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GENERALIZED PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH SYMBOL CLASSES INVOLVING FRACTIONAL FOURIER TRANSFORM

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Abstract. Generalized pseudo-differential operators $A(x, \Delta'_x)$ and $\mathcal{A}(x, \Delta'_x)$ involving fractional Fourier transform (FrFT) associated with symbol-classes are defined and respective symbol classes are introduced. Product and commutators of the generalized pseudo-differential operators are investigated.

1. Introduction and motivation. The fractional Fourier transform is a generalization of the Fourier transform with a parameter α. It has many applications in several areas including Communications, Optics, Quantum Physics and Signal processing. For more details of the fractional Fourier transform, see [1, 2, 15]. The one dimensional fractional Fourier transform [13, 14] with parameter α, $\varphi(x)$ is denoted by $(\mathscr{F}_\alpha\varphi)(\xi) = \widehat{\varphi}_\alpha(\xi)$ and given in $L_1(\mathbb{R})$ by

$$(\mathscr{F}_\alpha\varphi)(\xi) = \widehat{\varphi}_\alpha(\xi) = \int_{\mathbb{R}} K_\alpha(x, \xi)\varphi(x)dx \tag{1.1}$$

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where the kernel $K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(x^2+\xi^2)\cot\alpha}{2} - ix\xi\csc\alpha}, & \alpha \neq n\pi,\ n\epsilon\mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-ix\xi}, & \alpha = \frac{\pi}{2} \\ \delta(x-\xi), & \alpha = 2n\pi \\ \delta(x+\xi), & \alpha = (2n+1)\pi \end{cases}$

and

$$C_\alpha = (2\pi i \sin\alpha)^{-\frac{1}{2}} e^{\frac{i\alpha}{2}} = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}.$$

The corresponding inversion formula of $(\mathscr{F}_\alpha\varphi)(\xi)$ is given by

$$\varphi(x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}(\mathscr{F}_\alpha\varphi)(\xi)d\xi \tag{1.2}$$

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha}$$

and

$$C'_\alpha = \overline{C_\alpha} = \overline{(2\pi i \sin\alpha)^{-\frac{1}{2}}} e^{-\frac{i\alpha}{2}} = \sqrt{\frac{1 + i\cot\alpha}{2\pi}} = C_{-\alpha}.$$

Hence,

$$\overline{K_\alpha(x,\xi)} = K_{-\alpha}(x,\xi).$$

The inverse of a FrFT with the parameter α is the FrFT with the parameter $-\alpha$.

In 1960 to 1965, the pseudo-differential operators were studied by Kohn and Nirenberg [4] and Lars Hörmander [3] by using the theory of Fourier transform which established their importance in the theory of partial differential equations [9, 10, 11]. Our research work is motivated by the works of Zaidman [11] and Pathak, Prasad and Kumar [6, 7].

In this paper, we will define the generalized pseudo-differential operator $A(x,\Delta'_x)$ and $\mathscr{A}(x,\Delta'_x)$ involving the fractional Fourier transform for any function $\varphi \in \mathscr{G}$ and symbol $a(x,\xi) \in \Lambda$ (see Definitions 1 and 2 and Section 2 for more details). In addition we investigate product and commutators of these operators.

Definition 1. *A tempered distribution φ belongs to the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$ if its fractional Fourier transform $\mathscr{F}_\alpha\varphi$ corresponding to a locally integrable function $(\mathscr{F}_\alpha\varphi)(\xi)$ over $\mathbb{R}$ satisfies*

$$\|\varphi\|_s = \left(\int_{\mathbb{R}} \left|(1+|\xi|^2)^{\frac{s}{2}}(\mathscr{F}_\alpha\varphi)(\xi)\right|^2\right)^{\frac{1}{2}} < \infty. \tag{1.3}$$

This space is complete with respect to the norm $\| \ \|_s$.

Definition 2. *The space $\mathscr{G}$, the so-called space of smooth functions of rapid descent, is defined as follows: φ is member of $\mathscr{G}$ if and only if it is a complex valued C^∞-function on $\mathbb{R}$ and for all of non-negative integers β and γ it satisfies*

$$\Gamma_{\beta,\gamma}(\varphi) = \sup_{x\in\mathbb{R}} \left| x^\beta D^\gamma \varphi(x) \right| < \infty.$$

Lemma 1 (Peetre)**.** *For any real number s and for all $\xi, \eta \in \mathbb{R}$, the estimate*

$$\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^s \le 2^s \left(1+|\xi-\eta|^2\right)^{|s|}, \tag{1.4}$$

is satisfied.

P r o o f. See [12]. □

Theorem 1. *Let $K_\alpha(x,\xi)$ be the kernel of fractional Fourier transform and $\Delta_x = \left(\frac{\partial}{\partial x} - ix\cot\alpha\right)$. Then*

(i) $\Delta_x^n K_\alpha(x,\xi) = (-i\xi\csc\alpha)^n K_\alpha(x,\xi), \ \ \forall n \in \mathbb{N}.$

(ii) Let $\varphi \in \mathscr{G}$. Then

$$\left(\mathscr{F}_\alpha(\Delta'_x)^n \varphi(x)\right)(\xi) = (-i\xi\csc\alpha)^n (\mathscr{F}_\alpha\varphi(x))(\xi), \qquad \forall n \in \mathbb{N};$$

where $\Delta'_x = -\left(\frac{\partial}{\partial x} - ix\cot\alpha\right)$.

P r o o f. See [7]. □

The rest of the paper is organized as follows. In Sections 2 various useful definitions, lemmas, theorems, symbols and relations are given, which have been used throughout the paper. In Sections 3 and 4, two generalized pseudo-differential operators $A(x,\Delta'_x)$ and $\mathscr{A}(x,\Delta'_x)$ are introduced. Lastly, Section 5 deals with the product and commutators of these generalized pseudo-differential operators with some lemmas and theorems.

In [8], Prasad and Singh has defined a novel version of the pseudo-differential operators involving fractional Fourier cosine (sine) transform and discussed some of their properties. The fractional Fourier cosine (sine) transform is a generalization of ordinary Fourier cosine (sine) transform. The Fourier-cosine transform can be obtained by putting $\alpha = \beta = -\frac{1}{2}$ in the Fourier-Jacobi transform.

But, in our work, we have introduced pseudo-differential operators with the use of fractional Fourier transform. In [5], the fractional Fourier transform is an elegant generalization of ordinary Fourier transform.

2. Symbols. Let $a(x,\xi) : S \to \mathbb{C}$ be a complex valued function, where $S \subset \mathbb{R}\times\mathbb{R}$ and the $a(x,\xi) \in C^{\infty}(\mathbb{R}\times\mathbb{R})$ is said to be an element of the class Λ if and only if $a(x,t\xi) = a(x,\xi)$ for $t > 0$,

$$\lim_{|x|\to\infty} a(x,\xi) = a(\infty,\xi)$$

exists for $|\xi| = 1$ and $a(\infty,\xi)$ is C^{∞}-function. We define $e^{\frac{i}{2}x^2\cot\alpha}a'(x,\xi) = a(x,\xi) - a(\infty,\xi)$ assuming that $\forall\ l,p,q \in \mathbb{N}$, there exits $D_{l,p,q} > 0$ satisfuing

$$(1+|x|^2)^l \left| D_x^p D_\xi^q a'(x,\xi)\right| < D_{l,p,q}, \qquad \forall x \in \mathbb{R}. \tag{2.1}$$

Theorem 2. (*i*) *We have*

$$|a(\infty,\xi) - a(\infty,\eta)| \le C\left(\frac{|\xi-\eta|}{|\xi|+|\eta|}\right), \qquad \forall \xi,\eta \in \mathbb{R}.$$

(*ii*) *For any symbol* $e^{\frac{i}{2}x^2\cot\alpha}a'$ *and* $l \in \mathbb{N}$, *there exists a positive constant* D_l *such that*

$$|\hat{a}'_\alpha(\xi,\eta)| \le D_l\left(1+\xi^2\csc^2\alpha\right)^{\frac{-l}{2}}, \qquad \forall \xi,\eta \in \mathbb{R},$$

and
(*iii*)

$$|\hat{a}'_\alpha(\lambda,\xi) - \hat{a}'_\alpha(\lambda,\eta)| \le C_l\bigg(1+\lambda^2\csc^2\alpha\bigg)^{\frac{-l}{2}}, \qquad \forall \xi,\eta,\lambda \in \mathbb{R},$$

where C_l *is a positive constant.*

Proof. (i) See [11]. (ii) Similar proof of the Lemma 3.1 of [7]. (iii) It can be easily proved from (ii). This completes the proof. □

3. The operator $A(x,\Delta'_x)$. Here, we define the symbol $a(x,\xi) = a(\infty,\xi) + e^{\frac{ix^2\cot\alpha}{2}}a'(x,\xi)$ and

$$\widehat{a}'_\alpha(\lambda,\xi) = \int_{\mathbb{R}} K_\alpha(x,\lambda)a'(x,\xi)dx, \qquad \forall\lambda,\ \xi \in \mathbb{R}.$$

Since fractional Fourier transform is a continous linear map of $\mathscr{G}$ onto itself [13], this implies that $\hat{a}'_\alpha(\lambda,\xi) \in \mathscr{G}(\mathbb{R})$ uniformly for $\xi \in \mathbb{R}$. Let us define, for any $\varphi \in \mathscr{G}$ and $x \in \mathbb{R}$, a function $\vartheta(x) = [A(x,\Delta'_x)\varphi](x)$, by

$$[A(x,\Delta'_x)\varphi](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}G_\alpha(\xi)d\xi, \qquad \forall\ x \in \mathbb{R}. \tag{3.1}$$

The function $G_\alpha(\xi)$ is given by

$$G_\alpha(\xi) = a(\infty,\xi)\hat{\varphi}_\alpha(\xi) + C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\xi\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\hat{\varphi}_\alpha(\eta)d\eta. \tag{3.2}$$

Evidently, it has to be proved that $G_\alpha(\xi)$ is fractional Fourier transformable. In fact, we have $G_\alpha(\xi) \in L_1(\mathbb{R})$ as

$$|a(\infty,\xi)\widehat{\varphi}_\alpha\xi| \le \max_{|\xi|=1}|a(\infty,\xi)||\widehat{\varphi}_\alpha(\xi)| \in L_1(\mathbb{R}),$$

then obviously, it is sufficient to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)|d\eta d\xi < \infty.$$

Since

$$\hat{a}'_\alpha(\xi-\eta,\xi) = \int_{\mathbb{R}} K_\alpha(x,\xi-\eta)a'(x,\xi)dx,$$

then using Theorem 2 (ii), we obtain

$$\int_{\mathbb{R}} |\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)|d\eta \le D_l \int_{\mathbb{R}} (1+|\xi-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta.$$

The last expression is the convolution between $(1+|\xi|^2\csc^2\alpha)^{\frac{-l}{2}}$ and $|\widehat{\varphi}_\alpha(\xi)|$ both integrable for l sufficiently large.

Hence

$$\int_{\mathbb{R}} |\hat{a}'_\alpha(\xi-\eta,\eta)||\widehat{\varphi}_\alpha(\eta)|d\eta < \infty.$$

Thus $\widehat{A}_\alpha(x,\Delta'_x)\varphi$ is continous and bounded on $\mathbb{R}$. Hence we can say that

$$[\mathscr{F}_\alpha(A(x,\Delta'_x))\varphi](\xi) = a(\infty,\xi)\widehat{\varphi}_\alpha(\xi) + C'_\alpha \int_{\mathbb{R}} e^{-(\eta^2-\xi\eta)\cot\alpha}\widehat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta)d\eta,$$

is verified the fractional Fourier transform being in $\mathscr{G}'$.

Theorem 3. *If $a(x,\xi)$ is a symbol, we have*

$$[A(x,\Delta'_x)\varphi](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}\left\{\int_{\mathbb{R}} K_\alpha(y,\xi)a(y,\xi)\varphi(y)dy\right\}d\xi,$$

for every $\varphi \in \mathscr{G}$, $x \in \mathbb{R}$.

Proof. It will be sufficient to prove that

(i) The integral $\int_{\mathbb{R}} K_\alpha(x,\xi)a(x,\xi)\varphi(x)dx$ is absolutely convergent.

(ii) We have $G_\alpha(\xi) = \int_{\mathbb{R}} K_\alpha(y,\xi)a(y,\xi)\varphi(y)dy$, $\forall\xi \in \mathbb{R}$.

In fact to obtain (i), as $a(x,\xi) = a(\infty,\xi) + e^{\frac{ix^2 \cot\alpha}{2}} a'(x,\xi)$, it is sufficient to prove the absolute convergence of

$$\int_{\mathbb{R}} K_\alpha(x,\xi) a(\infty,\xi)\varphi(x)dx = a(\infty,\xi)\int_{\mathbb{R}} K_\alpha(x,\xi)\varphi(x)dx$$
$$= a(\infty,\xi)\widehat{\varphi}_\alpha(\xi), \quad \text{for} \quad \varphi \in \mathscr{G}$$

and

$$\int_{\mathbb{R}} K_\alpha(x,\xi) e^{\frac{ix^2 \cot\alpha}{2}} a'(x,\xi)\varphi(x)dx, \quad \text{for} \quad \varphi \in \mathscr{G}.$$

As

$$|a'(x,\xi)| \leq D_p(1+|x|^2)^{-p} \quad \text{for every} \quad p,$$

we have

$$\left|\int_{\mathbb{R}} K_\alpha(x,\xi) e^{\frac{ix^2 \cot\alpha}{2}} a'(x,\xi)\varphi(x)dx\right| \leq \int_{\mathbb{R}} |K_\alpha(x,\xi)||a'(x,\xi)||\varphi(x)|dx$$
$$\leq \frac{D_p}{\sqrt{2\pi}}\frac{1}{\sqrt{\sin\alpha}}\int_{\mathbb{R}} \frac{|\varphi(x)|}{(1+|x|^2)^p}dx.$$

It implies that $\int_{\mathbb{R}} K_\alpha(x,\xi) a(x,\xi)\varphi(x)dx$ is absolutely convergent. In order to prove (ii), we can use that

$$C'_\alpha \int_{\mathbb{R}} e^{-i[(\xi-\lambda)^2 - \xi(\xi-\lambda)]\cot\alpha} \hat{a}'_\alpha(\xi-\lambda,\xi)\widehat{\varphi_\alpha}(\lambda)d\lambda$$
$$= \int_{\mathbb{R}} K_\alpha(y,\xi) e^{\frac{iy^2 \cot\alpha}{2}} a'(y,\xi)\varphi(y)dy.$$

Now, we have

$$\int_{\mathbb{R}} K_\alpha(y,\xi) e^{\frac{iy^2 \cot\alpha}{2}} a'(y,\xi)\varphi(y)dy$$
$$= \int_{\mathbb{R}} K_\alpha(y,\xi) e^{\frac{iy^2 \cot\alpha}{2}} \left[\int_{\mathbb{R}} \overline{K_\alpha(y,\mu)}\hat{a}'_\alpha(\mu,\xi)d\mu\right]\varphi(y)dy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} C_\alpha C'_\alpha e^{\frac{i[(\xi-\mu)^2+y^2]\cot\alpha}{2} - iy(\xi-\mu)\csc\alpha} e^{i[\xi\mu-\mu^2]\cot\alpha}\hat{a}'_\alpha(\mu,\xi)\varphi(y)dyd\mu.$$

By making in the internal the substitution $\xi - \mu = \lambda$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} C_\alpha C'_\alpha e^{\frac{i[\lambda^2+y^2]\cot\alpha}{2} - iy\lambda\csc\alpha} e^{i[\xi(\xi-\lambda)\mu-(\xi-\lambda)^2]\cot\alpha}\hat{a}'_\alpha(\xi-\lambda,\xi)\varphi(y)dyd\lambda$$
$$= C'_\alpha \int_{\mathbb{R}} e^{-i(\lambda^2-\lambda\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\lambda,\xi)\widehat{\varphi}_\alpha(\lambda)d\lambda.$$

This completes the proof. □

Theorem 4. *We have the inequality*

$$\|A(x,\Delta'_x)\varphi\|_s \le C(\alpha)\|\varphi\|_s, \tag{3.3}$$

$\forall s \in \mathbb{R}$, $\forall \varphi \in \mathscr{G}(\mathbb{R})$, *for a certain constant* $C(\alpha)$.

Proof. We have in fact the immediate decompositition

$$A(x,\Delta'_x) = A(\infty,\Delta'_x) + A'(x,\Delta'_x).$$

We must remark that for $\varphi \in \mathscr{G}$, by definition we have

$$\left[\widehat{A}_\alpha(\infty,\Delta'_x)\varphi\right](\xi) = a(\infty,\xi)\widehat{\varphi}_\alpha(\xi),$$

and from [6], we have

$$[\mathscr{F}_\alpha(A'(x,\Delta'_x))\varphi](\xi) = C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta.$$

Then, we see that first of all

$$\begin{aligned}\|A(\infty,\Delta'_x)\varphi\|_s^2 &= \int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}|a(\infty,\xi)||\widehat{\varphi}_\alpha(\xi)|^2 d\xi\\ &\le \left(\sup_{|\xi|=1}|a(\infty,\xi)|\right)^2\int_{\mathbb{R}}[(1+|\xi|^2)^{\frac{s}{2}}|\widehat{\varphi}_\alpha(\xi)|]^2 d\xi\\ &= \left(\sup_{|\xi|=1}|a(\infty,\xi)|\right)^2\|\varphi\|_s^2.\end{aligned}$$

Therefore,

$$\|A(\infty,\Delta'_x)\varphi\|_s \le C_1\|\varphi\|_s.$$

Less trivial is estimate for $A'(x,\Delta'_x)\varphi$. Its fractional Fourier transform in $\mathscr{G}'$ equals

$$C'_\alpha \int_{\mathbb{R}} e^{-i(\eta-\xi)\xi\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta,$$

and then (using the definition of H^s), we will have to estimate the norm in $L_2(\mathbb{R})$ of the expression

$$(1+|\xi|^2)^{\frac{s}{2}}C'_\alpha \int_{\mathbb{R}} e^{-i(\eta-\xi)\xi\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta,$$

which is equal to $U_s(\xi)$. Now using Lemma 2 and Theorem 2(ii), then we have

$$\begin{aligned}&|U_s(\xi)|\\ &= |C'_\alpha|\left|\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}(1+|\eta|^2)^{\frac{-s}{2}}e^{-i(\eta-\xi)\xi\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)(1+|\eta|^2)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\eta)\right|d\eta\end{aligned}$$

$$\leq |C'_\alpha| D_l 2^{\frac{|s|}{2}} \int_{\mathbb{R}} (1+|\xi-\eta|^2)^{\frac{|s|}{2}} (1+|\xi-\eta|^2 \csc^2\alpha)^{\frac{-l}{2}} (1+|\eta|^2)^{\frac{s}{2}} |\widehat{\varphi}_\alpha(\eta)| d\eta$$

$$\leq |C'_\alpha| D_l 2^{\frac{|s|}{2}} \int_{\mathbb{R}} g_\alpha(\xi-\eta) f_\alpha(\eta) d\eta \quad \text{(say)}$$

$$= |C'_\alpha| D_l 2^{\frac{|s|}{2}} (g_\alpha * f_\alpha)(\xi).$$

If l is large, $g_\alpha \in L_1(\mathbb{R})$. Also since $\widehat{\varphi}_\alpha \in \mathscr{G}(\mathbb{R})$, $f_\alpha(\eta) \in L_2(\mathbb{R})$, then $(g_\alpha * f_\alpha)(\xi)$ belongs to $L_2(\mathbb{R})$ and following inequality holds:

$$\|g_\alpha * f_\alpha\|_{L_2(\mathbb{R})} \leq \|g_\alpha\|_{L_1(\mathbb{R})} \|f_\alpha\|_{L_2(\mathbb{R})}.$$

It implies that

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \leq |C_2(\alpha)| \|\varphi\|_s.$$

Now,

$$\|A(\infty,\Delta'_x)\varphi + A'(x,\Delta'_x)\varphi\|_s \leq \|A(\infty,\Delta'_x)\varphi\|_s + \|A'(x,\Delta'_x)\varphi\|_s$$

$$\|A(x,\Delta'_x)\varphi\|_s \leq C_1\|\varphi\|_s + |C_2(\alpha)|\|\varphi\|_s = C(\alpha)\|\varphi\|_s.$$

This completes the proof. □

4. The operator $\mathscr{A}(x,\Delta'_x)$. Let $a(x,\xi)$ be a symbol. We define an operator $\mathscr{A}(x,\Delta'_x)$ of $\mathscr{G}$ in $\mathscr{G}'$ by means of the formula

$$\mathscr{A}(x,\Delta'_x)\varphi = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)} H_\alpha(\xi) d\xi$$

where, for $\varphi \in \mathscr{G}$, the function $H_\alpha(\xi)$ is defined by the relation

$$H_\alpha(\xi) = a(\infty,\xi)\widehat{\varphi}_\alpha(\xi) + C'_\alpha \int_{\mathbb{R}} e^{-(\eta^2-\xi\eta)\cot\alpha} \hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta) d\eta,$$

$\forall \varphi \in \mathscr{G}$ and $\xi \in \mathbb{R} \setminus \{0\}$.

With the same proof used for $A(x,\Delta'_x)$ we have the function $\mathscr{A}(x,\Delta'_x)\varphi$ is continous and bounded for $x \in \mathbb{R}$. Besides, we see that if the symbol $a(x,\xi)$ does not depend on x, we have $A(\Delta'_x) = \mathscr{A}(\Delta'_x)$.

Theorem 5. *We have*

$$\left[\mathscr{A}(x,\Delta'_x)\varphi\right](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\eta)} a(x,\eta)\widehat{\varphi}_\alpha(\eta) d\eta, \quad \forall x \in \mathbb{R},\ \forall \varphi \in \mathscr{G}.$$

Proof. As $a(x,\eta) = a(\infty,\eta) + e^{\frac{ix^2\cot\alpha}{2}} a'(x,\eta)$ and $\widehat{\varphi}_\alpha(\eta) \in \mathscr{G}$, the integral is absolutely convergent.

Moreover, then:

$$\int_{\mathbb{R}} \overline{K_\alpha(x,\xi)} \left[C'_\alpha \int_{\mathbb{R}} \hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta) e^{i\eta(\xi-\eta)\cot\alpha} d\eta \right] d\xi, \tag{4.1}$$

is absolutely convergent because

$$\int_{\mathbb{R}} \overline{K_\alpha(x,\eta)} e^{\frac{ix^2 \cot\alpha}{2}} a'(x,\eta)\widehat{\varphi}_\alpha(\eta)d\eta = \int_{\mathbb{R}}\int_{\mathbb{R}} C'_\alpha \overline{K_\alpha(x,\xi)} \hat{a}'_\alpha(\xi-\eta,\eta) \times \widehat{\varphi}_\alpha(\eta) e^{i\eta(\xi-\eta)\cot\alpha} d\eta d\xi,$$

$$\begin{aligned}&\left|\int_{\mathbb{R}} \overline{K_\alpha(x,\eta)} e^{\frac{ix^2 \cot\alpha}{2}} a'(x,\eta)\widehat{\varphi}_\alpha(\eta)d\eta\right| \\ &\quad = \left|\int_{\mathbb{R}}\int_{\mathbb{R}} C'_\alpha \overline{K_\alpha(x,\xi)} \hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta) e^{i\eta(\xi-\eta)\cot\alpha} d\eta d\xi\right| \\ &\quad \le \int_{\mathbb{R}}\int_{\mathbb{R}} |C'_\alpha||\overline{K_\alpha(x,\xi)}||\hat{a}'_\alpha(\xi-\eta,\eta)||\widehat{\varphi}_\alpha(\eta)|d\eta d\xi \\ &\quad \le \frac{D_l}{2\pi|\sin\alpha|}\int_{\mathbb{R}} |\widehat{\varphi}_\alpha(\eta)| \left(\int_{\mathbb{R}} (1+|\xi-\eta|^2\csc^2\alpha)^{\frac{-l}{2}} d\xi\right) d\eta < \infty,\end{aligned}$$

for l large enough.

Furthermore, we see that (4.1) equals

$$\begin{aligned}&C'_\alpha \int_{\mathbb{R}} e^{\frac{-i}{2}[x^2+(\xi-\eta)^2]\cot\alpha + ix(\xi-\eta)\csc\alpha + \frac{i\eta}{2}(\eta-2\xi)\cot\alpha + ix\eta\csc\alpha} \\ &\times \left(C'_\alpha \int_{\mathbb{R}} \hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta) e^{i\eta(\xi-\eta)\cot\alpha} d\eta\right) d\xi \\ &= C'_\alpha \int_{\mathbb{R}} e^{\frac{ix^2\cot\alpha}{2}} \left(C'_\alpha \int_{\mathbb{R}} e^{\frac{-i}{2}[x^2+(\xi-\eta)^2]\cot\alpha + ix(\xi-\eta)\csc\alpha} a'_\alpha(\xi-\eta,\eta)d\xi\right) \\ &\times e^{\frac{-i}{2}(x^2+\eta^2)\cot\alpha + ix\eta\csc\alpha} \widehat{\varphi}_\alpha(\eta)d\eta \\ &= \int_{\mathbb{R}} e^{\frac{ix^2\cot\alpha}{2}} a'(x,\eta)\overline{K_\alpha(x,\eta)}\widehat{\varphi}_\alpha(\eta)d\eta.\end{aligned}$$

This proves Theorem 5. □

Theorem 6. *Let $a(x,\xi)$ be a symbol and $\overline{a(x,\xi)}$, its complex conjugate, operator $A(x,\Delta'_x)$ associated to $a(x,\xi)$, operator $\overline{\mathscr{A}(x,\Delta'_x)}$ associated to $\overline{a(x,\xi)}$. Then, we have the equality*

$$\left(A(x,\Delta'_x)\varphi,\phi\right)_{L_2(\mathbb{R})} = \left(\varphi, \overline{\mathscr{A}(x,\Delta'_x)\phi}\right)_{L_2(\mathbb{R})}, \qquad \forall\varphi,\ \phi\in L_2(\mathbb{R}).$$

P r o o f. It will be sufficient to show this for $\varphi,\phi\in\mathscr{G}$. We have first of all

$$\left[\overline{\mathscr{A}(x,\Delta'_x)\phi}\right](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\eta)}\ \overline{a(x,\eta)}\widehat{\phi}_\alpha(\eta)d\eta, \qquad \forall\phi\in\mathscr{G}, \qquad \text{(Theorem 5)}.$$

Hence we get, when

$$(\varphi, \phi)_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \varphi(x)\overline{\phi(x)}dx$$

the equality

$$\begin{aligned}\left(\varphi, \overline{\mathscr{A}(x, \Delta'_x)\phi}\right) &= \int_{\mathbb{R}} \varphi(x)\overline{\left(\int_{\mathbb{R}} \overline{K_\alpha(x,\eta)} \quad \overline{a(x,\eta)}\widehat{\phi}(\eta)d\eta\right)}dx \\ &= \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} K_\alpha(x,\eta)a(x,\eta)\overline{\widehat{\phi}_\alpha(\eta)}d\eta dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)K_\alpha(x,\eta)a(x,\eta)\overline{\widehat{\phi}_\alpha(\eta)}d\eta dx.\end{aligned}$$

Now by Parseval's formula and using also Theorem 3, we obtain

$$\begin{aligned}\left(A(x, \Delta'_x)\varphi, \phi\right)_{L_2(\mathbb{R})} &= \left(\mathscr{F}_\alpha\left[A(x,\Delta'_x)\varphi\right], \widehat{\phi}_\alpha\right)_{L_2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \mathscr{F}_\alpha\left[A(x,\Delta'_x)\varphi\right](\eta)\overline{\widehat{\phi}_\alpha(\eta)}d\eta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_\alpha(x,\eta)a(x,\eta)\varphi(x)\overline{\widehat{\phi}_\alpha(\eta)}d\eta dx.\end{aligned}$$

Therefore,

$$\left(A(x, \Delta'_x)\varphi, \phi\right)_{L_2(\mathbb{R})} = \left(\varphi, \overline{\mathscr{A}(x, \Delta'_x)\phi}\right)_{L_2(\mathbb{R})}.$$

This completes the proof. □

Remark 1. Let $a(x,\xi)$ be a symbol of special type: $a(x,\xi) = a(x)b(\xi)$. Then we have $\mathscr{A}(x,\Delta')\varphi = a(x)b(\Delta')\varphi$, $A(x,\Delta') = b(\Delta')(a(x)\varphi(x))$, $\forall \varphi \in \mathscr{G}$. In fact, we see that

$$\left[\mathscr{A}(x,\Delta'_x)\varphi\right](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\eta)}a(x)b(i\eta\csc\alpha)\widehat{\varphi}_\alpha(\eta)d\eta = a(x)b(\Delta'_x)\varphi$$

and

$$\begin{aligned}\mathscr{F}_\alpha\left[A(x,\Delta'_x)\varphi\right](\xi) &= \int_{\mathbb{R}} K_\alpha(y,\xi)a(y)b(i\xi\csc\alpha)\varphi(y)dy \\ &= b(i\xi\csc\alpha)\mathscr{F}_\alpha\left[a\varphi\right](\xi) \\ &= \mathscr{F}_\alpha\left[b(\Delta'_x)(a\varphi)\right](\xi).\end{aligned}$$

Theorem 7. *We have the relation*

$$\|\left[A(x,\Delta'_x) - \mathscr{A}(x,\Delta'_x)\right]\varphi\|_s \le C(\alpha)\|\varphi\|_s, \qquad \forall\varphi\in\mathscr{G},\ s\in\mathbb{R}.$$

P r o o f. We have

$$\left[\mathscr{F}_\alpha\left(A(x,\Delta'_x)\right)\varphi\right](\xi)$$
$$= a(\infty,\xi)\widehat{\varphi}_\alpha(\xi) + C'_\alpha\int_{\mathbb{R}} e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta,$$

(Fractional Fourier transform in $\mathscr{G}'$). The same is valid for $\mathscr{A}(x,\Delta'_x)\varphi$ and

$$\left[\mathscr{F}_\alpha\left(\mathscr{A}(x,\Delta'_x)\right)\varphi\right](\xi) = a(\infty,\xi)\widehat{\varphi}_\alpha(\xi) + C'_\alpha\int_{\mathbb{R}} e^{-i\xi(\xi-\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta)d\eta.$$

Hence, we obtain, with fractional Fourier transform in $\mathscr{G}'$

$$\left[\mathscr{F}_\alpha\left(A(x,\Delta'_x)-\mathscr{A}(x,\Delta'_x)\right)\varphi\right](\xi)$$
$$= C'_\alpha\int_{\mathbb{R}}\left[e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi) - e^{-i\xi(\xi-\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\right]\widehat{\varphi}_\alpha(\eta)d\eta.$$

Therefore, we will have to estimate the norm $L_2(\mathbb{R})$ of expression

$$U_s(\xi) = (1+|\xi|^2)^{\frac{s}{2}}C'_\alpha\int_{\mathbb{R}}\left[e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi) - e^{-i\xi(\xi-\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\right]$$
$$\times\widehat{\varphi}_\alpha(\eta)d\eta,$$
$$= C'_\alpha\int_{\mathbb{R}}\frac{(1+|\xi|^2)^{\frac{s}{2}}}{(1+|\eta|^2)^{\frac{s}{2}}}\left[e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi) - e^{-i\xi(\xi-\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\right]$$
$$\times(1+|\eta|^2)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\eta)d\eta,$$

$$|U_s(\xi)| \le |C'_\alpha|2^{\frac{|s|}{2}}\int_{\mathbb{R}}(1+|\xi-\eta|^2)^{\frac{|s|}{2}}|\hat{a}'_\alpha(\xi-\eta,\xi)||(1+|\eta|^2)^{\frac{s}{2}}||\widehat{\varphi}_\alpha(\eta)|d\eta$$
$$+|C'_\alpha|2^{\frac{|s|}{2}}\int_{\mathbb{R}}(1+|\xi-\eta|^2)^{\frac{|s|}{2}}|\hat{a}'_\alpha(\xi-\eta,\eta)||(1+|\eta|^2)^{\frac{s}{2}}||\widehat{\varphi}_\alpha(\eta)|d\eta$$
$$= 2|C'_\alpha|D_l2^{\frac{|s|}{2}}\int_{\mathbb{R}}(1+|\xi-\eta|^2)^{\frac{|s|}{2}}(1+|\xi-\eta|\csc^2\alpha)^{\frac{-l}{2}}(1+|\eta|^2)^{\frac{s}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta$$
$$= \int_{\mathbb{R}} h_\alpha(\xi-\eta)f_\alpha(\eta)d\eta \quad \text{(say)}$$
$$= (h_\alpha * f_\alpha)(\xi).$$

If l is large, $h_\alpha \in L_1(\mathbb{R})$. Also, since $\widehat{\varphi}_\alpha \in \mathscr{G}$, $f_\alpha(\eta) \in L_2(\mathbb{R})$. Then $(h_\alpha * f_\alpha)(\xi)$ belongs to $L_2(\mathbb{R})$ and the inequality

$$\|h_\alpha * f_\alpha\| \le \|h_\alpha\|_{L_1(\mathbb{R})}\|f_\alpha\|_{L_2(\mathbb{R})},$$

implies that

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \le C(\alpha)\|\varphi\|_s.$$

Hence,

$$\|[A(x,\Delta'_x)-\mathscr{A}(x,\Delta'_x)]\varphi\|_s \le C(\alpha)\|\varphi\|_s, \quad \forall \varphi\in\mathscr{G},\ s\in\mathbb{R}. \qquad \square$$

5. Product and commutators of generalized pseudo-differential operators.

Theorem 8. *Let $a(x,\xi)$, $b(x,\xi)$ be two symbols belonging to the class Λ. Then $c(x,\xi)=a(x,\xi)b(x,\xi)$ is also a symbol in the class Λ.*

Proof. We have $c(x,\xi)\in\Lambda$ as $a(x,\xi)$ and $b(x,\xi)$ are in the space. Besides, $\forall t>0$, we have $c(x,t\xi)=a(x,t\xi)b(x,t\xi)$. As

$$\lim_{|x|\to\infty} a(x,\xi)=a(\infty,\xi),$$

$$\lim_{|x|\to\infty} b(x,\xi)=b(\infty,\xi),$$

and the same is valid for $c(x,\xi)$,

$$\lim_{|x|\to\infty} c(x,\xi)=c(\infty,\xi),$$

which exists for $\xi\in\mathbb{R}$. If we put

$$\begin{aligned} c'(x,\xi) &= \big(a'(x,\xi)+a(\infty,\xi)\big)\big(b'(x,\xi)+b(\infty,\xi)\big)\\ &= a'(x,\xi)b'(x,\xi)+a(\infty,\xi)b'(x,\xi)+a'(x,\xi)b(\infty,\xi)+a(\infty,\xi)b(\infty,\xi), \end{aligned}$$

then we show that $c'(x,\xi)$ possesses the property (2.8) as

$$\begin{aligned} &(1+|x|^2)^l\left|D_x^pD_\xi^q c'(x,\xi)\right|\\ &\le (1+|x|^2)^l\left|D_x^pD_\xi^q a'(x,\xi)b'(x,\xi)\right| + (1+|x|^2)^l\left|D_x^pD_\xi^q a(\infty,\xi)b'(x,\xi)\right|\\ &+ (1+|x|^2)^l\left|D_x^pD_\xi^q a'(x,\xi)b(\infty,\xi)\right| + (1+|x|^2)^l\left|D_x^pD_\xi^q a(\infty,\xi)b(\infty,\xi)\right|\\ &\le \sum_{s=0}^p\binom{p}{s}\sum_{r=0}^q\binom{q}{r}(1+|x|^2)^l\left|D_x^{p-s}D_\xi^{q-r}a'(x,\xi)\right|\left|D_x^sD_\xi^r b'(x,\xi)\right|\\ &+ \sum_{s=0}^p\binom{p}{s}\sum_{r=0}^q\binom{q}{r}(1+|x|^2)^l\left|D_x^{p-s}D_\xi^{q-r}b'(x,\xi)\right|\left|D_x^sD_\xi^r a(\infty,\xi)\right|\\ &+ \sum_{s=0}^p\binom{p}{s}\sum_{r=0}^q\binom{q}{r}(1+|x|^2)^l\left|D_x^{p-s}D_\xi^{q-r}a'(x,\xi)\right|\left|D_x^sD_\xi^r b(\infty,\xi)\right|\\ &+ \sum_{s=0}^p\binom{p}{s}\sum_{r=0}^q\binom{q}{r}(1+|x|^2)^l\left|D_x^{p-s}D_\xi^{q-r}a(\infty,\xi)\right|\left|D_x^sD_\xi^r b(\infty,\xi)\right| \end{aligned}$$

$$(1+|x|^2)^l \left|D_x^p D_\xi^q c'(x,\xi)\right| \leq \sum_{s=0}^{p} \binom{p}{s} \sum_{r=0}^{q} \binom{q}{r} \left[C_{1,l,p,q} + C_{2,l,p,q} + C_{3,l,p,q} + C_{4,l,p,q}\right] \\ \leq D_{l,p,q}.$$

Therefore, $c'(x,\xi) \in \Lambda$. Let $C(x,\Delta'_x), A(x,\Delta'_x), B(x,\Delta'_x)$ be the operators corresponding to $c(x,\xi), a(x,\xi), b(x,\xi)$ respectively. We have

$$\begin{aligned} A(x,\Delta'_x) &= A(\infty,\Delta'_x) + A'(x,\Delta'_x) \\ B(x,\Delta'_x) &= B(\infty,\Delta'_x) + B'(x,\Delta'_x) \\ A(x,\Delta'_x)B(x,\Delta'_x) &= A(\infty,\Delta'_x)B(\infty,\Delta'_x) + A'(x,\Delta'_x)B(\infty,\Delta'_x) \\ &\quad + A(\infty,\Delta'_x)B'(x,\Delta'_x) + A'(x,\Delta'_x)B'(x,\Delta'_x). \end{aligned}$$

We denote

$$a(\infty,\xi)b(\infty,\xi) = \gamma(\xi); \quad e^{\frac{ix^2\cot\alpha}{2}} a'(x,\xi)b(x,\xi) = p(x,\xi)$$

$$a(\infty,\xi)e^{\frac{ix^2\cot\alpha}{2}} b'(x,\xi) = p_1(x,\xi), \quad b(\infty,\xi)e^{\frac{ix^2\cot\alpha}{2}} a'(x,\xi) = p_2(x,\xi).$$

Then,

$$C(x,\Delta'_x) = \Gamma(\Delta'_x) + P(x,\Delta'_x) + P_1(x,\Delta'_x) + P_2(x,\Delta'_x),$$

where, Γ, P, P_1 and P_2 are generalized pseudo-differential operators with symbols $\gamma(\xi), p(x,\xi), p_1(x,\xi)$ and $p_2(x,\xi)$, respectively. □

Theorem 9. *We have*

$$\Gamma(\Delta'_x)\varphi = A(\infty,\Delta'_x)B(\infty,\Delta'_x)\varphi, \quad \textit{for } \varphi \in \mathscr{G}.$$

P r o o f. In fact, we have

$$\begin{aligned} \mathscr{F}_\alpha\left[\Gamma(\Delta'_x)\varphi\right](\xi) &= \gamma(\xi)\widehat{\varphi}_\alpha(\xi), \\ &= \int_{\mathbb{R}} \gamma(\xi)K_\alpha(x,\xi)\varphi(x)dx, \\ &= a(\infty,\xi)b(\infty,\xi)\int_{\mathbb{R}} K_\alpha(x,\xi)\varphi(x)dx, \\ &= a(\infty,\xi)b(\infty,\xi)\widehat{\varphi}_\alpha(\xi), \\ &= a(\infty,\xi)\mathscr{F}_\alpha\left[B(\infty,\Delta'_x)\varphi\right](\xi), \\ &= \mathscr{F}_\alpha\left[A(\infty,\Delta'_x)B(\infty,\Delta'_x)\varphi\right](\xi), \end{aligned}$$

hence, by inversion formula of fractional Fourier transform valid in $\mathscr{G}'$, we arrive at Theorem 9. □

Theorem 10. *We have*

$$P_1(x,\Delta'_x) = A(\infty,\Delta'_x)B'(x,\Delta'_x).$$

Proof. In fact, we have for $\varphi \in \mathscr{G}$

$$\begin{aligned}\mathscr{F}_\alpha\left[P_1(x,\Delta'_x)\varphi\right](x) &= p_1(x,\xi)\widehat{\varphi}_\alpha(\xi)d\xi,\\ &= a(\infty,\xi)b'(x,\xi)e^{\frac{ix^2\cot\alpha}{2}}\widehat{\varphi}_\alpha(\xi)d\xi,\\ &= a(\infty,\xi)\mathscr{F}_\alpha\left[B(x,\Delta'_x)\varphi\right](\xi),\\ &= \mathscr{F}_\alpha\left[A(\infty,\Delta'_x)B(x,\Delta'_x)\varphi\right](\xi),\end{aligned}$$

hence, by inversion formula of fractional Fourier transform, we have the result of Theorem 10. □

Theorem 11. *We have*

$$P_2(x,\Delta'_x) = B(\infty,\Delta'_x)A'(x,\Delta'_x).$$

Proof. The proof is similar to that of Theorem 10. □

Theorem 12. *We have the relation*

$$\left\|\left[A'(x,\Delta'_x),B(\infty,\Delta'_x)\right]\right\|_s \le C(\alpha)\left\|\varphi\right\|_{s-1}, \quad \forall\varphi\in\mathscr{G}, \quad s\in\mathbb{R};$$

for a certain constant $C(\alpha)$, where $[\cdot,\cdot]$ denotes the commutator between the two operators.

Proof. In fact, we have the formula

$$\begin{aligned}&\mathscr{F}_\alpha\left[A'(x,\Delta'_x)B(\infty,\Delta'_x)\varphi\right](\xi)\\ &\quad = C'_\alpha\int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\mathscr{F}_\alpha\left[B(\infty,\Delta'_x)\varphi\right](\eta)d\eta\\ &\quad = C'_\alpha\int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)b(\infty,\eta)\widehat{\varphi}_\alpha(\eta)d\eta.\end{aligned}$$

Besides,

$$\begin{aligned}\mathscr{F}_\alpha\left[B(\infty,\Delta'_x)A'(x,\Delta'_x)\varphi\right](\xi) &= b(\infty,\xi)\mathscr{F}_\alpha\left[A'_\alpha(x,\Delta'_x)\varphi\right](\xi)\\ &= b(\infty,\xi)C'_\alpha\int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta;\end{aligned}$$

and hence

$$\begin{aligned}&\mathscr{F}_\alpha\left[A'(x,\Delta'_x),\ B(\infty,\Delta'_x)\varphi\right](\xi)\\ &\quad = C'_\alpha\int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)[b(\infty,\eta)-b(\infty,\xi)]\widehat{\varphi}_\alpha(\eta)d\eta,\end{aligned}$$

from [11], we have

$$|b(\infty,\xi)-b(\infty,\eta)| \le C|\xi-\eta|(|\xi|+|\eta|)^{-1} \le C(1+|\xi-\eta|^2)^{\frac{1}{2}}(1+|\eta|^2)^{\frac{-1}{2}}.$$

Hence we are obliged to estimate the norm $L_2(\mathbb{R})$ of the expression

$$U_s(\xi) = C'_\alpha (1+|\xi|^2)^{\frac{s}{2}} \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha} \hat{a}'_\alpha(\xi-\eta,\xi)[b(\infty,\eta)-b(\infty,\xi)]\widehat{\varphi}_\alpha(\eta)d\eta.$$

We have

$$\begin{aligned}
|U_s(\xi)| &\le |C'_\alpha| \int_{\mathbb{R}} (1+|\xi|^2)^{\frac{s}{2}} |\hat{a}'_\alpha(\xi-\eta,\xi)||b(\infty,\eta)-b(\infty,\xi)||\widehat{\varphi}_\alpha(\eta)|d\eta \\
&\le |C'_\alpha| D_l 2^{\frac{s}{2}} \int_{\mathbb{R}} (1+|\xi-\eta|^2)^{\frac{s+1}{2}} (1+|\xi-\eta|^2\csc\alpha)^{\frac{-l}{2}} (1+|\eta|^2)^{\frac{s-1}{2}} |\widehat{\varphi}_\alpha(\eta)|d\eta \\
&= \int_{\mathbb{R}} g_\alpha(\xi-\eta) h_\alpha(\eta) d\eta \quad \text{(say)} \\
&= (g_\alpha * h_\alpha)(\xi).
\end{aligned}$$

If l is large, $g_\alpha \in L_1(\mathbb{R})$. Also, since $\widehat{\varphi}_\alpha \in \mathscr{G}(\mathbb{R})$, $h_\alpha(\eta) \in L_2(\mathbb{R})$. Then $(g_\alpha * h_\alpha)(\xi)$ belongs to $L_2(\mathbb{R})$ and the inequality

$$\|g_\alpha * h_\alpha\|_{L_2(\mathbb{R})} \le \|g_\alpha\|_{L_1(\mathbb{R})} \|h_\alpha\|_{L_2(\mathbb{R})}.$$

It implies that

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \le C(\alpha) \|\varphi\|_{s-1}, \ \forall \varphi \in \mathscr{G}, \ \forall s \in \mathbb{R}.$$

Therefore,

$$\left\| \left[A'(x,\Delta'_x), B(\infty,\Delta'_x)\right] \varphi \right\|_s \le C(\alpha) \|\varphi\|_{s-1}, \ \forall \varphi \in \mathscr{G}, \ \forall s \in \mathbb{R}.$$

This completes the proof. □

Theorem 13. *We have the relation*

$$\left\| \left(A'(x,\Delta'_x)B'(x,\Delta'_x) - P(x,\Delta'_x)\right)\varphi \right\|_s \le C(\alpha) \|\varphi\|_s, \quad \forall \varphi \in \mathscr{G}, \quad \forall s \in \mathbb{R}.$$

Proof. Let us consider the operator $P(x,\Delta'_x)$ associated with $p(x,\xi)$, then

$$\left[\mathscr{F}_\alpha \left(P(x,\Delta'_x)\varphi\right)\right](\xi) = C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha} K_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta.$$

But we have for $p(x,\xi) = e^{\frac{ix^2\cot\alpha}{2}} a'(x,\xi)b'(x,\xi)$ that

$$(\mathscr{F}_\alpha p)(\lambda,\xi) = C'_\alpha \int_{\mathbb{R}} e^{-i(\mu^2-\mu\xi)\cot\alpha} \hat{a}'_\alpha(\lambda-\mu,\xi)\hat{b}'_\alpha(\mu,\xi)d\mu;$$

whence we arrive at

$$\begin{aligned}
&\left[\mathscr{F}_\alpha \left(P(x,\Delta'_x)\varphi\right)\right](\xi) \\
&= C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha} \left(\int_{\mathbb{R}} e^{-i(\mu^2-\mu\xi)\cot\alpha} \hat{a}'_\alpha(\xi-\eta-\mu,\xi)\hat{b}'_\alpha(\mu,\xi)d\mu \right) \widehat{\varphi}_\alpha(\eta)d\eta
\end{aligned}$$

$$= C'_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i[\eta^2+\mu^2-\xi(\eta+\mu)] \cot\alpha} \hat{a}'_\alpha(\xi-\eta-\mu,\xi)\hat{b}'_\alpha(\mu,\xi)\widehat{\varphi}_\alpha(\eta) d\eta d\mu.$$

In the interior integral, we make: $\eta+\mu=\tau$; $d\eta = d\tau$; it follows that

$$\left[\mathscr{F}_\alpha\left(P(x,\Delta'_x)\varphi\right)\right](\xi) \\ = C'_\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-[\mu^2+(\tau-\mu)^2-\xi\tau]\cot\alpha} \hat{a}'_\alpha(\xi-\tau,\xi)\widehat{\varphi}_\alpha(\tau-\mu)d\tau \right) \hat{b}'_\alpha(\mu,\xi)d\mu.$$

And once more in the interior, we make $\tau-\mu=\nu$; $-d\mu = d\nu$. We have now

$$\left[\mathscr{F}_\alpha\left(P(x,\Delta'_x)\varphi\right)\right](\xi) \\ = C'_\alpha \int_{\mathbb{R}} e^{-i[(\tau-\nu)^2+\nu^2-\xi\tau]\cot\alpha} \hat{a}'_\alpha(\xi-\tau,\xi) \left(\int_{\mathbb{R}} \hat{b}'_\alpha(\tau-\nu,\xi)\widehat{\varphi}_\alpha(\nu)d\nu \right) d\tau \\ = C'_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i[(\tau-\nu)^2+\nu^2-\xi\tau]\cot\alpha} \hat{a}'_\alpha(\xi-\tau,\xi)\hat{b}'_\alpha(\tau-\nu,\xi)\widehat{\varphi}_\alpha(\nu)d\nu d\tau.$$

Now we replace $\nu=\eta$. Then we get

$$\left[\mathscr{F}_\alpha\left(P(x,\Delta'_x)\varphi\right)\right](\xi) \\ = C'_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i[(\tau-\eta)^2+\eta^2-\xi\tau]\cot\alpha} \hat{a}'_\alpha(\xi-\tau,\xi)\hat{b}'_\alpha(\tau-\eta,\xi)\widehat{\varphi}_\alpha(\eta)d\eta d\tau.$$

On the other hand, we have

$$\left[\mathscr{F}_\alpha\left(A'(x,\Delta'_x)B'(x,\Delta'_x)\right)\varphi\right](\xi) \\ = C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha} \hat{a}'_\alpha(\xi-\eta,\xi)\mathscr{F}_\alpha\left[B'(x,\Delta'_x)\varphi\right](\eta)d\eta$$

and besides

$$\mathscr{F}_\alpha\left[B'(x,\Delta'_x)\varphi\right](\eta) = C'_\alpha \int_{\mathbb{R}} e^{-(\tau^2-\eta\tau)\cot\alpha} \hat{b}_\alpha(\eta-\tau,\eta)\widehat{\varphi}_\alpha(\tau)d\tau;$$

and hence we shall obtain

$$\left[\mathscr{F}_\alpha\left(A'(x,\Delta'_x)B'(x,\Delta'_x)\right)\varphi\right](\xi) = C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\eta\xi)\cot\alpha} \hat{a'}_\alpha(\xi-\eta,\xi) \\ \times \left(C'_\alpha \int_{\mathbb{R}} e^{-(\tau^2-\eta\tau)\cot\alpha} \hat{b'}_\alpha(\eta-\tau,\eta)\widehat{\varphi}_\alpha(\tau)d\tau \right) d\eta \\ = (C'_\alpha)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i[\eta^2+\tau^2-\eta(\xi+\tau)]\cot\alpha} \hat{a}'_\alpha(\xi-\eta,\xi) \\ \times \hat{b}'_\alpha(\eta-\tau,\eta)\widehat{\varphi}_\alpha(\tau)d\tau d\eta.$$

By making substitution $\eta=\tau$, $\tau=\eta$, we arrive at

$$\left[\mathscr{F}_\alpha\left(A'(x,\Delta'_x)B'(x,\Delta'_x)\right)\varphi\right](\xi) = (C'_\alpha)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i[\eta^2+\tau^2-\tau(\xi+\eta)]\cot\alpha} \hat{a}'_\alpha(\xi-\tau,\xi)$$

$$\times \hat{b}'_\alpha(\tau-\eta,\tau)\widehat{\varphi}_\alpha(\eta)d\tau d\eta.$$

The absolute convergence of the double integrals here, using the result from equation (2.1), estimates

$$|\hat{a}'_\alpha(\xi-\tau,\xi)| \le D_l(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}; \tag{5.1}$$

$$|\hat{b}'_\alpha(\tau-\eta,\tau)| \le D_l(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}; \tag{5.2}$$

where $l \in \mathbb{N}$.

Therefore, we can express the difference

$$\begin{aligned}&\left[\mathscr{F}_\alpha\left(A'(x,\Delta'_x)B'(x,\Delta'_x)-P(x,\Delta'_x)\right)\varphi\right](\xi)\\ &=(C'_\alpha)^2\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{a}'_\alpha(\xi-\tau,\xi)[e^{-i(\eta^2+\tau^2-\tau(\xi+\eta))\cot\alpha}\hat{b}'_\alpha(\tau-\eta,\tau)\\ &-e^{-i((\tau-\eta)^2+\eta^2-\xi\tau)\cot\alpha}\hat{b}'_\alpha(\tau-\eta,\xi)]\widehat{\varphi}_\alpha(\eta)d\tau d\eta.\end{aligned}$$

Let us examine here the norm $L_2(\mathbb{R})$ of expression

$$\begin{aligned}U_s(\xi)=(C'_\alpha)^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}\hat{a}'_\alpha(\xi-\tau,\xi)[e^{-i(\eta^2+\tau^2-\tau(\xi+\eta))\cot\alpha}\hat{b}'_\alpha(\tau-\eta,\tau)\\ -e^{-i((\tau-\eta)^2+\eta^2-\xi\tau)\cot\alpha}\hat{b}'_\alpha(\tau-\eta,\xi)]\widehat{\varphi}_\alpha(\eta)d\tau d\eta.\end{aligned}$$

Using equation (5.1) and (5.2), we get

$$\begin{aligned}&|U_s(\xi)|\\ &\le|C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}|\hat{a}'_\alpha(\xi-\tau,\xi)|[|\hat{b}'_\alpha(\tau-\eta,\tau)|+|b'_\alpha(\tau-\eta,\xi)|]|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\\ &\le D_l|C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\\ &+D_l|C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\\ &=2D_l|C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\\ &=T_s(\xi)\quad(say).\end{aligned}$$

Firstly, we want to show that

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \le \|T_s(\xi)\|_{L_2(\mathbb{R})}.$$

For,

$$\begin{aligned}&|T_s(\xi)|\\ &=2|D_l||C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta.\end{aligned}$$

Let us denote now,

$$H(\xi,\eta,\tau) = (1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}$$
$$L_s(\xi,\eta) = \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(1+|\eta|^2)^{\frac{s}{2}}}\int_{\mathbb{R}} H(\xi,\eta,\tau)d\tau.$$

We remark that it follows, $\forall \xi \in \mathbb{R}$

$$|T_s(\xi)| \leq 2|D_l||C'_\alpha|^2\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}H(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta.$$

Therefore, we have only to prove the inequality

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}H(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\right)^2 d\xi\right)^{\frac{1}{2}} \leq C(\alpha)\left\|\varphi\right\|_s, \quad \forall\varphi\in\mathscr{G}.$$

In order to do that we shall prove here a more general result, which is given in the following Lemma.

Lemma 2. *Let $r(\xi,\eta,\tau)>0$ be function such that $\int_{\mathbb{R}} r(\xi,\eta,\tau)d\tau < \infty$ for every ξ,η fixed in $\mathbb{R}$. Let us denote*

$$\rho_s(\xi,\eta) = \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(1+|\eta|^2)^{\frac{s}{2}}}\int_{\mathbb{R}} r(\xi,\eta,\tau)d\tau,$$

and suppose

$$\int_{\mathbb{R}}\rho_s(\xi,\eta)d\xi \leq L, \quad \int_{\mathbb{R}}\rho_s(\xi,\eta)d\eta \leq L, \ \forall\ \xi,\eta\in\mathbb{R}.$$

Then, there is a constant $C(\alpha)$ such that the inequality

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}r(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\right)^2 d\xi\right)^{\frac{1}{2}} \leq C(\alpha)\left\|\varphi\right\|_s,$$

$\forall\varphi\in\mathscr{G}$ for $s\in\mathbb{R}$ is verified.

P r o o f. We remark that in fact we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}}r(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta = \int_{\mathbb{R}}\rho_s(\xi,\eta)(1+|\eta|^2)^{\frac{s}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta.$$

Let us put $\vartheta(\eta) = (1+|\eta|^2)^{\frac{s}{2}}|\widehat{\varphi}_\alpha(\eta)|$. We get

$$\begin{aligned}\int_{\mathbb{R}}\rho_s(\xi,\eta)\vartheta(\eta)d\eta &= \int_{\mathbb{R}}\sqrt{\rho_s(\xi,\eta)}\sqrt{\rho_s(\xi,\eta)}\vartheta(\eta)d\eta \\ &\leq \left(\int_{\mathbb{R}}\rho_s(\xi,\eta)d\eta\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\rho_s(\xi,\eta)\vartheta^2(\eta)d\eta\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{L}\left(\int_{\mathbb{R}} \rho_s(\xi,\eta)\vartheta^2(\eta)d\eta\right)^{\frac{1}{2}}.$$

Hence, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}} r(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta \leq \sqrt{L}\left(\int_{\mathbb{R}} \rho_s(\xi,\eta)\vartheta^2(\eta)d\eta\right)^{\frac{1}{2}}$$

$$\begin{aligned}\left\|\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^{\frac{s}{2}} r(\xi,\eta,\tau)|\widehat{\varphi}_\alpha(\eta)|d\tau d\eta\right\|_{L_2(\mathbb{R})} &\\ &\leq \sqrt{L}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \rho_s(\xi,\eta)\vartheta^2(\eta)d\eta\right)d\xi\right)^{\frac{1}{2}} \\ &\leq L\left(\int_{\mathbb{R}}\vartheta^2(\eta)d\eta\right)^{\frac{1}{2}} \\ &= L\left\|\varphi\right\|_s.\end{aligned}$$ □

We shall apply Lemma 2, taking $r(\xi,\eta,\tau) = H(\xi,\eta,\tau)$ and $\rho_s(\xi,\eta) = \mathscr{L}_s(\xi,\eta)$. We see that $\int_{\mathbb{R}} H(\xi,\eta,\tau)d\tau < \infty$, and it remains to prove following

Lemma 3. *We have* $\int_{\mathbb{R}} \mathscr{L}_s(\xi,\eta)d\xi \leq \mathscr{L}$, $\int_{\mathbb{R}} \mathscr{L}_s(\xi,\eta)d\eta \leq \mathscr{L}$, *in fact*

$$\mathscr{L}_s(\xi,\eta) = \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(1+|\eta|^2)^{\frac{s}{2}}}\int_{\mathbb{R}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}d\tau.$$

P r o o f. It is given that

$$\begin{aligned}\mathscr{L}_s(\xi,\eta) &= \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(1+|\eta|^2)^{\frac{s}{2}}}\int_{\mathbb{R}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}d\tau \\ &\leq 2^{\frac{|s|}{2}}(1+|\xi-\eta|^2)^{\frac{|s|}{2}}\int_{\mathbb{R}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}} \\ &\times (1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}d\tau.\end{aligned}$$

Now from [11], we have

$$(1+|\xi-\eta|^2)^{\frac{|s|}{2}} \leq C(1+|\xi-\tau|^2)^{\frac{|s|}{2}}(1+|\tau-\eta|^2)^{\frac{|s|}{2}};$$

and hence

$$\begin{aligned}&\mathscr{L}_s(\xi,\eta) \\ &\leq 2^{\frac{|s|}{2}}C\int_{\mathbb{R}}(1+|\xi-\tau|^2)^{\frac{|s|}{2}}(1+|\tau-\eta|^2)^{\frac{|s|}{2}}(1+|\xi-\tau|^2\csc^2\alpha)^{\frac{-l}{2}}(1+|\tau-\eta|^2\csc^2\alpha)^{\frac{-l}{2}}d\tau.\end{aligned}$$

We denote at this stage:

$$\lambda(t) = \int_{\mathbb{R}} (1+|u|^2)^{\frac{|s|}{2}} (1+|u|^2 \csc\alpha)^{\frac{-l}{2}} (1+|t-u|^2)^{\frac{|s|}{2}} (1+|t-u|^2 \csc\alpha)^{\frac{-l}{2}} du, \qquad t \in \mathbb{R},$$

where l is large enough. We see that $\lambda(t) \in L_1(\mathbb{R})$ as convolution of two integrable functions; hence, putting $t = \xi - \eta$, we have

$$\lambda(\xi-\eta) = \int_{\mathbb{R}} (1+|u|^2)^{\frac{|s|}{2}} (1+|u|^2 \csc^2\alpha)^{\frac{-l}{2}} (1+|\xi-\eta-u|^2)^{\frac{|s|}{2}} (1+|\xi-\eta-u|^2 \csc^2\alpha)^{\frac{-l}{2}} du.$$

By putting $u = \xi - \tau$, we get

$$\lambda(\xi-\eta) = \int_{\mathbb{R}} (1+|\xi-\tau|^2)^{\frac{|s|}{2}} (1+|\xi-\tau|^2 \csc^2\alpha)^{\frac{-l}{2}} (1+|\tau-\eta|^2)^{\frac{|s|}{2}} (1+|\tau-\eta|^2 \csc^2\alpha)^{\frac{-l}{2}} d\tau.$$

Hence we get

$$\mathscr{L}_s(\xi,\eta) \le C_s \lambda(\xi-\eta);$$

and obviously:

$$\int_{\mathbb{R}} \lambda(\xi-\eta) d\xi < \infty, \quad \int_{\mathbb{R}} \lambda(\xi-\eta) d\eta < \infty,$$

which proves Lemma 3. □

Hence, by Lemma 3 we have that

$$\|T_s(\xi)\|_{L_2(\mathbb{R})} \le C(\alpha) \|\varphi\|_s .$$

Therefore,

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \le C(\alpha) \|\varphi\|_s , \quad \forall \varphi \in \mathscr{G} \text{ and } \forall s \in \mathbb{R}.$$

and this proves Theorem 13. □

Corollary 1. *If $A(x, \Delta'_x), B(x, \Delta'_x)$ are generalized pseudo-differential operators, the commutator $[A(x, \Delta'_x), B(x, \Delta'_x)]$ is of order ≤ 0.*

Proof. We have

$$A(x,\Delta'_x)B(x,\Delta'_x) - (ab)(x,\Delta'_x) = [A'(x,\Delta'_x), B(x,\Delta'_x)] - A'(x,\Delta'_x)B'(x,\Delta'_x) + P(x,\Delta'_x)$$

is of order ≤ 0 by Theorem 12 and Theorem 13. □

Remark 2. Let $A(x,\Delta'_x)$ be a generalized pseudo-differential operator associated with the symbol $a(x,\xi)$. We assume that $\rho_0 \in \mathbb{C}$ is an eigen-value of $A(x,\Delta'_x)$,such that$|a(x,\xi)-\rho_0| > \alpha > 0 \quad \forall x \in \mathbb{R}, \quad |\xi| = 1$. Then, any eigen-vector φ_o corresponding to eigen-value ρ_0 is a C^∞-function.

Therefore, $b(x,\xi) = (a(x,\xi)-\rho_0)^{-1}$ is also a symbol. If $B(x,\Delta'_x)$ is associated to it, we get $B(x,\Delta'_x)(A(x,\Delta'_x))-\rho_0 E = E+U$, where E is the identity operator and U has order ≤ 0. It follows that $\theta = B(A-\rho_0 E)\varphi_0 = \varphi_0 + U\varphi_0$, implies that $\varphi_0 = -U\varphi_0$, $\varphi_0 \in L^2(\mathbb{R})$. Therefore $U\varphi_0 \in H^1$ and $\varphi_0 \in H^1$ too.

Let us consider now the operator $\mathscr{I}_s = (1+|\Delta'_x|^2)^{\frac{s}{2}}$, defined by $\widehat{\mathscr{I}_s}\varphi(\xi) = (1+|\xi|^2\csc^2\alpha)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\xi), \quad \forall\varphi \in \mathscr{G}$.

Theorem 14. *Let $A(x,\Delta'_x)$ be a generalized pseudo-differential operator associated with a symbol $a(x,\xi)$ and the operator $\mathscr{I}_s = (1+|\Delta'_x|^2)^{\frac{s}{2}}$ is defined by $\widehat{\mathscr{I}_s}\varphi(\xi) = (1+|\xi|^2\csc^2\alpha)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\xi)$, $\forall\varphi \in \mathscr{G}$.*

Then

$$\left\|[A(x,\Delta'_x),\mathscr{I}_s]\right\|_{L_2(\mathbb{R})} \leq C\,\|\varphi\|_{s-1}, \qquad \forall\varphi \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}\mathscr{F}_\alpha[(A(x,\Delta'_x)\mathscr{I}_s)\varphi](\xi) &= a(\infty,\xi)(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\xi)\\ &\quad + C'_x\int_{\mathbb{R}} e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)(1+\eta^2\csc^2\alpha)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\eta)d\eta,\end{aligned}$$

and also

$$\begin{aligned}\mathscr{F}_\alpha[(\mathscr{I}_s A(x,\Delta'_x))\varphi](\xi) &= (1+\xi^2\csc^2\alpha)^{\frac{s}{2}}\widehat{A(x,\Delta'_x)}\varphi(\xi)\\ &= (1+\xi^2\csc^2\alpha)^{\frac{s}{2}}a(\infty,\xi)\widehat{\varphi}_\alpha(\xi)\\ &\quad + C'_\alpha\int_{\mathbb{R}} e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}\widehat{\varphi}_\alpha(\eta)d\eta\end{aligned}$$

$$\begin{aligned}\mathscr{F}_\alpha\left([A(x,\Delta'_x),\mathscr{I}_s]\varphi\right)(\xi) &= C'_\alpha\int_{\mathbb{R}} e^{-i\eta(\eta-\xi)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\\ &\quad\times\left[(1+\eta^2\csc^2\alpha)^{\frac{s}{2}}-(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}\right]\widehat{\varphi}_\alpha(\eta)d\eta\\ &= U_s(\xi), \qquad \xi \in \mathbb{R}\setminus\{0\}.\end{aligned}$$

Hence we need to estimate the norm $L_2(\mathbb{R})$ of the expression

$$(5.3)\quad |U_s(\xi)| \leq D_l\int_{\mathbb{R}}(1+(\xi-\eta)^2\csc^2\alpha)^{\frac{-l}{2}}\left|(1+|\eta|^2\csc^2\alpha)^{\frac{s}{2}}-(1+|\xi|^2\csc^2\alpha)^{\frac{s}{2}}\right||\widehat{\varphi}_\alpha(\eta)|\,d\eta$$

We consider the elementary inequality for $\dfrac{|s-1|}{2}$, $\quad 0<\theta<1$:

$$\left(1+\theta|\xi-\eta|^2\csc^2\alpha\right)^{\frac{|s-1|}{2}} \le (1+|\xi-\eta|^2\csc^2\alpha)^{\frac{|s-1|}{2}}$$

whence

$$\begin{aligned}&\left(1+|\eta\csc\alpha+\theta(\xi-\eta)\csc\alpha|^2\right)^{\frac{s-1}{2}}\\&\qquad\le 2^{\frac{|s-1|}{2}}(1+|\eta|^2\csc^2\alpha)^{\frac{|s-1|}{2}}(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{|s-1|}{2}}.\end{aligned}$$

By Taylor's formula, we have

$$(1+|\xi|^2\csc^2\alpha)^{\frac{s}{2}}-(1+|\eta|^2\csc^2\alpha)^{\frac{s}{2}}=\left((\xi-\eta)\csc\alpha,\operatorname{grad}(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}_{\xi=\zeta}\right),$$

where $\zeta=\eta\csc\alpha+\theta(\xi-\eta)\csc\alpha$,

$$\left|(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}-(1+\eta^2\csc^2\alpha)^{\frac{s}{2}}\right|=|\xi-\eta|\,|\csc\alpha|\left|\operatorname{grad}(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}_{\xi=\zeta}\right|.$$

Since we have

$$\frac{d}{d\xi}(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}=\frac{s}{2}(1+\xi^2\csc^2\alpha)^{\frac{s}{2}-1}2\xi\csc^2\alpha=s\xi\csc^2\alpha(1+\xi^2\csc^2\alpha)^{\frac{s}{2}-1},$$

one obtains

$$\left|\operatorname{grad}(1+\xi^2\csc^2\alpha)^{\frac{s}{2}}\right|=\left|\xi s\csc^2\alpha(1+\xi^2\csc^2\alpha)^{\frac{s}{2}-1}\right|\le|s|(1+\xi^2\csc^2\alpha)^{\frac{s-1}{2}}$$

and hence

$$\begin{aligned}&\left|(1+|\xi|\csc^2\alpha)^{\frac{s}{2}}-(1+|\eta|^2\csc^2\alpha)^{\frac{s}{2}}\right|\\&\qquad\le|\xi-\eta||\csc\alpha||s|(1+|\eta\csc\alpha+\theta(\xi-\eta)\csc\alpha|^2)^{\frac{s-1}{2}}\\&\qquad\le|s|(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{1}{2}}(1+|\eta\csc\alpha+\theta(\xi-\eta)\csc\alpha|^2)^{\frac{s-1}{2}}\\&\qquad\le|s|(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{1}{2}}2^{\frac{|s-1|}{2}}(1+|\eta|^2\csc^2\alpha)^{\frac{s-1}{2}}\\&\qquad\times(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{s-1}{2}}.\end{aligned}\tag{5.4}$$

Using (5.3) and (5.4), we get

$$\begin{aligned}&|U_s(\xi)|\\&\le C_{l,s}\int_{\mathbb{R}}(1+|\xi-\eta|^2\csc\alpha)^{\frac{-l}{2}}(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{|s-1|}{2}+\frac{1}{2}}(1+|\eta|^2\csc^2\alpha)^{\frac{s-1}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta\\&=C_{l,s}\int_{\mathbb{R}}(1+|\xi-\eta|^2\csc^2\alpha)^{\frac{-l+|s-1|+1}{2}}(1+|\eta|^2\csc^2\alpha)^{\frac{s-1}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta\\&=\int_{\mathbb{R}}h_\alpha(\xi-\eta)f_\alpha(\eta)d\eta\quad(say)\\&=(h_\alpha*f_\alpha)(\xi).\end{aligned}$$

If l is large, then $h_\alpha \in L_1(\mathbb{R})$. Also, $(h_\alpha * f_\alpha)(\xi)$ belongs to $L_2(\mathbb{R})$ and the inequality

$$\|h_\alpha * f_\alpha\|_{L_2(\mathbb{R})} \leq \|h_\alpha\|_{L_1(\mathbb{R})} \|f_\alpha\|_{L_2(\mathbb{R})}$$

holds. This implies that

$$\|U_s(\xi)\|_{L_2(\mathbb{R})} \leq C(\alpha) \|\varphi\|_{s-1},$$

hence

$$\|[A(x, \Delta'_x), \mathscr{I}_s]\|_{L_2(\mathbb{R})} \leq C(\alpha) \|\varphi\|_{s-1}, \ \forall \varphi \in \mathscr{G}, \ s \in \mathbb{R}.$$

Theorem 14 is proved. □

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