Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **48** (2022), 271–278

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

HARMONIOUS COLOURING OF LINE GRAPH OF COMMUTING AND NON-COMMUTING GRAPH OF D_{2n}

R. Divya, P. Chithra Devi

Communicated by V. Drensky

Abstract. Harmonious coloring of graph G is a proper vertex coloring, where each pair of colors occurs at most on one pair of adjacent vertices. Minimum number of colors required for Harmonious coloring of G is the harmonious chromatic number, $\chi_H(G)$. Here we determine the Harmonious chromatic number of the line graph of commuting graph and non-commuting graph of the dihedral group, D_{2n}.

1. Introduction. Graph coloring is a method of assigning colors to vertices of a graph so that no two vertices that are adjacent get the same color; this is called a vertex coloring. In the same way, an edge coloring is assignment of color to each edge so that no two adjacent edges get same color. Graph coloring has many practical appications as well as theoretical challenges. It is still an interesting field of research, with different types of coloring. One such type is Harmonious coloring.

2020 *Mathematics Subject Classification:* 05C25, 05C76.
Key words: harmonious coloring, harmonious chromatic number, Dihedral group, line graph.

Harmonious coloring of graph G is a proper vertex coloring, where each pair of colors appears at most on one pair of adjacent vertices. Minimum number of colors required for Harmonious coloring of G is the harmonious chromatic number, $\chi_H(G)$. It has many applications in radio Navigation systems, addressing the blocks, communication networks etc. This coloring was introduced by Hopcroft and Krishnamoorthy [5] in 1983.

The graphs considered here are simple without loops and multiple edges. The line graph of G, denoted by $L(G)$ has vertices as lines of G, and any two points of $L(G)$ are adjacent whenever the corresponding lines of G are adjacent. If x and y are the vertices adjacent in G, then the corresponding vertex in $L(G)$ is denoted by $\{x, y\}$. We refer [4] for definitions that are not defined here. For any integer $n \geq 3$, the Dihedral group D_{2n} is given by $D_{2n} = \langle r, s : s^2 = r^n = 1, rs = sr^{-1} \rangle$. For algebraic definitions not defined here, we refer [3].

In this paper we determine the Harmonious Chromatic number of the Line Graph of the Commuting graph and Non-Commuting graph of the Dihedral Group D_{2n}. Other properties of the Line Graph of the Commuting graph of D_{2n} were recently established by us in [2].

2. Harmonious coloring of line graph of commuting graph on D_{2n}. The commuting graph of D_{2n} and its properties were studied by T. Tamizh Chelvam, K. Selvakumar and S. Raja in [6]. The commuting graph of D_{2n} denoted by $C(D_{2n})$ has vertices as elements of D_{2n} with any two vertices are adjacent iff $xy = yx$, $x, y \in D_{2n}$. Line graph of Commuting Graph of the Dihedral Group denoted by $L(C(D_{2n}))$ has vertices as edges of $C(D_{2n})$ and any two vertices are adjacent whenever the corresponding edges of $C(D_{2n})$ are adjacent. In the following theorems, we find the Harmonious coloring of the Line graph of Commuting graph of the D_{2n}.

Theorem 2.1. *Let $G : L(C(D_{2n}))$ where $n \geq 3$ is an odd integer. Then*

$$\chi_H(G) = \frac{n(n+1)}{2}.$$

Proof. Let $G : L(C(D_{2n}))$ where $n \geq 3$ is an odd integer. The vertex set of the graph is

$$V = \left\{ \left\{1, r^k\right\}, \left\{r^i, r^j\right\}, \left\{1, sr^l\right\} \right\}$$

such that $1 \leq k \leq n-1$, $i < j$, $1 \leq i \leq n-2$, $2 \leq j \leq n-1$, $1 \leq l \leq n$.

Let

$$A = \left\{ \left\{1, r^k\right\} : 1 \leq k \leq n-1 \right\}.$$

The induced subgraph $\langle A \rangle \cong K_{n-1}$. Hence one needs $n-1$ colors to color that subgraph.

Now, let

$$A_1 = \{\{r, r^a\} : 2 \leq a \leq n-1\}.$$

The subgraph induced by the vertex set A_1 is complete and each vertex of A_1 is adjacent with exactly two vertices of the set A. Hence the vertices of the set A_1 cannot be colored with the above $n-1$ colors. So one needs $n-2$ colors different from the above $n-1$ colors to color the vertices in the set A_1.

Now, let

$$A_2 = \left\{\left\{r^2, r^b\right\} : 3 \leq b \leq n-1\right\}.$$

The subgraph induced by the set A_2 is also complete and each vertex of A_2 is adjacent with exactly two vertices of the set A and A_1. Hence the above $2n-3$ colors cannot be used to color the vertices of the set A_2. Hence one needs $n-3$ colors different from the above $2n-3$ colors to color the vertices in the set A_2.

Proceeding similarly, let

$$A_{n-2} = \left\{\left\{r^{n-2}, r^{n-1}\right\}\right\}.$$

The vertex of A_{n-2} is adjacent with exactly two vertices of each vertex sets A, A_1, A_2, ..., A_{n-3}. So the vertex of A_{n-2} requires a new color.

Now, consider

$$B = \left\{\left\{1, sr^l\right\} : 1 \leq l \leq n\right\}.$$

The induced subgraph $\langle A \cup B \rangle \cong K_{2n-1}$. Hence by definition of Harmonious coloring, the vertices of the set B needs n new colors.

$$\text{Hence, } \chi_H(G) = \frac{n(n+1)}{2}. \qquad \square$$

Theorem 2.2. *Let $G = L(C(D_{2n}))$ where $n \geq 3$ is an even integer. Then*

$$\chi_H(G) = \frac{n(n+3)}{2}.$$

Proof. Let $G : L(C(D_{2n}))$ where $n \geq 3$ is an even integer. The vertex set of the graph is

$$V = \left\{\left\{1, r^k\right\}, \left\{r^i, r^j\right\}, \left\{1, sr^l\right\}, \left\{r^{\frac{n}{2}}\right\}, \left\{sr^l, sr^{l \oplus_n \frac{n}{2}}\right\}\right\}$$

such that $1 \leq k \leq n-1$, $i < j$, $1 \leq i \leq n-2$, $2 \leq j \leq n-1$, $1 \leq l \leq n$.

Let

$$B = \left\{\left\{1, r^k\right\} : 1 \leq k \leq n-1\right\}.$$

The induced subgraph $\langle B\rangle \cong K_{n-1}$. Hence one needs $n-1$ colours to colour that subgraph.

Now, let

$$B_1 = \{\{r, r^a\} : 2 \le a \le n-1\}.$$

The subgraph induced by the set B_1 is complete and each vertex of B_1 is adjacent with exactly two vertices of the set B. Hence the vertices of the set B_1 cannot be colored with the above $n-1$ colours. So one needs $n-2$ colors different from the above $n-1$ colors to color the vertices in the set B_1.

Now, let

$$B_2 = \left\{\left\{r^2, r^b\right\} : 3 \le b \le n-1\right\}.$$

The subgraph induced by the vertex set B_2 is also complete and each vertex of B_2 is adjacent with exactly two vertices of the set B and B_1. Hence the above $2n-3$ colors cannot be used to color the vertices of the set A_2. Hence one needs $n-3$ colors different from the above $2n-3$ colors to color the vertices in the set B_2.

Proceeding similarly, let

$$B_{n-2} = \{\{r^{n-2}, r^{n-1}\}\}.$$

The vertex of B_{n-2} is adjacent with exactly two vertices of the sets B, B_1, B_2,..., B_{n-3}. So the vertex of B_{n-2} requires a new color.

Now, consider $C = \left\{\left\{1, sr^l\right\} : 1 \le l \le n\right\}$. The induced subgraph $\langle B \cup C\rangle \cong K_{2n-1}$. Hence by definition of Harmonious colouring, the vertices of the set C needs n new colours.

Now, consider the subset $D = \left\{\left\{r^{\frac{n}{2}}, sr^l\right\} : 1 \le l \le n\right\}$. The subgraph induced by the set D is complete. Also the vertices of the set D is adjacent with $\left\{1, r^{\frac{n}{2}}\right\}$, $\{r^i, r^j\}$ for $i = \frac{n}{2}$ or $j = \frac{n}{2}$ and with $\left\{1, sr^l\right\}$ for equal values of l. Let us consider the vertex set D_1=$\left\{\left\{r^c, r^d\right\} : a, b \ne \frac{n}{2}, 1 \le c \le n-2, 2 \le d \le n-1\right\}$, such vertices will be adjacent with $\left\{r^{\frac{n}{2}}, r^e\right\}$ for e=a or b. Hence by definition of Harmonious colouring, n new colours must be assigned to the vertices of the set D.

Let us consider the vertex set $E = \left\{\left\{sr^l, sr^{l\oplus\frac{n}{2}}\right\} : 1 \le l \le n\right\}$. The subgraph induced by the set E is totally disconnected and the vertices of E is not adjacent with $\left\{r^j, r^k\right\}$, $j < k$, $j, k \ne \frac{n}{2}$, $1 \le j \le n-2$, $2 \le k \le n-1$. Hence any one colour of $\left\{r^j, r^k\right\}$, $j < k$, $j, k \ne \frac{n}{2}$, $1 \le j \le n-2$, $2 \le k \le n-1$ can be assigned to these vertices.

$$\text{Hence, } \chi_H(G) = \frac{n(n+3)}{2}. \qquad \square$$

3. Harmonious colouring of line graph of non-commuting graph on D_{2n}. The non-commuting graph of D_{2n} and its properties were studied by A. Asghar Talebi in [1]. The non-commuting graph of D_{2n} denoted by $NC(D_{2n})$ has vertices as elements of $D_{2n} - Z(D_{2n})$ with any two vertices are adjacent iff $xy \neq yx$, $x, y \in D_{2n} - Z(D_{2n})$. Line graph of non-commuting graph of the Dihedral Group denoted by $L(NC(D_{2n}))$ has vertices as edges of $NC(D_{2n})$ and any two vertices are adjacent whenever the corresponding edges of $NC(D_{2n})$ are adjacent. In the following theorems, we find the Harmonious coloring of the Line graph of Non-Commuting graph of D_{2n}.

Theorem 3.1. *Let $G : L(NC(D_{2n}))$ where $n \geq 3$ is an odd integer. Then*

$$\chi_H(G) = \frac{3n(n-1)}{2}.$$

Proof. Let $G : L(NC(D_{2n}))$ where $n \geq 3$ is an odd integer. The vertex set of the graph is

$$V = \left\{ \left\{r^i, sr\right\}, \left\{r^i, sr^2\right\}, \ldots, \left\{r^i, sr^n\right\}, \left\{sr^j, sr^k\right\} \right\}$$

such that $1 \leq i \leq n-1$, $j < k$, $1 \leq j \leq n-1$, $2 \leq k \leq n$.

Let A_1=$\left\{\left\{r^i, sr\right\} : 1 \leq i \leq n-1\right\}$. The induced subgraph $\langle A_1 \rangle \cong K_{n-1}$. Hence one needs $n-1$ colors to color the vertices of the set A_1.

Now, let

$$A_2 = \left\{\left\{r^i, sr^2\right\} : 1 \leq i \leq n-1\right\}.$$

The subgraph induced by the set A_2 is complete and each vertex of A_2 is adjacent with exactly one vertex of the set A_1. Hence the vertices of the set A_2 cannot be colored with the above $n-1$ colors. So one needs $n-1$ colors different from the above $n-1$ colors to color the vertices in the set A_2.

Now, let

$$A_3 = \left\{\left\{r^i, sr^3\right\} : 1 \leq i \leq n-1\right\}.$$

The subgraph induced by the set A_3 is complete and each vertex of A_3 is adjacent with exactly one vertex of the set A_1 and A_2. Hence the above $2n-2$ colors cannot be used to color the vertices of the set A_3. So one needs $n-1$ colors different from the above $2n-2$ colors to color the vertices in the set A_3.

Proceeding similarly, let

$$A_n = \left\{ \left\{ r^i, sr^n \right\} : 1 \leq i \leq n-1 \right\}.$$

The subgraph induced by the set A_n is also complete and each vertex of A_n is adjacent with exactly one vertex of each vertex sets A_1, A_2, ..., A_{n-1}. Hence the above $(n-1)^2$ colors cannot be used to color the vertices of the set A_n. So one needs $n-1$ colors different from the above $(n-1)^2$ colors to color the vertices in the set A_n.

Now, let us consider the subset

$$B_1 = \left\{ \left\{ sr, sr^j \right\} : 2 \leq j \leq n \right\}.$$

The subgraph induced the vertices of the sets B_1 and A_1 is complete. Also the vertices of the set B_1 is adjacent with $\left\{ r^i, sr^j \right\}$, $1 \leq i \leq n-1$, $2 \leq j \leq n$ for equal values of j. Hence by definition of Harmonious coloring, the vertices of the set B_1 needs $n-1$ new colors.

Now, let us consider the subset

$$B_2 = \left\{ \left\{ sr^2, sr^k \right\} : 3 \leq k \leq n \right\}.$$

The subgraph induced the vertices of the sets B_2 and A_2 is complete. Also the vertices of the set B_2 is adjacent with $\left\{ r^i, sr^k \right\}$, $1 \leq i \leq n-1$, $3 \leq k \leq n$ for equal values of k. Hence by definition of Harmonious coloring, the vertices of the set B_2 needs $n-2$ new colors.

Proceeding similarly, let us consider the subset $B_{n-1} = \left\{ sr^{n-1}, sr^n \right\}$. The subgraph induced the vertices of the sets $B_{n-1} \cup A_{n-1} \cup A_n$ is complete. Also the vertex of B_{n-1} is adjacent with $\left\{ r^i, sr^l \right\}$; $1 \leq i \leq n-1$, $l = n-1$ and $l = n$. Hence by definition of Harmonious coloring, the vertex of B_{n-1} needs a new color.

$$\text{Hence, } \chi_H(G) = \frac{3n(n-1)}{2}. \qquad \square$$

Theorem 3.2. *Let $G : L(NC(D_{2n}))$ where $n \geq 3$ is an even integer. Then*

$$\chi_H(G) = \frac{3n(n-2)}{2}.$$

Proof. Let $G : L(NC(D_{2n}))$ where $n \geq 3$ is an even integer. The vertex set of the graph is

$$V = \left\{ \left\{ r^i, sr \right\}, \left\{ r^i, sr^2 \right\}, \ldots, \left\{ r^i, sr^n \right\}, \left\{ sr^j, sr^k \right\} \right\}$$

such that $1 \leq i \leq n-1$, $i \neq \frac{n}{2}$, $j < k$, $k \neq j \oplus \frac{n}{2}$, $1 \leq j \leq n-1$, $2 \leq k \leq n$.

Let

$$B_1 = \left\{ \{r^i, sr\} : 1 \le i \le n-1, i \ne \frac{n}{2} \right\}.$$

The induced subgraph $\langle B_1 \rangle \cong K_{n-1}$. Hence one needs $n-2$ colours to colour the vertices of the set B_1.

Now, let

$$B_2 = \left\{ \{r^i, sr^2\} : 1 \le i \le n-1, i \ne \frac{n}{2} \right\}.$$

The subgraph induced by the vertex set B_2 is complete and each vertex of B_2 is adjacent with exactly one vertex of B_1. Hence the vertices of the set B_2 cannot be coloured with the above $n-2$ colours. So one needs $n-2$ colours different from the above $n-2$ colours to colour the vertices in the set B_2.

Now, let

$$B_3 = \left\{ \{r^i, sr^3\} : 1 \le i \le n-1, i \ne \frac{n}{2} \right\}.$$

The subgraph induced by the vertex set B_3 is complete and each vertex of B_3 is adjacent with exactly one vertex of B_1 and B_2. Hence the above $2n-4$ colours cannot be used to colour the vertices of the set B_3. So one needs $n-2$ colours different from the above $2n-4$ colours to colour the vertices in the vertex set B_3.

Proceeding similarly, let

$$B_n = \left\{ \{r^i, sr^n\} : 1 \le i \le n-1, i \ne \frac{n}{2} \right\}.$$

The subgraph induced by the vertex set B_n is also complete and each vertex of B_n is adjacent with exactly one vertex of each vertex sets B_1, B_2,..., B_{n-1}. Hence the above $(n-1)(n-2)$ colours cannot be used to colour the vertices of the set B_n. So one needs $n-2$ colours different from the above $(n-1)(n-2)$ colours to colour the vertices in the set B_n.

Now, let us consider the subset

$$C_1 = \left\{ \left\{ sr, sr^l \right\} : 2 \le l \le n, l \ne 1 \oplus \frac{n}{2} \right\}.$$

The subgraph induced the vertices of the sets C_1 and B_1 is complete. Also the vertices of the set C_1 is adjacent with $\left\{ r^i, sr^l \right\}$, $1 \le i \le n-1$, $i \ne \frac{n}{2}$, $1 \le l \le n$ for equal values of l. Hence by definition of Harmonious colouring, the vertices of the set C_1 needs $n-2$ new colours.

Let us consider the subset

$$C_2 = \left\{ \{sr^2, sr^m\} : 3 \le m \le n, m \ne 2 \oplus \frac{n}{2} \right\}.$$

The subgraph induced the vertices of the sets C_2 and B_2 is complete. Also the vertices of the set C_2 is adjacent with $\{r^i, sr^m\}$, $1 \le i \le n-1$, $i \ne$

$\frac{n}{2}$, $1 \leq m \leq n$ for equal values of m. Hence by definition of Harmonious colouring, the vertices of the set C_2 needs $n-3$ new colours.

Proceeding similarly, let us consider the subset $C_{n-1} = \{sr^{n-1}, sr^n\}$. The subgraph induced the vertices of the sets $C_{n-1} \cup B_{n-1} \cup B_n$ is complete. Also the vertex of C_{n-1} is adjacent with $\{r^i, sr^t\}$, $1 \leq i \leq n-1$, $i \neq \frac{n}{2}$, $t = n-1$ and $t = n$. Hence by definition of Harmonious colouring, the vertex of C_{n-1} needs a new colour.

$$\text{Hence, } \chi_H(G) = \frac{3n(n-2)}{2}. \qquad \square$$

REFERENCES

[1] A. Asghar Talebi. On the non-commuting graphs of group D_{2n}. *Int. J. Algebra* **2**, *17–20* (2008), 957–961.

[2] R. Divya, P. Chithra Devi. The line graph of a commuting graph on the dihedral group D_{2n}. *Sarajevo J. Math.* **17(30)**, *2* (2021), 143–150.

[3] D. S. Dummit, R. M. Foote. Abstract Algebra, 2nd edi. Singapore, John Wiley and Son, Inc(Asia) Pvt. Ltd, 2005.

[4] F. Harary. Graph Theory. Mass.-Menlo Park, Calif.-London, Addison-Wesley Publishing Co., Reading, 1969.

[5] J. E. Hopcroft, M. S. Krishnamoorthy. On the harmonious coloring of graphs. *SIAM J. Algebraic Discrete Methods* **4**, *3* (1983), 306–311.

[6] T. Tamizh Chelvam, K. Selvakumar, S. Raja. Commuting graphs on dihedral group. *J. Math. Comput. Sci. (JMCS)* **2**, *2*, (2011), 402–406, http://dx.doi.org/10.22436/jmcs.002.02.20.

Department of Mathematics
Sri Parasakthi College for Women, Courtallam
Affiliated to Manonmaniam Sundaranar University
Tirunelveli – 627 012, Tamil Nadu, India
e-mail: divyaramakrishnan1224@gmail.com *(R. Divya)*
Designation: Research Scholar, Reg. No.: 18214222092007
e-mail: chithradevi095@gmail.com *(P. Chithra Devi)*

Received October 10, 2022
Accepted May 09, 2023