Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **48** (2022), 211–218

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A STUDY OF THE MATRIX CLASSES $(\ell_\alpha, \ell_\alpha)$ AND (ℓ_α, c), $0 < \alpha \leq 1$

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Communicated by V. Drensky

Abstract. In this paper, entries of sequences, infinite series and infinite matrices are real or complex numbers. The present paper is a continuation of [8], where we established some properties of the matrix class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$. In this paper, we also record some properties of the class (ℓ_α, c) and the sequence space ℓ_α, $0 < \alpha \leq 1$.

1. Introduction. To make the paper self-contained, we recall the following notions and concepts. If X, Y are sequence spaces and $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ is an infinite matrix, we write $A \in (X, Y)$ if

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,\ n = 0, 1, 2, \ldots$$

is defined and the sequence $A(x) = \{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. $A(x)$ is called the A-transform of x. We now give a brief account of the research

2020 *Mathematics Subject Classification:* 40C05, 40H05.

Key words: A-transform, α-norm, α-normed linear space, convex, semigroup, convolution product.

done so far regarding the characterization of the matrix class $(\ell_\alpha, \ell_\beta)$, where the sequence space ℓ_α is defined by:

$$\ell_\alpha = \left\{ x = \{x_k\} \Big/ \sum_{k=0}^{\infty} |x_k|^\alpha < \infty \right\}, \ \alpha > 0.$$

A complete characterization of the matrix class $(\ell_\alpha, \ell_\beta)$, $\alpha, \beta \geq 2$, does not seem to be available in the literature. The latest result in this direction [3] characterizes only non-negative matrices in $(\ell_\alpha, \ell_\beta)$, $\alpha \geq \beta > 1$. A known simple sufficient condition ([5], p. 174, Theorem 9) for $A \in (\ell_\alpha, \ell_\alpha)$ is

$$A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1).$$

Sufficient conditions or necessary conditions for $A \in (\ell_\alpha, \ell_\beta)$ are available in the literature (see, for instance, [10]). Necessary and sufficient conditions for $A \in (\ell_1, \ell_1)$ are due to Mears [6] (for alternate proofs, see Knopp and Lorentz [2], Fridy [1]). Natarajan [8] obtained necessary and sufficient conditions for $A \in (\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, in the following theorem.

Theorem 1.1. *$A = (a_{nk}) \in (\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, if and only if*

$$\sup_{k \geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\alpha < \infty. \tag{1.1}$$

2. More properties of the class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$. In this section, in continuation of [8], we obtain some more properties of the matrix class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$.

Continuing our discussion on the complete, α-normed linear space $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, we have the following result:

Theorem 2.1. *$(\ell_\alpha, \ell_\alpha; P)$, as a subset of $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, is a closed, convex semigroup with identity, the multiplication being the usual matrix multiplication, where, we recall that $(\ell_\alpha, \ell_\alpha; P)$ is the subclass of $(\ell_\alpha, \ell_\alpha)$ such that*

$$\sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k, \ x = \{x_k\} \in \ell_\alpha.$$

Proof. Let $\lambda, \mu \geq 0$ be such that $\lambda + \mu = 1$. Let $A = (a_{nk})$, $B = (b_{nk}) \in (\ell_\alpha, \ell_\alpha; P)$. First, we note that $\lambda A + \mu B \in (\ell_\alpha, \ell_\alpha)$. Since $A, B \in (\ell_\alpha, \ell_\alpha; P)$,

$$\sum_{n=0}^{\infty} a_{nk} = 1 = \sum_{n=0}^{\infty} b_{nk}, \ k = 0, 1, 2, \ldots.$$

Now,

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda a_{nk}+\mu b_{nk}) &= \lambda\sum_{n=0}^{\infty}a_{nk}+\mu\sum_{n=0}^{\infty}b_{nk}\\ &=\lambda(1)+\mu(1)\\ &=\lambda+\mu\\ &=1,\ k=0,1,2,\ldots,\end{aligned}$$

so that $\lambda A+\mu B\in(\ell_\alpha,\ell_\alpha;P)$. Consequently, $(\ell_\alpha,\ell_\alpha;P)$ is a convex subset of $(\ell_\alpha,\ell_\alpha)$. Let, now, $A=(a_{nk})\in\overline{(\ell_\alpha,\ell_\alpha;P)}$. Then there exist

$$A^{(m)}=(a_{nk}^{(m)})\in(\ell_\alpha,\ell_\alpha;P),\ m=0,1,2,\ldots$$

such that

$$\|A^{(m)}-A\|_\alpha\to 0,\ m\to\infty,$$

where

$$\|A\|_\alpha=\sup_{k\ge 0}\sum_{n=0}^{\infty}|a_{nk}|^\alpha,$$

$A=(a_{nk})\in(\ell_\alpha,\ell_\alpha)$, is the α-norm in $(\ell_\alpha,\ell_\alpha)$, $0<\alpha\le 1$. So, for $\epsilon>0$, there exists a positive integer N such that

$$\|A^{(m)}-A\|_\alpha<\epsilon,\ m\ge N,$$

$$\text{(2.1)}\qquad i.e.,\ \sup_{k\ge 0}\sum_{n=0}^{\infty}|a_{nk}^{(m)}-a_{nk}|^\alpha<\epsilon^\alpha,\ m\ge N.$$

Now, for $k=0,1,2,\ldots,$

$$\begin{aligned}\left|\sum_{n=0}^{\infty}a_{nk}-1\right|^\alpha &= \left|\sum_{n=0}^{\infty}a_{nk}-\sum_{n=0}^{\infty}a_{nk}^{(N)}\right|^\alpha,\\ &\qquad \sum_{n=0}^{\infty}a_{nk}^{(N)}=1,\ k=0,1,2,\ldots,\ \text{since}\\ &\qquad\qquad A^{(N)}\in(\ell_\alpha,\ell_\alpha;P)\\ &=\left|\sum_{n=0}^{\infty}(a_{nk}-a_{nk}^{(N)})\right|^\alpha\\ &\le\sum_{n=0}^{\infty}|a_{nk}-a_{nk}^{(N)}|^\alpha,\ \text{since } 0<\alpha\le 1\\ &\le\sup_{k\ge 0}\sum_{n=0}^{\infty}|a_{nk}-a_{nk}^{(N)}|^\alpha\end{aligned}$$

$$< \epsilon^{\alpha}, \quad \text{using (2.1)}.$$

So

$$\left| \sum_{n=0}^{\infty} a_{nk} - 1 \right| < \epsilon, \ k = 0, 1, 2, \dots .$$

Since $\epsilon > 0$ is arbitrary,

$$\sum_{n=0}^{\infty} a_{nk} = 1, \ k = 0, 1, 2, \dots .$$

Hence $A \in (\ell_\alpha, \ell_\alpha; P)$ and so $(\ell_\alpha, \ell_\alpha; P)$ is a closed subset of $(\ell_\alpha, \ell_\alpha)$. To complete the proof, we have to check closure under matrix multiplication. If $A = (a_{nk})$, $B = (b_{nk}) \in (\ell_\alpha, \ell_\alpha; P)$, it is clear that $AB \in (\ell_\alpha, \ell_\alpha)$ in the first instance. Now, for $k = 0, 1, 2, \dots$,

$$\begin{aligned}
\sum_{n=0}^{\infty} (AB)_{nk} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right) \\
&= \sum_{i=0}^{\infty} b_{ik} \left(\sum_{n=0}^{\infty} a_{ni} \right), \quad \text{interchange of} \\
&\qquad \text{summation is possible because the series} \\
&\qquad \text{on the right side is absolutely convergent} \\
&= \sum_{i=0}^{\infty} b_{ik}, \quad \text{since } \sum_{n=0}^{\infty} a_{ni} = 1, \ i = 0, 1, 2, \dots, \\
&\qquad A \in (\ell_\alpha, \ell_\alpha; P) \\
&= 1, \ \text{since } \sum_{i=0}^{\infty} b_{ik} = 1, \ k = 0, 1, 2, \dots, \\
&\qquad B \in (\ell_\alpha, \ell_\alpha; P),
\end{aligned}$$

$$\text{i.e.,} \ \sum_{n=0}^{\infty} (AB)_{nk} = 1, \ k = 0, 1, 2, \dots$$

and so $AB \in (\ell_\alpha, \ell_\alpha; P)$, completing the proof of the theorem. □

For further study of the class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, we need the following definition (see [7]).

Definition 2.1. *For $A = (a_{nk})$, $B = (b_{nk})$, define*

$$(A \circ B)_{nk} = \sum_{i=0}^{n} a_{ik} b_{n-i,k}, \ k = 0, 1, 2, \dots . \tag{2.2}$$

$A \circ B = ((A \circ B)_{nk})$ *is called the 'convolution product' of* A *and* B.

Keeping the α-norm structure in $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$ and replacing the matrix product by the convolution product $\circ$, we prove the following result.

Theorem 2.2. *$(\ell_\alpha, \ell_\alpha; P)$, as a subset of $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, is a closed, convex semigroup with identity.*

Proof. First, we will prove closure under the convolution product $\circ$. Let $A = (a_{nk})$, $B = (b_{nk}) \in (\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} |(A \circ B)_{nk}|^\alpha &= \sum_{n=0}^{\infty} \left| \sum_{i=0}^{n} a_{ik} b_{n-i,k} \right|^\alpha \\
&\leq \sum_{n=0}^{\infty} \sum_{i=0}^{n} |a_{ik}|^\alpha |b_{n-i,k}|^\alpha, \\
&\qquad \text{since } 0 < \alpha \leq 1 \\
&= \left(\sum_{n=0}^{\infty} |a_{nk}|^\alpha \right) \left(\sum_{n=0}^{\infty} |b_{nk}|^\alpha \right) \\
&\leq \|A\|_\alpha^\alpha \|B\|_\alpha^\alpha \\
&< \infty, \ k = 0, 1, 2, \ldots,
\end{aligned}$$

so that

$$\sup_{k \geq 0} \sum_{n=0}^{\infty} |(A \circ B)_{nk}|^\alpha < \infty.$$

Hence $A \circ B \in (\ell_\alpha, \ell_\alpha)$. Also

$$\begin{aligned}
\|A \circ B\|_\alpha^\alpha &\leq \|A\|_\alpha^\alpha \|B\|_\alpha^\alpha, \\
\text{i.e., } \|A \circ B\|_\alpha &\leq \|A\|_\alpha \|B\|_\alpha.
\end{aligned}$$

It is clear that $\circ$ is commutative. The identity element is the matrix $E = (e_{nk})$, whose first row consists of 1's and 0's elsewehere,

$$\begin{aligned}
\text{i.e., } e_{\circ k} &= 1, \ k = 0, 1, 2, \ldots; \\
e_{nk} &= 0, \ n = 1, 2, \ldots; k = 0, 1, 2, \ldots.
\end{aligned}$$

We also note that $\|E\|_\alpha = 1$ and $E \in (\ell_\alpha, \ell_\alpha; P)$, since $\sum_{n=0}^{\infty} e_{nk} = 1$, $k = 0, 1, 2, \ldots$. To complete the proof, it suffices to prove that $(\ell_\alpha, \ell_\alpha; P)$, $0 < \alpha \leq 1$, is closed under $\circ$. Let $A = (a_{nk})$, $B = (b_{nk}) \in (\ell_\alpha, \ell_\alpha; P)$.

Now,

$$\begin{aligned}\sum_{n=0}^{\infty}(A\circ B)_{nk} &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_{ik}b_{n-i,k}\right)\\ &= \left(\sum_{n=0}^{\infty} b_{nk}\right)\left(\sum_{n=0}^{\infty} a_{nk}\right)\\ &= (1)(1), \quad \text{since } A, B \in (\ell_\alpha, \ell_\alpha; P)\\ &= 1, \ k = 0, 1, 2, \ldots,\end{aligned}$$

so that $A \circ B \in (\ell_\alpha, \ell_\alpha; P)$. Convexity can be proved as in Theorem 2.1, completing the proof of the theorem. □

Remark 2.1. Theorem 2.1 and Theorem 2.2, for the case $\alpha = 1$, were already proved in [7].

3. Some properties of the matrix class (ℓ_α, c), $0 < \alpha \leq 1$.

c, as usual, denotes the Banach space of all convergent sequences. The following result is well-known (see [4], Theorem 5).

Theorem 3.1. *$A = (a_{nk}) \in (\ell_\alpha, c)$, $0 < \alpha \leq 1$ if and only if*

$$\sup_{n,k} |a_{nk}| < \infty; \tag{3.1}$$

and

$$\lim_{n\to\infty} a_{nk} = a_k \quad \text{exists}, k = 0, 1, 2, \ldots. \tag{3.2}$$

Since $0 < \alpha \leq 1$, we note that $\ell_\alpha \subset \ell_1$ and so if $x = \{x_k\} \in \ell_\alpha$, then $\sum_{k=0}^{\infty} x_k$ converges. We write $A = (a_{nk}) \in (\ell_\alpha, c; P')$ if $A \in (\ell_\alpha, c)$ and

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k, \ x = \{x_k\} \in \ell_\alpha.$$

As in [9], we have the following characterization of $(\ell_\alpha, c; P')$.

Theorem 3.2. *$A = (a_{nk}) \in (\ell_\alpha, c; P')$ if and only* (3.1) *and* (3.2) *hold and further*

$$\lim_{n\to\infty} a_{nk} = 1, \ k = 0, 1, 2, \ldots. \tag{3.3}$$

We recall the following definition from ([5], [9]).

Definition 3.1. *Given the infinite matrices* $A = (a_{nk})$, $B = (b_{nk})$, *we define*

$$(A**B)_{nk} = \frac{1}{k+1} \sum_{i=0}^{k} a_{ni} b_{n,k-i}, \ n, k = 0, 1, 2, \dots . \tag{3.4}$$

$A**B = ((A**B)_{nk})$ *is called the second convolution of* A *and* B.

As noted in ([5], p. 183), the set S of all infinite matrices is a groupoid under the second convolution $**$ defined by (3.4), i.e., S is closed under $**$. Also S is commutative, non-associative and S has no identity. As in [9], we have

Theorem 3.3. $(\ell_\alpha, c; P')$ *is a subgroupoid of* S *under* $**$ *defined by* (3.4).

$(\ell_\alpha, c)'$ denotes the subclass of (ℓ_α, c) such that for $0 < \alpha \leq 1$

$$a_{nk} \to 0, \ k \to \infty, n = 0, 1, 2, \dots, A = (a_{nk}) \in (\ell_\alpha, c). \tag{3.5}$$

Again, as in [9], we have

Theorem 3.4. $(\ell_\alpha, c)'$ *is an ideal of* (ℓ_α, c) *under the second convolution* $**$.

4. Conclusion. We conclude the paper recording a property of the sequence space ℓ_α, $0 < \alpha \leq 1$.

Given two sequences $\{a_k\}$, $\{b_k\}$, define

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0, \ k = 0, 1, 2, \dots . \tag{4.1}$$

Then $\{c_k\}$ is called the Cauchy product of $\{a_k\}$ and $\{b_k\}$.

Theorem 4.1. *If* $\{a_k\}$, $\{b_k\} \in \ell_\alpha$, $0 < \alpha \leq 1$, *then* $\{c_k\} \in \ell_\alpha$ *too, where* c_k, $k = 0, 1, 2, \dots$ *is defined by* (4.1).

Proof.

$$A \equiv \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Note that the A-transform of $\{b_k\}$ is $\{c_k\}$. Since $\{a_k\} \in \ell_\alpha$, $A \in (\ell_\alpha, \ell_\alpha)$, in view of Theorem 1.1. Since $\{b_k\} \in \ell_\alpha$, it follows that $\{c_k\} \in \ell_\alpha$, completing the proof of the theorem. □

Remark 4.1. Theorem 4.1 can be stated formally in the following form too: If a sequence $\{a_k\}$ is given, then $\{c_k\} \in \ell_\alpha$ for every sequence $\{b_k\} \in \ell_\alpha$ if and only if $\{a_k\} \in \ell_\alpha$, where c_k, $k = 0, 1, 2, \dots$ is given by (4.1) and $0 < \alpha \leq 1$.

Remark 4.2. We can also reformulate Theorem 4.1 as follows:
If $\sum_{k=0}^{\infty} |a_k|^\alpha < \infty$ and $\sum_{k=0}^{\infty} |b_k|^\alpha < \infty$, then $\sum_{k=0}^{\infty} |c_k|^\alpha < \infty$, where c_k, $k = 0, 1, 2, \ldots$ is given by (4.1) and $0 < \alpha \leq 1$. In this context, we note that the case $\alpha = 1$ is the well-known Cauchy's theorem.

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Received March 4, 2022
Accepted February 14, 2023