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COEFFICIENT ESTIMATES FOR SOME GENERALIZED SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS

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Abstract. In this paper, we introduce certain unified subclasses of close-to-convex functions and quasi-convex functions with respect to symmetric and conjugate points in the unit disc $E = \{z \in \mathbb{C} :\mid z \mid< 1\}$ and establish the upper bounds of the first four coefficients for these classes. This study will work as a motivation for the other researchers in this field to study some more similar classes.

1. Introduction. Let $\mathcal{A}$ be the class of analytic functions f in the unit disc $E = \{z :\mid z \mid< 1\}$ and which are of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. The class of functions $f \in \mathcal{A}$, which are univalent in E, is denoted by $\mathcal{S}$. Let $\mathcal{U}$ be the class

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of Schwarzian functions of the form $u(z)=\sum_{k=1}^{\infty}c_kz^k$, which are analytic in the unit disc E and satisfying the conditions $u(0)=0$ and $\mid u(z)\mid< 1$.

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f\prec g$) if there exists a Schwarzian function $u\in\mathcal{U}$ such that $f(z)=g(u(z))$.

Firstly, let's have an overview of some fundamental classes of univalent functions, which are relevant to the study of this paper:

$\mathcal{S}^*=\left\{f\in\mathcal{A}: Re\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in E\right\}$, the class of starlike functions.

$\mathcal{K}=\left\{f\in\mathcal{A}: Re\left(\frac{(zf'(z))'}{f'(z)}\right)>0,\ z\in E\right\}$, the class of convex functions.

The classes $\mathcal{S}^*$ and $\mathcal{K}$ are related by the Alexander relation [3] as $f\in\mathcal{K}$ if and only if $zf'\in\mathcal{S}^*$.

A function $f\in\mathcal{A}$ is said to be close-to-convex if there exists a convex function h such that $Re\left(\frac{f'(z)}{h'(z)}\right)>0$ or equivalently there exists a starlike function g such that $Re\left(\frac{zf'(z)}{g(z)}\right)>0$. The class of close-to-convex functions is denoted by $\mathcal{C}$ and was introduced by Kaplan [9].

Sakaguchi [12] established the class $\mathcal{S}_s^*$ of the functions $f\in\mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Clearly, the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions, as $\frac{f(z)-f(-z)}{2}$ is a starlike function [5] in E.

Later on, Das and Singh [5] introduced the class $\mathcal{K}_s$ of the functions $f\in\mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)>0.$$

The functions in the class $\mathcal{K}_s$ are called convex functions with respect to symmetric points. Clearly $f\in\mathcal{K}_s$ if and only if $zf'\in\mathcal{S}_s^*$.

Further, El-Ashwah and Thomas [4] established the class $\mathcal{S}_c^*$, the class of starlike functions with respect to conjugate points and $\mathcal{K}_c$, the class of convex functions with respect to conjugate points, as follows:

$$\mathcal{S}_c^* = \left\{ f \in \mathcal{A} : Re\left(\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K}_c = \left\{ f \in \mathcal{A} : Re\left(\frac{2(zf'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'}\right) > 0, z \in E \right\}.$$

Janteng et al. [7] studied the following classes:

$$\mathcal{C}_s^* = \left\{ f \in \mathcal{A} : Re\left(\frac{2zf'(z)}{g(z)-g(-z)}\right) > 0, g \in \mathcal{S}_s^*, z \in E \right\},$$

the class of close-to-convex functions with respect to symmetric points.

$$\mathcal{C}_c = \left\{ f \in \mathcal{A} : Re\left(\frac{2zf'(z)}{g(z)+\overline{g(\overline{z})}}\right) > 0, g \in \mathcal{S}_c, z \in E \right\},$$

the class of close-to-convex functions with respect to conjugate points.

$$\mathcal{Q}_s^* = \left\{ f \in \mathcal{A} : Re\left(\frac{2(zf'(z))'}{(h(z)-h(-z))'}\right) > 0, h \in \mathcal{K}_s, z \in E \right\},$$

the class of quasi-convex functions with respect to symmetric points.

$$\mathcal{Q}_c = \left\{ f \in \mathcal{A} : Re\left(\frac{2(zf'(z))'}{\left(h(z)+\overline{h(\overline{z})}\right)'}\right) > 0, h \in \mathcal{K}_c, z \in E \right\},$$

the class of quasi-convex functions with respect to conjugate points.

The class $\mathcal{P}[C,D]$ consists of the functions p analytic in E with $p(0)=1$ and subordinate to $\frac{1+Cz}{1+Dz}, (-1 \le D < C \le 1)$. This class was established by Janowski [6].

Kanas and Ronning [8] introduced an interesting class $\mathcal{A}(w)$ of analytic functions of the form

$$f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k \tag{1.1}$$

and normalized by the conditions $f(w)=0, f'(w)=1$, where w is a fixed point in E.

Also the classes of w-starlike functions and w-convex functions were defined in [8] as follows:

$$\mathcal{S}^*(w)=\left\{f\in\mathcal{A}(w): Re\left(\frac{(z-w)f'(z)}{f(z)}\right)>0, z\in E\right\},$$

and

$$\mathcal{K}(w)=\left\{f\in\mathcal{A}(w): 1+Re\left(\frac{(z-w)f''(z)}{f'(z)}\right)>0, z\in E\right\}.$$

The class $\mathcal{S}^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $\mathcal{K}(w)$ is defined by the property that the image of any circular arc centered at w is convex. For $w=0$, the classes $\mathcal{S}^*(w)$ and $\mathcal{K}(w)$ agree with the well known classes of starlike and convex functions, respectively. Also it is obvious that $f\in\mathcal{K}(w)$ if and only if $(z-w)f'\in\mathcal{S}^*(w)$. Various authors such as Acu and Owa [1], Al-Hawary et al. [2], Olatunji and Oladipo [11] and Singh and Singh [13] have worked on the classes of analytic functions with fixed point.

For $-1\le B<A\le 1$, Oladipo [10] studied the following subclasses of $\mathcal{A}(w)$:

$$\mathcal{S}_s^*(w;A,B)=\left\{f\in\mathcal{A}(w): \frac{2(z-w)f'(z)}{f(z)-f(-z)}\prec\frac{1+A(z-w)}{1+B(z-w)}, z\in E\right\},$$

$$\mathcal{K}_s(w;A,B)=\left\{f\in\mathcal{A}(w): \frac{2((z-w)f'(z))'}{(f(z)-f(-z))'}\prec\frac{1+A(z-w)}{1+B(z-w)}, z\in E\right\},$$

$$\mathcal{S}_c^*(w;A,B)=\left\{f\in\mathcal{A}(w): \frac{2(z-w)f'(z)}{f(z)+\overline{f(\overline{z})}}\prec\frac{1+A(z-w)}{1+B(z-w)}, z\in E\right\},$$

$$\mathcal{K}_c(w;A,B)=\left\{f\in\mathcal{A}(w): \frac{2((z-w)f'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'}\prec\frac{1+A(z-w)}{1+B(z-w)}, z\in E\right\}.$$

For $A=1, B=-1$, the classes $\mathcal{S}_s^*(w;A,B)$, $\mathcal{K}_s(w;A,B)$, $\mathcal{S}_c^*(w;A,B)$ and $\mathcal{K}_c(w;A,B)$, reduce to $\mathcal{S}_s^*(w)$, $\mathcal{K}_s(w)$, $\mathcal{S}_c^*(w)$ and $\mathcal{K}_c(w)$, respectively.

To avoid repetition throughout this paper, we assume that $-1\le D<C\le 1$, $-1\le B<A\le 1$, $z\in E$.

Motivated and stimulated by the above defined classes, we now introduce the following subclasses of $\mathcal{A}(w)$, associated with Janowski function:

Definition 1.1. *A function $f \in \mathcal{A}(w)$ is said to be in the class $\phi_s(w;\alpha;A, B;C,D)$ if*

$$\frac{2(z-w)f'(z)+2\alpha(z-w)^2 f''(z)}{(1-\alpha)(g(z)-g(-z))+\alpha(z-w)(g(z)-g(-z))'} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $g(z)=(z-w)+\sum_{k=2}^{\infty} b_k(z-w)^k \in \mathcal{S}_s^*(w;A,B)$.

The following points are to be noted:

(i) $\phi_s(0;\alpha;A,B;C,D) \equiv \phi_s(\alpha;A,B;C,D)$.

(ii) $\phi_s(0;0;1,-1;C,D) \equiv \mathcal{C}_s^*(C,D)$, the subclass of close-to-convex functions with respect to symmetric points.

(iii) $\phi_s(w;0;1,-1;C,D) \equiv \mathcal{C}_s^*(w;C,D)$, the subclass of w-close-to-convex functions with respect to symmetric points.

(iv) $\phi_s(w;0;1,-1;1,-1) \equiv \mathcal{C}_s^*(w)$, the class of w-close-to-convex functions with respect to symmetric points.

(v) $\phi_s(0;0;1,-1;1,-1) \equiv \mathcal{C}_s^*$, the class of close-to-convex functions with respect to symmetric points.

Definition 1.2. *A function $f \in \mathcal{A}(w)$ is said to be in the class $\phi_s'(w;\alpha;A, B;C,D)$ if*

$$\frac{2(z-w)f'(z)+2\alpha(z-w)^2 f''(z)}{(1-\alpha)(h(z)-h(-z))+\alpha(z-w)(h(z)-h(-z))'} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $h(z)=(z-w)+\sum_{k=2}^{\infty} d_k(z-w)^k \in \mathcal{K}_s(w;A,B)$.

The following observations are obvious:

(i) $\phi_s'(0;\alpha;A,B;C,D) \equiv \phi_s'(\alpha;A,B;C,D)$.

(ii) $\phi_s'(0;1;1,-1;C,D) \equiv \mathcal{Q}_s^*(C,D)$, the subclass of quasi-convex functions with respect to symmetric points.

(iii) $\phi_s'(w;1;1,-1;C,D) \equiv \mathcal{Q}_s^*(w;C,D)$, the subclass of w-quasi-convex functions with respect to symmetric points.

(iv) $\phi_s'(w;1;1,-1;1,-1) \equiv \mathcal{Q}_s^*(w)$, the class of w-quasi-convex functions with respect to symmetric points.

(v) $\phi_s'(0;1;1,-1;1,-1) \equiv \mathcal{Q}_s^*$, the class of quasi-convex functions with respect to symmetric points.

Definition 1.3. *A function $f \in \mathcal{A}(w)$ is said to be in the class $\psi_c(w;\alpha;A,B;C,D)$ if*

$$\frac{2(z-w)f'(z)+2\alpha(z-w)^2f''(z)}{(1-\alpha)(g(z)+\overline{g(\overline{z})})+\alpha(z-w)(g(z)+\overline{g(\overline{z})})'}\prec\frac{1+C(z-w)}{1+D(z-w)},$$

where $g(z)=(z-w)+\sum_{k=2}^{\infty}b_k(z-w)^k\in\mathcal{S}_c^*(w;A,B)$.

We have the following observations:

(i) $\psi_c(0;\alpha;A,B;C,D)\equiv\psi_c(\alpha;A,B;C,D)$.

(ii) $\psi_c(0;0;1,-1;C,D)\equiv\mathcal{C}_c(C,D)$, the subclass of close-to-convex functions with respect to conjugate points.

(iii) $\psi_c(w;0;1,-1;C,D)\equiv\mathcal{C}_c(w;C,D)$, the subclass of w-close-to-convex functions with respect to conjugate points.

(iv) $\psi_c(w;0;1,-1;1,-1)\equiv\mathcal{C}_c$, the class of w-close-to-convex functions with respect to conjugate points.

(v) $\psi_c(0;0;1,-1;1,-1)\equiv\mathcal{C}_c$, the class of close-to-convex functions with respect to conjugate points.

Definition 1.4. *A function $f \in \mathcal{A}(w)$ is said to be in the class $\psi_c'(w;\alpha;A,B;C,D)$ if*

$$\frac{2(z-w)f'(z)+2\alpha(z-w)^2f''(z)}{(1-\alpha)(h(z)+\overline{h(\overline{z})})+\alpha(z-w)(h(z)+\overline{h(\overline{z})})'}\prec\frac{1+C(z-w)}{1+D(z-w)},$$

where $h(z)=(z-w)+\sum_{k=2}^{\infty}d_k(z-w)^k\in\mathcal{K}_c(w;A,B)$.

The following points are obvious:

(i) $\psi_c'(0;\alpha;A,B;C,D)\equiv\psi_c'(\alpha;A,B;C,D)$.

(ii) $\psi_c'(0;1;1,-1;C,D)\equiv\mathcal{Q}_c(C,D)$, the subclass of quasi-convex functions with respect to conjugate points.

(iii) $\psi_c'(w;1;1,-1;C,D)\equiv\mathcal{Q}_c(w;C,D)$, the subclass of w-quasi-convex functions with respect to conjugate points.

(iv) $\psi_c'(w;1;1,-1;1,-1)\equiv\mathcal{Q}_c(w)$, the class of w-quasi-convex functions with respect to conjugate points.

(v) $\psi_c'(0;1;1,-1;1,-1)\equiv\mathcal{Q}_c$, the class of quasi-convex functions with respect to conjugate points.

In this paper, we establish the upper bounds of the first four coefficients for the functions belonging to the classes $\phi_s(w;\alpha;A,B;C,D)$, $\phi_s'(w;\alpha;A,B;C,D)$,

$\psi_c(w;\alpha;A,B;C,D)$ and $\psi'_c(w;\alpha;A,B;C,D)$. This paper will motivate the other researchers to investigate some more interesting classes.

2. Preliminary results.

Lemma 2.1 ([11]). *For* $u(z)=\sum_{k=1}^{\infty}c_k(z-w)^k$ *and* $p(z)=\frac{1+Cu(z)}{1+Du(z)}=1+\sum_{k=1}^{\infty}p_k(z-w)^k$, *we have,*

$$|p_n|\le\frac{(C-D)}{(1+d)(1-d)^n},n\ge 1,|w|=d.$$

Lemma 2.2 ([10]). *If* $g(z)=(z-w)+\sum_{k=2}^{\infty}b_k(z-w)^k\in\mathcal{S}_s^*(w;A,B)$, *then*

$$|b_2|\le\frac{(A-B)}{2(1-d^2)},\tag{2.1}$$

$$|b_3|\le\frac{(A-B)}{2(1-d^2)(1-d)},\tag{2.2}$$

$$|b_4|\le\frac{(A-B)[(A-B)+2(1+d)]}{8(1-d^2)^2(1-d)},\tag{2.3}$$

and

$$|b_5|\le\frac{(A-B)[(A-B)+2(1+d)]}{8(1-d^2)^2(1-d)^2}.\tag{2.4}$$

Lemma 2.3 ([10]). *If* $h(z)=(z-w)+\sum_{k=2}^{\infty}d_k(z-w)^k\in\mathcal{K}_s(w;A,B)$, *then*

$$|d_2|\le\frac{(A-B)}{4(1-d^2)},$$

$$|d_3|\le\frac{(A-B)}{6(1-d^2)(1-d)},$$

$$|d_4|\le\frac{(A-B)[(A-B)+2(1+d)]}{32(1-d^2)^2(1-d)},$$

and

$$|d_5|\le\frac{(A-B)[(A-B)+2(1+d)]}{40(1-d^2)^2(1-d)^2}.$$

Lemma 2.4 ([10]). *If* $g(z) = (z-w)+\sum_{k=2}^{\infty} b_k(z-w)^k \in \mathcal{S}_c^*(w;A,B)$, *then*

$$|b_2| \le \frac{(A-B)}{1-d^2}, \tag{2.5}$$

$$|b_3| \le \frac{(A-B)[(A-B)+(1+d)]}{2(1-d^2)^2}, \tag{2.6}$$

$$|b_4| \le \frac{(A-B)[(A-B)^2+3(A-B)(1+d)+2(1+d)^2]}{6(1-d^2)^3}, \tag{2.7}$$

and

$$|b_5| \le \frac{(A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3]}{24(1-d^2)^4}. \tag{2.8}$$

Lemma 2.5 ([10]). *If* $h(z) = (z-w)+\sum_{k=2}^{\infty} d_k(z-w)^k \in \mathcal{K}_c(w;A,B)$, *then*

$$|d_2| \le \frac{(A-B)}{2(1-d^2)},$$

$$|d_3| \le \frac{(A-B)[(A-B)+(1+d)]}{6(1-d^2)^2},$$

$$|d_4| \le \frac{(A-B)[(A-B)^2+3(A-B)(1+d)+2(1+d)^2]}{24(1-d^2)^3},$$

and

$$|d_5| \le \frac{(A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3]}{120(1-d^2)^4}.$$

3. Main results.

Theorem 3.1. *If* $f \in \phi_s(w;\alpha;A,B;C,D)$, *then*

$$|a_2| \le \frac{(C-D)}{2(1+\alpha)(1-d^2)}, \tag{3.1}$$

$$|a_3| \le \frac{[(1+2\alpha)(A-B)+2(C-D)]}{6(1+2\alpha)(1-d)(1-d^2)}, \tag{3.2}$$

$$|a_4| \le \frac{(C-D)[(1+2\alpha)(A-B)+2(1+d)]}{8(1+3\alpha)(1-d^2)^2(1-d)}, \tag{3.3}$$

and

(3.4)
$$|a_5| \leq \frac{(1+4\alpha)(A-B)[(A-B)+2(1+d)] + (C-D)[4(1+2\alpha)(A-B)+2(1+d)]}{40(1+4\alpha)(1-d^2)^2(1-d)^2}.$$

Proof. Using concept of subordination in Definition 1.1, we have

(3.5)
$$\frac{2(z-w)f'(z) + 2\alpha(z-w)^2 f''(z)}{(1-\alpha)(g(z)-g(-z)) + \alpha(z-w)(g(z)-g(-z))'} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1 + \sum_{k=1}^{\infty} p_k(z-w)^k,$$

where $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$.

Expansion of (3.5) leads to

$$\begin{aligned}
&(z-w) + 2(1+\alpha)a_2(z-w)^2 + 3(1+2\alpha)a_3(z-w)^3 \\
&+4(1+3\alpha)a_4(z-w)^4 + 5(1+4\alpha)a_5(z-w)^5 + \ldots \\
&= (z-w) + (1+2\alpha)b_3(z-w)^3 + (1+4\alpha)b_5(z-w)^5 + \ldots \\
&+p_1(z-w)^2 + (1+2\alpha)p_1b_3(z-w)^4 + (1+4\alpha)p_1b_5(z-w)^6 + \ldots \\
&+p_2(z-w)^3 + (1+2\alpha)p_2b_3(z-w)^5 + (1+4\alpha)p_2b_5(z-w)^7 + \ldots \\
&+p_3(z-w)^4 + (1+2\alpha)p_3b_3(z-w)^6 + (1+4\alpha)p_3b_5(z-w)^8 + \ldots \\
&+p_4(z-w)^5 + (1+2\alpha)p_4b_3(z-w)^7 + (1+4\alpha)p_4b_5(z-w)^9 + \ldots \\
&+p_5(z-w)^6 + (1+2\alpha)p_5b_3(z-w)^8 + \ldots
\end{aligned}$$

On equating the coefficients of $(z-w)^2, (z-w)^3, (z-w)^4$ and $(z-w)^5$ in the above expansion, it yields

(3.6)
$$2(1+\alpha)a_2 = p_1,$$

(3.7)
$$3(1+2\alpha)a_3 = p_2 + (1+2\alpha)b_3,$$

(3.8)
$$4(1+3\alpha)a_4 = p_3 + (1+2\alpha)b_3p_1,$$

and

(3.9)
$$5(1+4\alpha)a_5 = p_4 + (1+2\alpha)b_3p_2 + (1+4\alpha)b_5.$$

On taking modulus and application of triangle inequality, the equations (3.6), (3.7), (3.8) and (3.9) transform to

(3.10)
$$2(1+\alpha)|a_2| = |p_1|,$$

$$3(1+2\alpha)|a_3| \le |p_2| + (1+2\alpha)|b_3|, \tag{3.11}$$
$$4(1+3\alpha)|a_4| \le |p_3| + (1+2\alpha)|b_3||p_1|, \tag{3.12}$$

and

$$5(1+4\alpha)|a_5| \le |p_4| + (1+2\alpha)|b_3||p_2| + (1+4\alpha)|b_5|. \tag{3.13}$$

Using Lemma 2.1 and inequality (2.1) in (3.10), the result (3.1) is obvious.

Again using Lemma 2.1 and inequality (2.2) in (3.11), the simplification leads to the result (3.2).

Further using the inequality (2.3) and applying Lemma 2.1, the result (3.3) can be easily obtained from (3.12).

On using the inequalities (2.2) and (2.4) and application of Lemma 2.1 in (3.13), it leads to the result (3.4). □

For $\alpha = 0$, Theorem 3.1 yields the following result:

Corollary 3.1. *If $f \in \phi_s(w; A, B; C, D)$, then*

$$|a_2| \le \frac{(C-D)}{2(1-d^2)},$$
$$|a_3| \le \frac{[(A-B)+2(C-D)]}{6(1-d)(1-d^2)},$$
$$|a_4| \le \frac{(C-D)[(A-B)+2(1+d)]}{8(1-d)(1-d^2)^2},$$

and

$$|a_5| \le \frac{(A-B)[(A-B)+2(1+d)]+(C-D)[4(A-B)+2(1+d)]}{40(1-d)^2(1-d^2)^2}.$$

On putting $\alpha = 1$, Theorem 3.1 agrees with the following result:

Corollary 3.2. *If $f \in \phi_{s1}(w; A, B; C, D)$, then*

$$|a_2| \le \frac{(C-D)}{4(1-d^2)},$$
$$|a_3| \le \frac{[3(A-B)+2(C-D)]}{18(1-d)(1-d^2)},$$
$$|a_4| \le \frac{(C-D)[3(A-B)+2(1+d)]}{32(1-d)(1-d^2)^2},$$

and

$$|a_5| \le \frac{[(A-B)+2(1+d)][(A-B)+4(C-D)]}{40(1-d)^2(1-d^2)^2}.$$

Theorem 3.2. *If $f \in \phi'_s(w;\alpha;A,B;C,D)$, then*

$$|a_2| \leq \frac{(C-D)}{2(1+\alpha)(1-d^2)},$$
$$|a_3| \leq \frac{[(1+2\alpha)(A-B)+6(C-D)]}{18(1+2\alpha)(1+d)(1-d)^2},$$
$$|a_4| \leq \frac{(C-D)[(1+2\alpha)(A-B)+6(1+d)]}{24(1+3\alpha)(1-d^2)^2(1-d)},$$

and

$$|a_5| \leq \frac{3(1+4\alpha)(A-B)[(A-B)+2(1+d)]+(C-D)[20(1+2\alpha)(A-B)+120(1+d)]}{600(1+4\alpha)(1-d^2)^2(1-d)^2}.$$

P r o o f. Following the procedure of Theorem 3.1, and using Lemma 2.1 and Lemma 2.3, the proof of Theorem 3.2 is obvious. □

For $\alpha = 0$, Theorem 3.2 yields the following result:

Corollary 3.3. *If $f \in \phi'_s(w;A,B;C,D)$, then*

$$|a_2| \leq \frac{(C-D)}{2(1-d^2)},$$
$$|a_3| \leq \frac{[(A-B)+6(C-D)]}{18(1-d)(1-d^2)},$$
$$|a_4| \leq \frac{(C-D)[(A-B)+6(1+d)]}{24(1-d)(1-d^2)^2},$$

and

$$|a_5| \leq \frac{3(A-B)[(A-B)+2(1+d)]+20(C-D)[(A-B)+6(1+d)]}{600(1-d)^2(1-d^2)^2}.$$

On putting $\alpha = 1$, Theorem 3.2 agrees with the following result:

Corollary 3.4. *If $f \in \phi'_{s1}(w;A,B;C,D)$, then*

$$|a_2| \leq \frac{(C-D)}{4(1-d^2)},$$
$$|a_3| \leq \frac{[3(A-B)+6(C-D)]}{54(1-d)(1-d^2)},$$
$$|a_4| \leq \frac{(C-D)[3(A-B)+6(1+d)]}{96(1-d)(1-d^2)^2},$$

and

$$|a_5| \leq \frac{[(A-B)+2(1+d)][(A-B)+4(C-D)]}{200(1-d)^2(1-d^2)^2}.$$

Theorem 3.3. *If $f \in \psi_c(w;\alpha;A,B;C,D)$, then*

$$|a_2| \leq \frac{(1+\alpha)(A-B)+(C-D)}{2(1+\alpha)(1-d^2)}, \tag{3.14}$$

$$|a_3| \leq \frac{1}{6(1+2\alpha)(1-d^2)^2}\begin{bmatrix} (1+2\alpha)(A-B)[(A-B)+(1+d)] \\ +2(C-D)[(1+\alpha)(A-B)+(1+d)] \end{bmatrix}, \tag{3.15}$$

$$|a_4| \leq \frac{1}{24(1+3\alpha)(1-d^2)^3}\begin{bmatrix} (1+3\alpha)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +3(C-D)(A-B)[(1+2\alpha)(A-B) \\ +(1+d)(3+4\alpha)]+6(1+d)^2(C-D) \end{bmatrix} \tag{3.16}$$

and

(3.17)

$$|a_5| \leq \frac{1}{120(1+4\alpha)(1-d^2)^4}\begin{bmatrix} (1+4\alpha)(A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +4(1+3\alpha)(C-D)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[(1+2\alpha)(A-B) \\ +(3+4\alpha)(1+d)]+24(1+d)^3(C-D) \end{bmatrix}.$$

Proof. On applying the concept of subordination in Definition 1.3, it yields

$$\begin{aligned} \frac{2(z-w)f'(z)+2\alpha(z-w)^2f''(z)}{(1-\alpha)(g(z)+\overline{g(\overline{z})})+\alpha(z-w)(g(z)+\overline{g(\overline{z})})'} &= p(z) = \frac{1+Cu(z)}{1+Du(z)} \\ &= 1+\sum_{k=1}^{\infty} p_k(z-w)^k, \end{aligned} \tag{3.18}$$

where $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$.

Expansion of (3.18) leads to

$$
\begin{aligned}
&(z-w)+2(1+\alpha)a_2(z-w)^2+3(1+2\alpha)a_3(z-w)^3\\
&\qquad+4(1+3\alpha)a_4(z-w)^4+5(1+4\alpha)a_5(z-w)^5+\cdots\\
&=(z-w)+(1+\alpha)b_2(z-w)^2+(1+2\alpha)b_3(z-w)^3\\
&\qquad+(1+3\alpha)b_4(z-w)^4+(1+4\alpha)b_5(z-w)^5+\cdots\\
&\quad+p_1(z-w)^2+(1+\alpha)p_1b_2(z-w)^3+(1+2\alpha)p_1b_3(z-w)^4\\
&\qquad+(1+3\alpha)p_1b_4(z-w)^5+\cdots\\
&\quad+p_2(z-w)^3+(1+\alpha)p_2b_2(z-w)^4+(1+2\alpha)p_2b_3(z-w)^5\\
&\qquad+(1+3\alpha)p_2b_4(z-w)^6+(1+4\alpha)p_2b_5(z-w)^7+\cdots\\
&\quad+p_3(z-w)^4+(1+\alpha)p_3b_2(z-w)^5+(1+2\alpha)p_3b_3(z-w)^6\\
&\qquad+(1+3\alpha)p_3b_4(z-w)^7+(1+4\alpha)p_3b_5(z-w)^8+\cdots\\
&\qquad+p_4(z-w)^5+\cdots
\end{aligned}
\tag{3.19}
$$

On equating the coefficients of $(z-w)^2,(z-w)^3,(z-w)^4$ and $(z-w)^5$ in (3.19), it yields

$$2(1+\alpha)a_2=p_1+(1+\alpha)b_2, \tag{3.20}$$

$$3(1+2\alpha)a_3=p_2+(1+\alpha)p_1b_2+(1+2\alpha)b_3, \tag{3.21}$$

$$4(1+3\alpha)a_4=p_3+(1+\alpha)p_2b_2+(1+2\alpha)b_3p_1+(1+3\alpha)b_4, \tag{3.22}$$

and

$$5(1+4\alpha)a_5=p_4+(1+\alpha)p_3b_2+(1+2\alpha)b_3p_2+(1+3\alpha)p_1b_4+(1+4\alpha)b_5. \tag{3.23}$$

Taking modulus and applying triangle inequality, the equations (3.20), (3.21), (3.22) and (3.23) transform to

$$2(1+\alpha)|a_2|\le|p_1|+(1+\alpha)|b_2|, \tag{3.24}$$

$$3(1+2\alpha)|a_3|\le|p_2|+(1+\alpha)|p_1||b_2|+(1+2\alpha)|b_3|, \tag{3.25}$$

$$4(1+3\alpha)|a_4|\le|p_3|+(1+\alpha)|p_2||b_2|+(1+2\alpha)|b_3||p_1|+(1+3\alpha)|b_4|, \tag{3.26}$$

and

$$5(1+4\alpha)|a_5| \le |p_4| + (1+\alpha)|b_2||p_3| + (1+2\alpha)|b_3||p_2| + (1+3\alpha)|b_4||p_1| + (1+4\alpha)|b_5|. \tag{3.27}$$

Using Lemma 2.1 and inequality (2.5) in (3.24), the result (3.14) is obvious.

Again using Lemma 2.1 and inequalities (2.5) and (2.6) in (3.25), the simplification leads to the result (3.15).

Further using the inequalities (2.5), (2.6) and (2.7) and applying Lemma 2.1, the result (3.16) can be easily obtained from (3.26).

On using inequalities (2.5), (2.6), (2.7) and (2.8) and application of Lemma 2.1 in (3.27), it leads to the result (3.17). □

For $\alpha = 0$, Theorem 3.3 yields the following result:

Corollary 3.5. *If $f \in \psi_c(w; A, B; C, D)$, then*

$$|a_2| \le \frac{(A-B)+(C-D)}{2(1-d^2)},$$

$$|a_3| \le \frac{[(A-B)+(1+d)][(A-B)+2(C-D)]}{6(1-d^2)^2},$$

$$|a_4| \le \frac{1}{24(1-d^2)^3}\begin{bmatrix} (A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +3(C-D)(A-B)[(A-B)+3(1+d)]+6(1+d)^2(C-D) \end{bmatrix}$$

and

$$|a_5| \le \frac{1}{120(1-d^2)^4}\begin{bmatrix} (A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +4(C-D)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[(A-B)+3(1+d)] \\ +24(1+d)^3(C-D) \end{bmatrix}.$$

On putting $\alpha = 1$, Theorem 3.3 agrees with the following result:

Corollary 3.6. *If $f \in \psi_{c1}(w; A, B; C, D)$, then*

$$|a_2| \le \frac{2(A-B)+(C-D)}{4(1-d^2)},$$

$$|a_3| \leq \frac{3(A-B)[(A-B)+(1+d)]+2(C-D)[2(A-B)+(1+d)]}{18(1-d^2)^2},$$

$$|a_4| \leq \frac{1}{96(1-d^2)^3}\left[\begin{array}{r} 4(A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +3(C-D)(A-B)[3(A-B)+3(1+d)] \\ +6(1+d)^2(C-D) \end{array}\right]$$

and

$$|a_5| \leq \frac{1}{600(1-d^2)^4}\left[\begin{array}{r} 5(A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +16(C-D)(A-B)[(A-B) \\ +(1+d)][(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[3(A-B)+7(1+d)] \\ +24(1+d)^3(C-D) \end{array}\right].$$

Theorem 3.4. *If* $f \in \psi'_c(w;\alpha;A,B;C,D)$, *then*

$$|a_2| \leq \frac{(1+\alpha)(A-B)+2(C-D)}{4(1+\alpha)(1-d^2)},$$

$$|a_3| \leq \frac{}{18(1+2\alpha)(1-d^2)^2}\left[\begin{array}{r} (1+2\alpha)(A-B)[(A-B)+(1+d)] \\ +3(C-D)[(1+\alpha)(A-B)+2(1+d)] \end{array}\right],$$

$$|a_4| \leq \frac{1}{96(1+3\alpha)(1-d^2)^3}\left[\begin{array}{r} (1+3\alpha)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +4(C-D)(A-B)[(1+2\alpha)(A-B) \\ +(1+d)(4+5\alpha)]+24(1+d)^2(C-D) \end{array}\right],$$

and

$$|a_5| \leq \frac{1}{600(1+4\alpha)(1-d^2)^4}\left[\begin{array}{r} (1+4\alpha)(A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +5(1+3\alpha)(C-D)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +20(C-D)(A-B)(1+d)[(1+2\alpha)(A-B) \\ +(4+5\alpha)(1+d)]+120(1+d)^3(C-D) \end{array}\right].$$

P r o o f. Following the procedure of Theorem 3.3, and using Lemma 2.1 and Lemma 2.4, the proof of Theorem 3.4 is obvious. □

For $\alpha = 0$, Theorem 3.4 yields the following result:

Corollary 3.7. *If* $f \in \psi'_c(w; A, B; C, D)$, *then*

$$|a_2| \leq \frac{(A-B)+2(C-D)}{4(1-d^2)},$$

$$|a_3| \leq \frac{(A-B)[(A-B)+(1+d)]+3(C-D)[(A-B)+2(1+d)]}{18(1-d^2)^2},$$

$$|a_4| \leq \frac{1}{96(1-d^2)^3}\begin{bmatrix} (A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +4(C-D)(A-B)[(A-B)+4(1+d)] \\ +24(1+d)^2(C-D) \end{bmatrix},$$

and

$$|a_5| \leq \frac{1}{600(1-d^2)^4}\begin{bmatrix} (A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +5(C-D)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +20(C-D)(A-B)(1+d)[(A-B) \\ +4(1+d)]+120(1+d)^3(C-D) \end{bmatrix}.$$

On putting $\alpha = 1$, Theorem 3.4 agrees with the following result:

Corollary 3.8. *If* $f \in \psi'_{c1}(w; A, B; C, D)$, *then*

$$|a_2| \leq \frac{(A-B)+(C-D)}{4(1-d^2)},$$

$$|a_3| \leq \frac{[(A-B)+(1+d)][(A-B)+2(C-D)]}{18(1-d^2)^2},$$

$$|a_4| \leq \frac{1}{96(1-d^2)^3}\begin{bmatrix} (A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +3(C-D)(A-B)[(A-B)+3(1+d)] \\ +6(1+d)^2(C-D) \end{bmatrix},$$

and

$$|a_5| \leq \frac{1}{600(1-d^2)^4} \begin{bmatrix} (A-B)[(A-B)^3+6(1+d)(A-B)^2 \\ +11(1+d)^2(A-B)+6(1+d)^3] \\ +4(C-D)(A-B)[(A-B)+(1+d)] \\ \times[(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[(A-B)+3(1+d)] \\ +24(1+d)^3(C-D) \end{bmatrix}.$$

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