, \varphi : \mathbb{C}^n \to \mathbb{C}$ be 2 analytic functions, $n \geq 1$. Denote by $v(z,w) = |g(w - \overline{z}) + \overline{\varphi}(w - \overline{z})|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. We have*

(I) *v is strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$ if and only if $n = 1$, g and φ are affine functions on $\mathbb{C}$ and $2|\frac{\partial g}{\partial z}(z)\frac{\partial \varphi}{\partial z}(z)| < |\frac{\partial g}{\partial z}(z)|^2 + |\frac{\partial \varphi}{\partial z}(z)|^2$, for each $z \in \mathbb{C}$.*

(II) *Suppose that*

- *(a) g is not an affine function, and*
- *(b) φ is nonconstant on $\mathbb{C}^n$.*

Then v is not psh on $\mathbb{C}^n \times \mathbb{C}^n$.

P r o o f. Obvious by Theorem 4.6 and Theorem 6.2. □

In the sequel, this theorem gives an important family of functions u psh and strictly $n-$ subharmonic on $\mathbb{C}^n$, but v is not psh on $\mathbb{C}^n \times \mathbb{C}^n$. Where $u(z) = |g(z) + \overline{\varphi}(z)|^2$, $v(z,w) = u(w - \overline{z})$ for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$, $g, \varphi : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions; g is not affine and φ is nonconstant. Although, using Theorem 6.2, we can characterize all the holomorphic functions $g_1, f_1, \dots, g_N, f_N : \mathbb{C}^n \to \mathbb{C}$, such that $u_1, \dots, u_N$ are convex functions on $\mathbb{C}^n$ and u is strictly psh on $\mathbb{C}^n$. Where $u_j = |g_j + \overline{f_j}|^{2s}$, $u = (u_1 + \cdots + u_N)$, $N, s \in \mathbb{N}\backslash\{0\}$.

Question. Let $N, s \in \mathbb{N}\backslash\{0,1\}$. Find all the pluriharmonic functions $h_1, \dots, h_N, k_1, \dots, k_s : \mathbb{C}^n \to \mathbb{C}$, such that $u = (u_1 + u_2)$ is psh on $\mathbb{C}^n \times \mathbb{C}$. Where

$$u_1(z,w) = |w^N + h_{N-1}(z)w^{N-1} + \cdots + h_1(z)w + h_0(z)| \text{ and}$$
$$u_2(z,w) = |w^s + k_{s-1}(z)w^{s-1} + \cdots + k_1(z)w + k_0(z)|, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

7. Holomorphic partial differential equations and the convexity of the modulus of a special class of pluriharmonic functions in several complex variables. We have

Theorem 7.1. *Let $\beta, \gamma \in \mathbb{C}$. Consider $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Put $h = (g_1 + \overline{g_2})$.*
Suppose that h is nonconstant and

$$\begin{cases} \dfrac{\partial^2 h}{\partial z^2} h = \gamma \left(\dfrac{\partial h}{\partial z}\right)^2 & \textit{and} \\ \dfrac{\partial^2 h}{\partial \overline{z}^2} h = \beta \left(\dfrac{\partial h}{\partial \overline{z}}\right)^2 \end{cases}$$

on $\mathbb{C}$.

Then $|g_1 + \overline{g_2}(0)|$ and $|g_2 + \overline{g_1}(0)|$ are convex functions on $\mathbb{C}$, β or $\gamma \in S$, where $S = \left\{\dfrac{s-1}{s}, 1 \ / \ s \in \mathbb{N}\backslash\{0\}\right\}$.

P r o o f. If h is affine on $\mathbb{C}$. Since h is nonconstant, then by [2], β or $\gamma \in S$. In fact, if g_1 and g_2 are nonconstant functions, then $\beta = \gamma = 0 \in S$. Assume now h is not affine in all the rest of this proof.

Case 1. $\dfrac{\partial h}{\partial z} = g_1' \neq 0$, on $\mathbb{C}$. Therefore, the differential equation $\dfrac{\partial^2 h}{\partial z^2} h = \gamma \left(\dfrac{\partial h}{\partial z}\right)^2$ implies that

$$g_1''(z)(g_1(z) + \overline{g_2}(z)) = \gamma(g_1'(z))^2,$$

for each $z \in \mathbb{C}$. We prove that $g_1'' \neq 0$. Assume that $g_1'' = 0$ on $\mathbb{C}$. Then g_1 is affine nonconstant. By the differential equation $\dfrac{\partial^2 h}{\partial \overline{z}^2} h = \beta \left(\dfrac{\partial h}{\partial \overline{z}}\right)^2$, we obtain $\overline{g_2''}(g_1 + \overline{g_2}) = \beta(\overline{g_2'})^2$ on $\mathbb{C}$. If now $g_2'' \neq 0$, then $\overline{g_1}$ is holomorphic on $\mathbb{C}$. Since now g_1 is holomorphic on $\mathbb{C}$, it follows that g_1 is constant on $\mathbb{C}$. A contradiction. Then $g_2'' = 0$ on $\mathbb{C}$. Therefore h is affine on $\mathbb{C}$. A contradiction.

Consequently, $g_1'' \neq 0$ on $\mathbb{C}$. Then $\overline{g_2}$ is a holomorphic function on $\mathbb{C}$. Thus g_2 is constant. By [2], (see also [3]), since g_2 is constant, we conclude that $\gamma \in S$, where exactly $S = \left\{\dfrac{s-1}{s}, 1 \ / \ s \in \mathbb{N}\backslash\{0\}\right\}$. In this case $|g_1 + \overline{g_2}(0)|$ is convex on $\mathbb{C}$.

Case 2. $\dfrac{\partial h}{\partial \overline{z}} = \overline{g_2'} \neq 0$, on $\mathbb{C}$. This case is obvious by the above situation. We prove that $\beta \in S$ and $|g_2 + \overline{g_1}(0)|$ is convex on $\mathbb{C}$. □

Theorem 7.2. *Let $\beta, \gamma \in \mathbb{C}$. Given $h : \mathbb{C} \to \mathbb{C}$ be harmonic and not affine. Suppose that*

$$\begin{cases} \dfrac{\partial^2 h}{\partial z^2} h = \gamma \left(\dfrac{\partial h}{\partial z}\right)^2 & \text{and} \\ \dfrac{\partial^2 h}{\partial \overline{z}^2} h = \beta \left(\dfrac{\partial h}{\partial \overline{z}}\right)^2 \end{cases}$$

on $\mathbb{C}$. Then

$$h(\xi) = \begin{cases} (a_1\xi + b_1)^{m_1}, & \text{or} \\ (\overline{a_2\xi + b_2})^{m_2}, & \text{or} \\ e^{\alpha_1\xi + \beta_1}, & \text{or} \\ e^{\overline{\alpha_2\xi + \beta_2}} \end{cases}$$

and β or $\gamma \in S = \left\{0, \dfrac{s-1}{s}, 1/s \in \mathbb{N}\backslash\{0,1\}\right\}$, for each $\xi \in \mathbb{C}$, where $a_1, b_1, a_2, b_2, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$, $(a_1 a_2 \alpha_1 \alpha_2 \neq 0)$, $m_1, m_2 \in \mathbb{N}\backslash\{0,1\}$.

Corollary 7.1. *Let $\gamma, \beta \in \mathbb{C}$. The problem*

$$\begin{cases} h : \mathbb{C} \to \mathbb{C} \text{ be not affine} \\ u(z,w) = |w - h(z)| \text{ is psh on } \mathbb{C}^2; \\ \dfrac{\partial^2 h}{\partial z^2} h = \beta \left(\dfrac{\partial h}{\partial z}\right)^2, \quad \text{and} \\ \dfrac{\partial^2 h}{\partial \overline{z}^2} h = \gamma \left(\dfrac{\partial h}{\partial \overline{z}}\right)^2 \quad \text{on } \mathbb{C} \end{cases}$$

have only one the solution described in the above theorem.

This problem is in fact equivalent with the following problem

$$\begin{cases} h : \mathbb{C} \to \mathbb{C} \text{ be not affine} \\ u(z,w) = |w - h(z)| \text{ is psh on } \mathbb{C}^2; \quad \text{and} \\ v(z,w) = |h(w - \overline{z})| \text{ is psh on } \mathbb{C}^2 \end{cases}$$

which is a technical result in complex analysis.

Let $n \geq 1$. Recall that, the problem

$$\begin{cases} g : \mathbb{C}^n \to \mathbb{C} \text{ be nonconstant} \\ u_1(z,w) = \log|w - g(z)| \text{ is psh on } \mathbb{C}^n \times \mathbb{C}, \text{and} \\ v_1(z,\zeta) = |g(\zeta - \overline{z})| \text{ is psh on } \mathbb{C}^n \times \mathbb{C}^n \end{cases}$$

is equivalent to the problem

$$\begin{cases} g : \mathbb{C}^n \to \mathbb{C} \text{ be analytic nonconstant, and} \\ |g| \text{ is convex on } \mathbb{C}^n. \end{cases}$$

But this is equivalent to the problem

$$\begin{cases} g : \mathbb{C}^n \to \mathbb{C} \text{ be analytic nonconstant, and} \\ \left| \sum_{j,k=1}^{n} \frac{\partial^2 g}{\partial z_j \partial z_k}(z)\overline{g}(z)\alpha_j\alpha_k \right| \le \left| \sum_{j=1}^{n} \frac{\partial g}{\partial z_j}(z)\alpha_j \right|^2, \end{cases}$$

$\forall z \in \mathbb{C}^n, \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.

Suppose that $n = 1$. Then we can solve the system

$$\begin{cases} g : \mathbb{C} \to \mathbb{C} \text{ be holomorphic nonconstant; and} \\ g''(g + c) = \gamma(g')^2 \text{ on } \mathbb{C}, \end{cases}$$

($(c, \gamma) \in \mathbb{C}^2$, γ is to describe exactly). In this case we solve the holomorphic differential equation $g''(g + c) = \gamma(g')^2$, g is nonconstant. From the paper Abidi [2], we have exactly $\gamma \in S = \left\{0, \frac{s-1}{s}, 1 \ / \ s \in \mathbb{N}\backslash\{0, 1\}\right\}$. Using this technical holomorphic differential equation, in fact we can solve several holomorphic (or antiholomorphic) partial differential equations in $\mathbb{C}^N$, ($N \ge 1$).

Example. (I) Let $(a, b) \in \mathbb{C}^2$. Find all the holomorphic functions $g : \mathbb{C} \to \mathbb{C}$ such that u is psh on $\mathbb{C}^2$, where

$$u(z, w) = |g(w + \overline{z}) + ag'(w + \overline{z}) + bg''(w + \overline{z})|, \quad \text{for } (z, w) \in \mathbb{C}^2.$$

(II) Find all the analytic functions $g : \mathbb{C} \to \mathbb{C}$ such that $g^{(4)}(g'' + c) = \gamma(g^{(3)})^2$, where $c \in \mathbb{C}$ and ($\gamma \in \mathbb{C}$ is to describe exactly). Prove that $|g^{(3)}|$ is convex on $\mathbb{C}$.

(III) Let $N \in \mathbb{N}\backslash\{0\}$ and $(A_0, \dots, A_N) \in \mathbb{C}^{N+1}$. Find all the holomorphic functions $k : \mathbb{C} \to \mathbb{C}$ such that φ is psh on $\mathbb{C}^2$, here

$$\varphi(z, w) = |A_N k^{(N)}(w + \overline{z}) + \cdots + A_0 k(w + \overline{z})|, \quad \text{for } (z, w) \in \mathbb{C}^2.$$

Remark 7.1. Let $n \ge 2$ and $\gamma_j, \beta_j \in \mathbb{C}$, for $1 \le j \le n$. The problem

$$\begin{cases} h : \mathbb{C}^n \to \mathbb{C} \text{ be nonconstant} \\ u(z, w) = |w - h(z)| \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \\ \dfrac{\partial^2 h}{\partial z_j^2} h = \gamma_j \left(\dfrac{\partial h}{\partial z_j}\right)^2, \text{ and} \\ \dfrac{\partial^2 h}{\partial \overline{z}_j^2} h = \beta_j \left(\dfrac{\partial h}{\partial \overline{z}_j}\right)^2 \text{ on } \mathbb{C}^n \end{cases}$$

does not imply that $|h|^2$ is convex on $\mathbb{C}^n$. We can consider $h(z) = z_1 z_2$, for $z = (z_1, z_2) \in \mathbb{C}^2$. Therefore for the above system, it follows to add another condition for the study of the real convexity.

8. The comparison between real and complex convexity.

We begin this study by the following.

Lemma 8.1. *Let $s \in \mathbb{R}_+ \backslash \{0\}$ and D a convex domain of $\mathbb{C}^n$, $n \geq 1$.*

(a) Suppose that there exist a nonconstant function $f_1 : D \to \mathbb{C}$ such that u_1 and u_2 are psh on $D \times \mathbb{C}$,where $u_1(z,w) = |w + f_1(z)|^s$ and $u_2(z,w) = |w + \overline{f}_1(z)|^s$, for $(z,w) \in D \times \mathbb{C}$. Then $s \geq 1$.

(b) Suppose that the condition $u(z,w) = |w + f(z)|^s$ is psh on $D \times \mathbb{C}$ implies that $v(z,w) = |w + \overline{f}(z)|^s$ is psh on $D \times \mathbb{C}$ (for all functions $f : D \to \mathbb{C}$). Then $s \geq 1$.

(c) Let $s \in]0,1[$. Denote by $v_1(z,w) = |w + f(z)|^s$, $v_2(z,w) = |w + \overline{f}(z)|^s$, $(z,w) \in D \times \mathbb{C}$. In general v_1 is psh on $D \times \mathbb{C}$ does not imply that the function v_2 is psh on $D \times \mathbb{C}$ (for each $f : D \to \mathbb{C}$), (and conversely).

We have now the following technical tool comparison.

Proposition 8.1. *Let $\alpha \in \mathbb{R}_+$ and D a convex domain of $\mathbb{C}^n$, $n \geq 1$. Given $f : D \to \mathbb{C}$ be a function. Let $u(z,w) = |w + f(z)|^\alpha$ and $v(z,w) = |w + \overline{f}(z)|^\alpha$, for $(z,w) \in D \times \mathbb{C}$. We have*

(a) u is convex on $D \times \mathbb{C}$ if and only if v is convex on $D \times \mathbb{C}$. In this case we have $\alpha \geq 1$.

(b) Let $s \in]0,1[$. Put $g(z) = z$, $u_1(z,w) = |w + z|^s$, $v_1(z,w) = |w + \overline{z}|^s$, $(z,w) \in \mathbb{C}^2$. Then u_1 is psh on $\mathbb{C}^2$. But v_1 is not psh on $\mathbb{C}^2$.

(c) u is strictly psh on $D \times \mathbb{C}$ if and only if $n = 1$, $\alpha = 2$, f is harmonic on D and $\left|\dfrac{\partial f}{\partial \overline{z}}\right| > 0$ on D. In this case v is not in general strictly psh on $D \times \mathbb{C}$. Therefore the condition u is strictly psh on $D \times \mathbb{C}$ does not implies that the function v is strictly psh on $D \times \mathbb{C}$. But u is convex on $D \times \mathbb{C}$ if and only if v is convex on $D \times \mathbb{C}$.

Proof. We can see [1], [2] and [4]. □

In the theory of holomorphic and convex functions, we have

Theorem 8.1. *For each $k \in \mathbb{N}$, let $f_k : \mathbb{C}^n \to \mathbb{C}$ be analytic, where $n \geq 1$. Put*

$$u(z,w) = e^{e^{\left(\left|\sum_{k \geq 0} f_k(z) w^k\right|\right)}},$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. *Then we have*

(I) *Suppose that for each* $k_0 \in \mathbb{N}$, $\exists N \geq k_0$, *such that* $f_N \neq 0$. *Then* u *is convex on* $\mathbb{C}^n \times \mathbb{C}$ *if and only if*

$$f_k(z) = \frac{b^k}{k!} e^{(\langle z/a \rangle + c)}$$

for all $z \in \mathbb{C}^n$, *where* $a \in \mathbb{C}^n$ *and* $b, c \in \mathbb{C}$.

(II) *Let*

$$v(z, w) = \left| \sum_{k \geq 1} f_k(z) w^k + \overline{f_0}(z) \right|,$$

for every $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. *Then there exists several cases where* v *is convex on* $\mathbb{C}^n \times \mathbb{C}$.

Proof. (I). Obvious by [2].

(II). *Case 1.* f_0 is constant on $\mathbb{C}^n$. Put $g_0 = \overline{f_0}$ and $g_k = f_k$, for each $k \geq 1$. Let $\sum_{k \geq 0} g_k(z) w^k = F(z, w)$ and $v = |F|$, where $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is holomorphic. Since v is convex on $\mathbb{C}^n \times \mathbb{C}$, then by [2], we have two possible states.

State 1. There exists $N \geq 0$, such that $g_k = 0$, for each $k \geq N + 1$. If $g_k = 0$, for each $k \in \mathbb{N}$. The proof is finished. Suppose that $g_N \neq 0$. Then $F(z, w) = \sum_{k=0}^{N} g_k(z) w^k$, for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Since $|F|$ is convex on $\mathbb{C}^n \times \mathbb{C}$, then $F(z, w) = (a_1 z_1 + \cdots + a_n z_n + bw + c)^N$, where $a_1, \ldots, a_n, b, c \in \mathbb{C}$. Put $(a_1, \ldots, a_n) = a \in \mathbb{C}^n$. Note that we can calculate $f_0, f_1, \ldots, f_N$ by their expressions.

State 2. For each $k_0 \in \mathbb{N}$, there exists $k_1 \geq k_0$, $g_{k_1} \neq 0$. In this case, by Abidi [2], $F(z, w) = e^{(<z/a> + bw + c)}$, $a \in \mathbb{C}^n$, $b, c \in \mathbb{C}$. Since $F(z, 0) = \overline{f_0}(z)$, then f_k are constant functions and we can calculate f_k, for each $k \in \mathbb{N}$.

Case 2. f_0 is nonconstant on $\mathbb{C}^n$. Define $F_1(z, w) = \sum_{k \geq 1} f_k(z) w^k$, $F_0(z, w) = f_0(z)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. F_1 and F_0 are holomorphic functions on $\mathbb{C}^n \times \mathbb{C}$. $v = |F_1 + \overline{F_0}|$, on $\mathbb{C}^n \times \mathbb{C}$. Since F_1 is holomorphic in (z, w), then $F_1(z, w)$ is independent of $(\overline{z}, \overline{w})$. If $F_1(z, w) = 0$, for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. This case is obvious.

Now assume that $F_1 \neq 0$. Since F_0 is holomorphic, then $\overline{F_0}(z,w)$ is independent of the variable (z,w). Now since the variables (z,w) and $(\overline{z},\overline{w})$ are $\mathbb{C}-$ linearly independent and the condition

$$|\alpha F_1(z,w) + \beta\overline{F_0}(z,w)| = 0,$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, where $(\alpha,\beta) \in \mathbb{C}^2$, implies that $\alpha = \beta = 0$. It follows that the function v_1, defined by

$$v_1(z,w,\xi,\zeta) = |F_1(z,w) + \overline{F_0}(\xi,\zeta)|,$$

is convex on $(\mathbb{C}^n \times \mathbb{C}) \times (\mathbb{C}^n \times \mathbb{C})$. Therefore φ is a convex function on $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$, where
$\varphi(z,w,\xi) = |\sum_{k\geq 1} f_k(z)w^k + \overline{f_0}(\overline{\xi})|$. Now, put $\psi(\xi) = \overline{f_0}(\overline{\xi})$, for $\xi \in \mathbb{C}^n$. Then ψ is a holomorphic function on $\mathbb{C}^n$ and we have the following two states.

State 1. ψ is affine on $\mathbb{C}^n$. Since $F_1 \neq 0$, then F_1 is nonconstant on $\mathbb{C}^n \times \mathbb{C}$. Consequently, F_1 is affine on $\mathbb{C}^n \times \mathbb{C}$. $F_1(z,w) = f_1(z)w$, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The condition F_1 is affine implies that f_1 is constant on $\mathbb{C}^n$. Therefore the function, defined by

$$\sum_{k\geq 1} f_k(z)w^k + \overline{f_0}(z) = (Aw + \overline{f_0}(z))$$

is affine on $\mathbb{C}^n \times \mathbb{C}$, where $A = f_1(z) \in \mathbb{C}\backslash\{0\}$.

State 2. ψ is not affine on $\mathbb{C}^n$. Since $|F_1(z,w) + \psi(\overline{\xi})|$ is convex on $\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$, then F_1 is constant and consequently, $F_1 = 0$ on $\mathbb{C}^n \times \mathbb{C}$.
It follows that $f_0(\xi) = (< \xi/a > +b)^m$, where $a \in \mathbb{C}^n\backslash\{0\}$, $b \in \mathbb{C}$ and $m \in \mathbb{N}\backslash\{0\}$. $\square$

9. On convex and plurisubharmonic or subharmonic functions. In this section we study many technical investigations between subharmonic, plurisubharmonic and convex functions on $\mathbb{C}^n$, $n \geq 1$.

We begin this section by

Remark 9.1. Let $u : \mathbb{C}^n \to \mathbb{R}$ be a function. Put $v_1(z,w) = u(w + \overline{z})$ and $v_{(\alpha,\beta)}(z) = u(\langle z/\alpha_1\rangle + \langle \beta_1/z\rangle, \ldots, < z/\alpha_n > + \langle \beta_n/z\rangle)$, for $(z,w), (\alpha_j,\beta_j) \in \mathbb{C}^n \times \mathbb{C}^n$, $j \in \{1,\ldots,n\}$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\beta = (\beta_1,\ldots,\beta_n)$. Suppose that v_1 is subharmonic on $\mathbb{C}^n \times \mathbb{C}^n$. Then we can not conclude that u is convex on $\mathbb{C}^n$.

Example. $u_0(z) = |z|^{\frac{1}{4}}$, for $z \in \mathbb{C}$. u_0 is continuous and not convex on $\mathbb{C}$. But v_0 is subharmonic on $\mathbb{C}^2$, where $v_0(z,w) = u_0(w + \overline{z})$, for $(z,w) \in \mathbb{C}^2$. But if

$v_{(\alpha,\beta)}$ is subharmonic (respectively n-sh) on $\mathbb{C}^n$, for each $(\alpha,\beta) \in (\mathbb{C}^n)^n \times (\mathbb{C}^n)^n$, then u is convex on $\mathbb{C}^n$. If v_1 is psh on $\mathbb{C}^n \times \mathbb{C}^n$, then we can prove that u is convex on $\mathbb{C}^n$. Note that if for each $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}^n$, the function $v_{(\alpha,\beta)}$ is psh on $\mathbb{C}^n \times \mathbb{C}^n$. We can prove that u is convex on $\mathbb{C}^n$.

We study in this section several problems like the remark and some related topics.

Theorem 9.1. *Let* $u : \mathbb{C}^n \to [-\infty, +\infty[$ *be a function. Put*

$$u_{(\alpha,\beta)}(z,) = u(\langle z/\alpha_1\rangle + \langle \beta_1/z\rangle, \ldots, \langle z/\alpha_n\rangle + \langle \beta_n/z\rangle),$$

for $z \in \mathbb{C}^n$, $(\alpha_1, \ldots, \alpha_n) = \alpha \in (\mathbb{C}^n)^n$, $(\beta_1, \ldots, \beta_n) = \beta \in (\mathbb{C}^n)^n$. *The following are equivalents*

(I) $u(\mathbb{C}^n) \subset \mathbb{R}$ *and* u *is convex on* $\mathbb{C}^n$;

(II) *For each* $\alpha, \beta \in (\mathbb{C}^n)^n$, $u_{(\alpha,\beta)}$ *is psh on* $\mathbb{C}^n$;

(III) *For every* $\alpha, \beta \in (\mathbb{C}^n)^n$, $u_{(\alpha,\beta)}$ *is subharmonic (sh) on* $\mathbb{C}^n$.

Proof. (I) implies (II). u is convex on $\mathbb{C}^n$, then for all $\alpha, \beta \in (\mathbb{C}^n)^n$, $u_{(\alpha,\beta)}$ is convex on $\mathbb{C}^n$. Therefore $u_{(\alpha,\beta)}$ is psh on $\mathbb{C}^n$.

(II) implies (I). Suppose that there exists $z_0 \in \mathbb{C}^n$, such that $u(z_0) = -\infty$.

Case 1. $n = 1$. We prove that there exists $\alpha_0, \beta_0 \in \mathbb{C}$ such that $\{z \in \mathbb{C} \ / \ \alpha_0 z + \beta_0 \overline{z} = z_0\}$ is not polar on $\mathbb{C}$.

Step 1. $z_0 \in \mathbb{R}$. We take $\alpha_0 = \beta_0 = 1$.

Step 2. $z_0 \in i\mathbb{R}$. We take $\alpha_0 = -\beta_0 = 1$.

Step 3. $z_0 = (a + ib)$, $a, b \in \mathbb{R}$, with $ab \neq 0$.

Put $z = x_1 + ix_2$, $x_1 = \mathrm{Re}(z)$. Let $\alpha = (\alpha_1 + i\alpha_2)$, $\beta = (\beta_1 + i\beta_2) \in \mathbb{C}$, $\alpha_1 = \mathrm{Re}(\alpha)$, $\beta_1 = \mathrm{Re}(\beta)$. We solve the equation $\alpha z + \beta \overline{z} = z_0$. Then

$$\begin{cases} \alpha_1 x_1 - \alpha_2 x_2 + \beta_1 x_1 + \beta_2 x_2 = a \\ \alpha_2 x_1 + \alpha_1 x_2 - \beta_1 x_2 + \beta_2 x_1 = b. \end{cases}$$

Note that the system

$$\begin{cases} (\alpha_1 + \beta_1) x_1 + (\beta_2 - \alpha_2) x_2 = a \\ (\alpha_2 + \beta_2) x_1 + (\alpha_1 - \beta_1) x_2 = b \end{cases}$$

have an infinite number of solutions if

$$(\alpha_2 + \beta_2) x_1 + (\alpha_1 - \beta_1) x_2 = K[(\alpha_1 + \beta_1) x_1 + (\beta_2 - \alpha_2) x_2]$$

and $b = Ka$, where $K \in \mathbb{R}$.
Therefore,

$$\alpha_1^2 - \beta_1^2 + \alpha_2^2 - \beta_2^2 = 0.$$

Put $\alpha_1 = \beta_1 = 1$. Therefore

$$\begin{cases} 2x_1 + (\beta_2 - \alpha_2)x_2 = a \\ (\alpha_2 + \beta_2)x_1 = b \end{cases}$$

Take $\beta_2 = \alpha_2$. Then $x_1 = \dfrac{a}{2}$ and $\alpha_2 + \beta_2 = 2K$. Therefore $\alpha_2 = \beta_2 = K$. Now let $\alpha_0 = 1 + iK = \beta_0$, where $K = \dfrac{b}{a}$. It follows that $\{z \in \mathbb{C}/\ \alpha_0 z + \beta_0 \overline{z} = z_0\} = A$ is not polar on $\mathbb{C}$, because $A = \{z \in \mathbb{C}/\ 2x_1 = a\} = \{\dfrac{a}{2} + ix_2/\ x_2 \in \mathbb{R}\}$. But we know that the real line $\{\dfrac{a}{2} + ix_2/\ x_2 \in \mathbb{R}\}$ is not polar on $\mathbb{C}$. Consequently, $\psi(z) = u(\alpha_0 z + \beta_0 \overline{z})$ is not sh on $\mathbb{C}$. A contradiction. Consequently, $u(z) > -\infty$, for every $z \in \mathbb{C}$.

State 1. u is a function of class C^2 on $\mathbb{C}$. Let $\alpha, \beta \in \mathbb{C}$. Put $v(z) = u(\alpha z + \beta \overline{z})$, for $z \in \mathbb{C}$. v is a function of class C^2 on $\mathbb{C}$.

$$\frac{\partial^2 v}{\partial \overline{z} \partial z}(z) = \frac{\partial^2 u}{\partial \overline{\zeta} \partial \zeta}(\alpha z + \beta \overline{z})\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial \overline{\zeta} \partial \zeta}(\alpha z + \beta \overline{z})\beta\overline{\beta} + 2\mathrm{Re}[\frac{\partial^2 u}{\partial \zeta^2}(\alpha z + \beta \overline{z})\alpha\overline{\beta}] \geq 0$$

for each $(\alpha, \beta) \in \mathbb{C}^2$. Deduce that

$$\left|\frac{\partial^2 u}{\partial \zeta^2}\right| \leq \frac{\partial^2 u}{\partial \overline{\zeta} \partial \zeta} = \frac{1}{4}\Delta(u)$$

on $\mathbb{C}$. Consequently, u is convex on $\mathbb{C}$.

State 2. $u : \mathbb{C} \to \mathbb{R}$ be a function. Let $\rho : \mathbb{C} \to \mathbb{R}_+$, ρ is a C^∞ radial function, $\mathrm{supp}(\rho) \subset D(0,1)$ and $\int \rho(\xi) dm_2(\xi) = 1$. Put $\rho_\delta(z) = \dfrac{1}{\delta^2}\rho(\dfrac{z}{\delta})$, for $\delta > 0$ and $z \in \mathbb{C}$. Since u is sh on $\mathbb{C}$, then the convolution $u * \rho_\delta$ is also subharmonic, for every $\delta > 0$. Let $\alpha, \beta \in \mathbb{C}$. Put $\theta(z) = u(\alpha z + \beta \overline{z})$, for $z \in \mathbb{C}$. Then θ is subharmonic on $\mathbb{C}$. Therefore for each $\xi \in \mathbb{C}$, θ_1 is sh on $\mathbb{C}$, where $\theta_1(z) = u(\alpha z + \beta \overline{z} - \xi)$. Therefore the function θ_2, $\theta_2(z) = \theta_1(z)\rho_\delta(\xi)$, is also sh on $\mathbb{C}$. Let $\varphi_{(\alpha,\beta)}(z) = \int \theta_1(z)\rho_\delta(\xi) dm_2(\xi)$, for $z \in \mathbb{C}$. Then $\varphi_{(\alpha,\beta)}$ is a C^∞ subharmonic function on $\mathbb{C}$. Note that $\varphi_{(\alpha,\beta)}(z) = u * \rho_\delta(\alpha z + \beta \overline{z})$, for $z \in \mathbb{C}$. Since $u * \rho_\delta(\zeta) = \int u(\zeta - \xi)\rho_\delta(\xi) dm_2(\xi)$, then $u * \rho_\delta$ is a function of class C^∞ and decreases to u on $\mathbb{C}$ (if δ decreases to 0).

It follows that $\psi_\delta = u * \rho_\delta$ is a convex function on $\mathbb{C}$, for each $\delta > 0$, by the state 1. Now the sequence $(\psi_{\frac{1}{j}})_{j \in \mathbb{N} \setminus \{0\}}$ decreases to u if j increases to $+\infty$.

Since $u(\mathbb{C}) \subset \mathbb{R}$, then u is convex on $\mathbb{C}$. If $n = 1$, the proof of the theorem is now finished.

Case 2. $n \geq 2$. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. If $u(a) = -\infty$, by the case 1, there exists $\alpha_{0j}, \beta_{0j} \in \mathbb{C}$, such that $A_j = \{z_j \in \mathbb{C} \ / \ \alpha_{0j} z_j + \beta_{0j} \overline{z_j} = a_j\}$ is not polar on $\mathbb{C}$, for each $j \in \{1, \ldots, n\}$. Denote by $A = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \ / \ z_j \in A_j, \forall j = 1, \ldots, n\}$. We have $A = A_1 \times \cdots \times A_n$. Then A is not pluripolar in $\mathbb{C}^n$. Let now $\psi(z) = u(\langle z/\overline{\alpha_0}\rangle + \langle \beta_0/z\rangle)$, for $z \in \mathbb{C}^n$, where $\alpha_0 = (\alpha_{01}, \ldots, \alpha_{0n})$, $\beta_0 = (\beta_{01}, \ldots, \beta_{0n}) \in \mathbb{C}^n$. ψ is psh on $\mathbb{C}^n$ and $A \subset \{z \in \mathbb{C}^n \ / \ \psi(z) = -\infty\}$ is not pluripolar. A contradiction.

Therefore $u(a) > -\infty$ and we have $u(\mathbb{C}^n) \subset \mathbb{R}$.

State 1. u is a function of class C^2 on $\mathbb{C}^n$. Let $v = u_{(\alpha,\beta)}$, $\alpha, \beta \in (\mathbb{C}^n)^n$, $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n)$. $v(z) = u(T(z) + \overline{\tau}(z))$,

$$T(z) = (\langle z/\alpha_1\rangle, \ldots, \langle z/\alpha_n\rangle), \quad \tau(z) = (\langle z/\beta_1\rangle, \ldots, \langle z/\beta_n\rangle),$$

for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. For $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{C}^n$, we have

$$\begin{aligned}
&\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z)\delta_j \overline{\delta_k} \\
=&2\mathrm{Re}\left[\sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \xi_m}(T(z) + \overline{\tau}(z)) \left(\sum_{j=1}^{n} \delta_j \frac{\partial T_m}{\partial z_j}(z)\right) \overline{(\sum_{k=1}^{n} \delta_k \frac{\partial \tau_s}{\partial z_k}(z))}\right] \\
&+ \sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \overline{\xi_m}}(T(z) + \overline{\tau}(z)) \left(\sum_{j=1}^{n} \delta_j \frac{\partial T_s}{\partial z_j}(z)\right) \overline{\left(\sum_{k=1}^{n} \delta_k \frac{\partial T_m}{\partial z_k}(z)\right)} \\
&+ \sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \overline{\xi_m}}(T(z) + \overline{\tau}(z)) \left(\sum_{j=1}^{n} \delta_j \frac{\partial \tau_m}{\partial z_j}(z)\right) \overline{\left(\sum_{k=1}^{n} \delta_k \frac{\partial \tau_s}{\partial z_k}(z)\right)} \geq 0.
\end{aligned}$$

Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, where for $1 \leq m \leq n$,

$$\gamma_m = \sum_{j=1}^{n} \delta_j \frac{\partial T_m}{\partial z_j}(z) = \sum_{k=1}^{n} \delta_k \frac{\partial \tau_m}{\partial z_k}(z) \in \mathbb{C}.$$

Note that $\dfrac{\partial T_m}{\partial z_j}(z)$, $\dfrac{\partial \tau_m}{\partial z_k}(z)$ are independent of z, because T and τ are $\mathbb{C}-$ linear on $\mathbb{C}^n$. Therefore we have

$$2\mathrm{Re}\left[\sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \xi_m}(T(z) + \overline{\tau}(z))\gamma_m \gamma_s\right] + 2\sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \overline{\xi_m}}(T(z) + \overline{\tau}(z))\overline{\gamma_m}\gamma_s \geq 0$$

for each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$. Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then if we replace γ by $\lambda\gamma$, we have

$$-\mathrm{Re}\left(\lambda^2 \sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \xi_m}(T(z) + \overline{\tau}(z))\gamma_m \gamma_s\right) \leq \sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \overline{\xi_m}}(T(z) + \overline{\tau}(z))\overline{\gamma_m}\gamma_s$$

for each $\lambda \in \partial D(0,1)$. Therefore, by the next lemma we have

$$\left|\sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \xi_m}(T(z) + \overline{\tau}(z))\gamma_m \gamma_s\right| \leq \sum_{s,m=1}^{n} \frac{\partial^2 u}{\partial \xi_s \partial \overline{\xi_m}}(T(z) + \overline{\tau}(z))\overline{\gamma_m}\gamma_s$$

for each $z, \gamma \in \mathbb{C}^n$. Then u is convex on $\mathbb{C}^n$.

Lemma 9.1. *Let $a, b \in \mathbb{C}$. Assume that $Re(\lambda a) \leq b$, for each $\lambda \in \partial D(0,1)$. Then $b \geq 0$ and $|a| \leq b$.*

Proof. Obvious. □

State 2. The function u is not of class C^2. Fix $A, B \in M_n(\mathbb{C})$. We have for each $\xi \in \mathbb{C}^n$, the function u_1 is psh on $\mathbb{C}^n$, where $u_1(z) = u(Az + B\overline{z} - \xi)$, for $z \in \mathbb{C}^n$. Let $\rho : \mathbb{C}^n \to \mathbb{R}_+$ be a function of class C^∞, $\mathrm{supp}(\rho) \subset B(0,1)$, ρ is a radial function, $\int \rho(\xi) dm_{2n}(\xi) = 1$. Put $\rho_\delta(z) = \frac{1}{\delta^{2n}} \rho(\frac{z}{\delta})$, for $\delta > 0$. We have $\rho_\delta \in C_c^\infty(\mathbb{C}^n)$. Therefore the new function ψ is psh on $\mathbb{C}^n$, where $\psi(z) = \int u(Az + B\overline{z} - \xi)\rho_\delta(\xi) dm_{2n}(\xi)$, $z \in \mathbb{C}^n$. Now let $\varphi_\delta(z) = \int u(z-\xi)\rho_\delta(\xi) dm_{2n}(\xi)$. φ_δ is a function of class C^∞ and psh on $\mathbb{C}^n$. The sequence $(\varphi_{\frac{1}{j}})_{j \in \mathbb{N} \setminus \{0\}}$ decreases to u if j increases to $+\infty$. Let $\psi_\delta(z) = \varphi_\delta(Az + B\overline{z})$, $\delta > 0$. Since ψ is psh on $\mathbb{C}^n$, then ψ_δ is psh on $\mathbb{C}^n$. φ_δ is a function of class C^∞ implies that ψ_δ is also of class C^∞ on $\mathbb{C}^n$. By state 1, φ_δ is convex on $\mathbb{C}^n$. Since $u(\mathbb{C}^n) \subset \mathbb{R}$, then u is convex on $\mathbb{C}^n$. □

In complex dimension 1, we have the following

Theorem 9.2. *Let $u : \mathbb{C} \to \mathbb{R}$ be a function. The following two conditions are equivalent*

(I) *u is convex on $\mathbb{C}$;*

(II) *For each $\alpha, \beta \in \mathbb{C}$, v is sh on $\mathbb{C}$, where $v(z) = u(\alpha z + \beta \overline{z})$, for $z \in \mathbb{C}$.*

But we can see that we have two observations:

Proposition 9.1. *Let $u_1 : \mathbb{C} \to [-\infty, +\infty[$ be a function. The following are equivalent*

(I) $u_1(\mathbb{C}) \subset \mathbb{R}$ *and* u_1 *is convex on* $\mathbb{C}$;
(II) *For each* $\alpha, \beta \in \mathbb{C}$, v_1 *is sh on* $\mathbb{C}$, *where* $v_1(z) = u(\alpha z + \beta \overline{z})$, *for* $z \in \mathbb{C}$.

The second observation is the following: On $\mathbb{C}^n$, $n \geq 2$, Theorem 9.2 is not true.

Example. Let $u(z) = |z_1(z_2 + 1)|$, $z = (z_1, z_2) \in \mathbb{C}^2$. $u(\alpha z + \beta \overline{z}) = |(\alpha z_1 + \beta \overline{z_1})(\alpha z_2 + \beta \overline{z_2} + 1)| = v(z)$, for $\alpha, \beta \in \mathbb{C}$. u is continuous on $\mathbb{C}^2$. We have $v(., z_2)$ and $v(z_1, .)$ are convex functions on $\mathbb{C}$, for each z_2, z_1 fixed on $\mathbb{C}$. Therefore v is continuous and subharmonic on $\mathbb{C}$ and consequently, v is sh on $\mathbb{C}^2$, for each $\alpha, \beta \in \mathbb{C}$. But u is not convex on $\mathbb{C}^2$, because ψ is not convex on $\mathbb{C}$; $\psi(z_1) = u(z_1, z_1)$, for $z_1 \in \mathbb{C}$. Therefore it should be enable to replace this property by another in several variables. We have

Proposition 9.2. *Let* $n \in \mathbb{C} \backslash \{0\}$. *Let* $u : \mathbb{C}^n \to \mathbb{R}$ *be a continuous function* ($u \in C(\mathbb{C}^n)$). *Given* $T_1, T_2 : \mathbb{C}^n \to \mathbb{C}^n$ *be two* $\mathbb{C}-$ *linear bijective transformations on* $\mathbb{C}^n$. *Put* $v(z, w) = u(w - \overline{z})$, $\varphi(z, w) = u(T_2(w) - \overline{T}_1(z))$ *and* $K_{(\alpha,\beta)}(z, w) = u(\langle w/\alpha_1 \rangle + \langle \beta_1/z \rangle, \ldots, \langle w/\alpha_n \rangle + \langle \beta_n/z \rangle)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{C}^n)^n$. *The following conditions are equivalent*
(I) u *is convex on* $\mathbb{C}^n$;
(II) v *is convex on* $\mathbb{C}^n \times \mathbb{C}^n$;
(III) v *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$;
(IV) φ *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$;
(V) *for each* $\alpha, \beta \in (\mathbb{C}^n)^n$, $K_{(\alpha,\beta)}$ *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$.

We have also the same result for $u \in C^k(\mathbb{C}^n)$, $k \in \mathbb{N} \cup \{\infty\} \backslash \{0\}$.

Proof. Obvious by Theorem 9.1. $\square$

Remark 9.2. Let $g : \mathbb{C}^n \to \mathbb{C}$ be holomorphic, $|g|$ is convex on $\mathbb{C}^n$. Observe that for each $\psi : \mathbb{C}^m \to \mathbb{C}^n$ prh, the function $|g(\psi)|$ is psh on $\mathbb{C}^m$, $n, m \geq 1$. This proves a new technical tool in complex function theory.

In general, we can prove in the next corollary the importance of the point $(-\infty)$ in real convexity and in several complex variables.

Corollary 9.1. *Let* $u : \mathbb{C}^n \to [-\infty, +\infty[$ *be a function,* $n \geq 1$. *Define* $v_{(\alpha,\beta)}(z, w) = u(\langle w/\alpha_1 \rangle + \langle \beta_1/z \rangle, \ldots, \langle w/\alpha_n \rangle + \langle \beta_n/z \rangle)$, $\psi_{(\alpha,\beta)}(z) = v_{(\alpha,\beta)}(z, z)$, $K_{(a,b)}(z, w) = u(a_1 w_1 + b_1 \overline{z}_1, \ldots, a_n w_n + b_n \overline{z}_n)$, *for* $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$, $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$, $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{C}^n)^n$. *The following are equivalents*
(I) $u(\mathbb{C}^n) \subset \mathbb{R}$ *and* u *is convex on* $\mathbb{C}^n$;
(II) *For every* $(a, b) \in \mathbb{C}^n \times \mathbb{C}^n$, *the function* $K_{(a,b)}$ *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$;

(III) $v_{(\alpha,\beta)}$ *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$, *for each* $\alpha, \beta \in (\mathbb{C}^n)^n$;
(IV) $\psi_{(\alpha,\beta)}$ *is subharmonic (sh) on* $\mathbb{C}^n$, *for every* $\alpha, \beta \in (\mathbb{C}^n)^n$.

Corollary 9.2. *Let* $u : \mathbb{C}^n \to \mathbb{R}$ *be a continuous function,* $T_1, T_2 : \mathbb{C}^n \to \mathbb{C}^n$ *be two* $\mathbb{C}-$ *linear bijective transformations,* $n \geq 1$. *Put* $v(z,w) = u(T_2(w) - \overline{T_1}(z))$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. *The following conditions are equivalent*
(I) u *is convex on* $\mathbb{C}^n$;
(II) v *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$;
(III) $u_{(A,B)}$ *is sh on* $\mathbb{C}^n$, *for each* $A, B \in M_n(\mathbb{C})$, *where* $u_{(A,B)}(z) = u(Az + B\overline{z})$.

Proof. Obvious. We can also see the proof of theorem 9.1. □

We have

Corollary 9.3. *Let* $u : \mathbb{C}^n \to [-\infty, +\infty[$, $n \geq 1$ *and* $a, b \in \mathbb{C} \backslash \{0\}$. *Put* $v(z,w) = u(aw - b\overline{z})$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. *Suppose that* v *is psh on* $\mathbb{C}^n \times \mathbb{C}^n$. *Then* $u(\mathbb{C}^n) \subset \mathbb{R}$ *and* u *is continuous on* $\mathbb{C}^n$. *Consequently, the function* v *is continuous on* $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. Obvious by the proof of Theorem 9.1. □

In the sequel, we have

Theorem 9.3. *Let* $u : \mathbb{C}^n \to \mathbb{C}$ *be a function and* D *a domain of* $\mathbb{C}^m$, $n, m \geq 1$. *The following assertions are equivalent*
(I) *For each* $\varphi : D \to \mathbb{C}^n$ *prh, the function* $u(\varphi)$ *is psh on* D;
(II) u *is convex on* $\mathbb{C}^n$.

Now for the strict inequality, we have then

Theorem 9.4. *Let* $u : \mathbb{C}^n \to \mathbb{R}$ *be a function,* $n \geq 1$ *and* $a, b \in \mathbb{C} \backslash \{0\}$. *Put* $v(z,w) = u(aw + b\overline{z})$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. *The following assertions are equivalent*
(1) u *is strictly convex on* $\mathbb{C}^n$;
(2) v *is strictly psh on* $\mathbb{C}^n \times \mathbb{C}^n$;
(3) $u(Az + B\overline{z})$ *is strictly sh on* $\mathbb{C}^n$, *for each* $A, B \in M_n(\mathbb{C})$, *with the determinant* $det(AB) \neq 0$.

Proof. The proof is based on the proof of Theorem 9.1 and the paper [4]. □

10. Holomorphic polynomials on real and complex convexity. Consider the polynomial $p(z) = z^3 + z + 1$, $z \in \mathbb{C}$. There does not exists two real constants $A, B \in \mathbb{R}_+\backslash\{0\}$ such that v is psh on $\mathbb{C}^2$, where $v(z,w) = A|p(w-\overline{z})|^2 + B|p'(w-\overline{z})|^2$ for $(z,w) \in \mathbb{C}^2$. But we have the following

Theorem 10.1. *Let p be a analytic polynomial on $\mathbb{C}$, deg$(p) \leq 2$. Then there exists $A > 0$, $B > 0$ such that u is psh on $\mathbb{C}^2$, where*

$$u(z,w) = A|p(w-\overline{z})|^2 + B|p'(w-\overline{z})|^2 \quad \textit{for} \quad (z,w) \in \mathbb{C}^2.$$

Proof. If $\deg(p) = 1$, we take $A = B = 1$. Now suppose that $\deg(p) = 2$. We have $p(z) = az^2 + bz + c$, $a \in \mathbb{C}\backslash\{0\}$, $b, c \in \mathbb{C}$. Let $B \in \mathbb{R}_+\backslash\{0\}$. We study the convexity of v, $v(z) = |az^2 + bz + c|^2 + B|2az + b|^2$, for $z \in \mathbb{C}$. We have v is convex on $\mathbb{C}$ if

$$\left|2a^2\left[\left(z + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]\right| \leq |2az + b|^2 + 4B|a|^2$$

for each $z \in \mathbb{C}$. Take $z_0 = \dfrac{-b}{2a}$. Therefore we choose B satisfying the condition

$$8B|a|^2 \geq |b^2 - 4ac|.$$

In this situation, by the triangle inequality, we have

$$\begin{aligned}\left|\frac{\partial^2 u}{\partial z^2}(z)\right| &= |2a(az^2 + bz + c)| = \left|2a^2\left[\left(z + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]\right| \\ &\leq \left|2a^2\left(z + \frac{b}{2a}\right)^2\right| + 2|a|^2\left|\frac{b^2 - 4ac}{4a^2}\right| \leq \left|4a^2\left(z + \frac{b}{2a}\right)^2\right| + 2|a|^2(2B) \\ &= |2az + b|^2 + 4B|a|^2 = \frac{\partial^2 v}{\partial z \partial \overline{z}}(z),\end{aligned}$$

for each $z \in \mathbb{C}$. Consequently, v is convex on $\mathbb{C}$. □

Observe that the above property described in Theorem 10.1 is not true if p is a holomorphic polynomial on $\mathbb{C}$ with $\deg(p) \geq 3$. But only for holomorphic polynomials on $\mathbb{C}$, we have

Proposition 10.1. *Let p be a holomorphic polynomial on $\mathbb{C}$. Then there exists a constant $A \in \mathbb{C}$, such that $u = (|p|^2 + |q|^2)$ is convex on $\mathbb{C}$, where $q(z) = Az$, for $z \in \mathbb{C}$.*

Proof. Assume that $|p|^2$ is not convex on $\mathbb{C}$. Let $n = \deg(p) \geq 1$. We have

$$\lim_{|z|\to+\infty} |\frac{p''(z)p(z)}{(p'(z))^2}| = \frac{n-1}{n} < 1.$$

Therefore, there exists $B > 0$, such that $|p''(z)p(z)| < |p'(z)|^2$, for each $z \in \mathbb{C}\backslash D(0,B)$. Now since the function $|p''p|$ is continuous on the compact set $\overline{D}(0,B)$, then there exists $A > 0$ such that $|p''(\xi)p(\xi)| < A^2$, for each $\xi \in \overline{D}(0,B)$. Therefore $|p''(z)p(z)| < A^2 + |p'(z)|^2$, for each $z \in \mathbb{C}$. Now put $q(z) = Az$, for $z \in \mathbb{C}$. Then q is a holomorphic polynomial on $\mathbb{C}$ and satisfy $u = (|p|^2 + |q|^2)$ is a convex function on $\mathbb{C}$. □

Corollary 10.1. *Let p be a analytic polynomial on $\mathbb{C}$, $|p|$ is not convex and deg$(p) = 2$. Then there exist a constant $c > 0$ such that $u = (|p|^2 + c|p'|^2)$ is convex on $\mathbb{C}$ and for all $t \in [0, c[$, the function $v = (|p|^2 + t|p'|^2)$ is not convex on $\mathbb{C}$. This constant c is called the constant of the convex stability of the polynomial p. Moreover, $c = \min\{\tau > 0 \ / \ (|p|^2 + \tau|p'|^2)$ is a convex function on $\mathbb{C}\}$.*

Proof. Obvious by Theorem 10.1. □

Finally we have

Theorem 10.2. *Let p be a analytic polynomial on $\mathbb{C}$, deg$(p) \leq 2$ and $m \in \mathbb{N}\backslash\{0,1\}$. Then we have exactly one of the following assertions.*

(I) *For each $a, b \in \mathbb{R}_+\backslash\{0\}$, the function u is psh on $\mathbb{C}^2$, where $u(z,w) = a|p(w-\overline{z})|^{2m} + b|p'(w-\overline{z})|^{2m}$, for $(z,w) \in \mathbb{C}^2$, or*

(II) *For each $c, d \in \mathbb{R}_+\backslash\{0\}$, the function v is not psh in $\mathbb{C}^2$; where $v(z,w) = c|p(w-\overline{z})|^{2m} + d|p'(w-\overline{z})|^{2m}$, for $(z,w) \in \mathbb{C}^2$.*

Proof. If $\deg(p) = 1$, or $|p|$ is convex on $\mathbb{C}$, then for each $a, b \in \mathbb{R}_+\backslash\{0\}$, the function $(a|p|^2 + b|p'|^2)$ is convex on $\mathbb{C}$. Now assume that $|p|$ is not convex on $\mathbb{C}$. In this case we prove that $u = (a|p|^{2m} + b|p'|^{2m})$ is not convex on $\mathbb{C}$, for $a, b \in \mathbb{R}_+\backslash\{0\}$ and $m \in \mathbb{N}\backslash\{0,1\}$. In fact u is a function of class C^∞ on $\mathbb{C}$. Assume that u is convex on $\mathbb{C}$. Then

$$\left|\frac{\partial^2 u}{\partial z^2}(z)\right| \leq \frac{\partial^2 u}{\partial z \partial \overline{z}}(z)$$

for each $z \in \mathbb{C}$. Therefore,

$$|a[mp''p^{m-1} + m(m-1)(p')^2 p^{m-2}](\overline{p})^m + (m-1)(m-2)(p'')^2(p')^{m-3}(\overline{p'})^m| \\ \leq a|mp'p^{m-1}|^2 + b|p''(p')^{m-1}|^2, \quad \text{on } \mathbb{C}.$$

Case 1. $m \geq 3$. Since $\deg(p) = 2$. There exists $\alpha \in \mathbb{C}$, such that $p'(\alpha) = 0$, $p(\alpha) \neq 0$ and $p''(\alpha) \neq 0$. Then we have $|amp''(\alpha)p^{m-1}(\alpha)p^m(\alpha)| = 0$. A contradiction.

Case 2. $m = 2$. Let $\beta \in \mathbb{C}$, the only zero of p'. We have $p(\beta) \neq 0$ and $p''(\beta) \neq 0$. Then $|amp''(\beta)p^{m-1}(\beta)p^m(\beta)| = 0$. A contradiction. Consequently, $u = u_{(a,b)}$ is not convex on $\mathbb{C}$, for each $a, b \in \mathbb{R}_+\backslash\{0\}$. □

The question posed now is to characterize all the holomorphic polynomials p and q on $\mathbb{C}$, such that $(|p|^2 + |q|^2)$ is a convex function on $\mathbb{C}$.

Example. Let $g(z) = e^{e^{e^z}}$, $z \in \mathbb{C}$. Then for each holomorphic function $k : \mathbb{C} \to \mathbb{C}$, we have $(|g|^2 + |k|^2)$ is not convex on $\mathbb{C}$. Another example. Let $g_1(z) = ze^z$, $z \in \mathbb{C}$. Then $(|g_1|^2 + |k_1|^2)$ is not convex on $\mathbb{C}$, for each holomorphic function $k_1 : \mathbb{C} \to \mathbb{C}$. Note that on $\mathbb{C}^n$, $(n \geq 2)$, we have the same property. For example we can see the following.

Question. characterize all the holomorphic polynomials p and q on $\mathbb{C}$, such that $\deg(p) \geq 2$, $\deg(q) \geq 2$ and $(|p|^2 + |q|^2)$ is a convex function on $\mathbb{C}$. The general question is now. Question. Let $N \in \mathbb{N}\backslash\{0, 1\}$. Find all the holomorphic functions $g_1, \ldots, g_N : \mathbb{C}^n \to \mathbb{C}$, such that v is psh on $\mathbb{C}^n \times \mathbb{C}^n$, where $v(z, w) = (|g_1(w + \overline{z})|^2 + \cdots + |g_N(w + \overline{z})|^2)$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$.

REFERENCES

[1] J. Abidi. Sur quelques problèmes concernant les fonctions holomorphes et plurisousharmoniques. *Rend. Circ. Mat. Palermo (2)* **51**, *3* (2002), 411–424.

[2] J. Abidi. Contribution à l'étude des fonctions plurisousharmoniques convexes et analytiques. *Serdica Math. J.* **40**, *3–4* (2014), 329–388.

[3] J. Abidi. Partial differential equations and strictly plurisubharmonic functions in several variables. *Extr. Math.* **33**, *1* (2018), 67–108.

[4] J. Abidi. Contribution to real and complex convexity in complex domains. *J. Appl.& Pure Math.* **2**, *5–6* (2020), 215–277, `http://japm.or.kr/out/02262236077195865.pdf`.

[5] M. Andersson, M. Passare, R. Sigurdsson. Complex convexity and analytic functionals. Progress in Mathematics, vol. **225**. Basel, Birkhäuser Verlag, 2004.

[6] D. Coman, V. Guedj, A. Zeriahi. On the extension of quasiplurisubharmonic functions. *Anal. Math.* **48**, *2* (2022), 411–426.

[7] S. Dinew, Ż. Dinew. On a problem of Chirka. *Proc. Amer. Math. Soc.* **150**, *5* (2022), 2115–2119.

[8] L. Hörmander. Notions of convexity. Progress in Mathematics, vol. **127**. Boston, MA, Birkhauser Boston, Inc., 1994.

[9] M. Jarnicki, P. Pflug. Extension of holomorphic functions. De Gruyter Expositions in Mathematics, vol. **34**. Berlin, Walter de Gruyter & Co., 2000.

[10] M. Klimek. Pluripotential theory. London Mathematical Society Monographs. New Series, vol. **6**. New York, Oxford Science Publications. The Clarendon Press, Oxford University Press, 1991.

[11] S. G. Krantz. Function theory of several complex variables. Reprint of the 1992 edition. Providence, RI, AMS Chelsea Publishing, 2001.

[12] P. Lelong. Définition des fonctions plurisousharmoniques. *C. R. Acad. Sci. Paris* **215** (1942), 398–400.

[13] P. Lelong. Ensembles singuliers impropres des fonctions plurisousharmoniques. *J. Math. Pures Appl. (9)* **36** (1957), 263–303.

[14] M. Nilsson, F. Wikström. Quasibounded plurisubharmonic functions. *Internat. J. Math.* **32**, *9* (2021), Paper No. 2150068, 16 pp.

[15] A. Sadullaev. Continuation of analytic and pluriharmonic functions in a given direction by E. M. Chirka's method (a survey). *Sovrem. Mat. Fundam. Napravl.* **65**, *1* (2019), 83–94 (in Russuan); English translation in: *J. Math. Sci. (N.Y.)* **265**, *1* (2022), 79–89.

[16] S. Yao, Z. Li, X. Zhou. On the optimal L^2 extension theorem and a question of Ohsawa. *Nagoya Math. J.* **245**, (2022), 154–165.

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