МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2001
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2001
Proceedings of Thirtieth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 8–11, 2001

LINEAR PROGRAMMING BOUNDS FOR SPHERICAL CODES AND DESIGNS*

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We describe linear programming (LP) techniques used for obtaining upper/lower bounds on the size of spherical codes/spherical designs. A survey of universal bounds is presented together with description of necessary and sufficient conditions for their optimality. If improvements are possible, we describe methods for finding new bounds. In both cases we are able to find new bounds in great ranges of parameters. Investigations on the possibilities for attaining the universal bounds are described. Some investigations which lead beyond the pure LP are also presented.

1. Introduction. A *spherical* (n, M, s)*-code* is a finite subset C of the n-dimensional Euclidean sphere $\mathbf{S}^{n-1}$ with cardinality $|C| = M$ and a maximal inner product $s = \max\{\langle x, y\rangle : x, y \in C, x \neq y\}$. The concept of spherical designs was introduced in 1977 by Delsarte, Goethals and Seidel [18].

Definition 1. *A code* $C \subset \mathbf{S}^{n-1}$ *is called a* spherical $\tau-$design *if and only if the equality*

$$\int_{\mathbf{S}^{n-1}} f(x)\, d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x) \tag{1}$$

($(\mu(\cdot)$ *is the usual Lebesgue measure on* $\mathbf{S}^{n-1}$*, normalized for* $\mu(\mathbf{S}^{n-1}) = 1)$ *holds for all polynomials* $f(x) = f(x_1, x_2, \ldots, x_n)$ *of degree at most* τ*.*

LP techniques for estimating the size of codes and designs on $\mathbf{S}^{n-1}$ were introduced after 1975. The result was immediate and new bounds appeared to be, in a sense, universal, and relatively easy for calculation. Apart from the theoretical point of view, spherical codes and designs were proved to be important at least in two practical areas: codes are used in communications (modems, mobile devices, etc.) and designs in approximations.

The background for the LP approach was developed by Delsarte, Goethals and Seidel [18] and Kabatiansky and Levenshtein [23]. They proved that real polynomials having certain properties can be used for obtaining upper bounds on the size of codes (for fixed dimension and maximal inner product) and lower bounds on designs (for fixed dimension and strength). Moreover, suitable polynomials were proposed and universal bounds have been obtained.

*This research was partially suppoqerted by UNESCO under contract UVO-ROSTE 875.695.0 and by Bulgarian NSF under contract MM-901/99.

The aim of this survey is to describe the present situation along with the universal bounds and their possible LP improvements and to propose some directions for future investigations. The paper is organized as follows.

In section 2 we discuss the logic of the LP bounds and the logic of the universal bounds, i.e. the Levenshtein bound for codes [24, 26] and the Delsarte-Goethals-Seidel bound for designs [18]. The distance distributions of codes and designs attaining the universal bounds can be computed. This can be used for proving nonexistence results [5, 6, 8, 16, 13]. In section 3 we describe necessary and sufficient conditions for existence of better than the universal bounds. Despite extremal in some sense and best possible in (infinitely) many cases, these bounds appear not to give the full strength of the LP [30, 7, 12, 17]. Properties of test functions introduced in [12] need to be studied. Section 4 describes the search of new LP bounds for codes and designs. When improvements are possible, we apply certain methods for calculating new bounds. In section 5 we go beyond the pure LP. Indeed, LP techniques can be combined with other (geometric, for example) ideas for investigations on the structure of putative good codes and designs. The problem for designs is easier and we describe improvements (either asymptotic and in concrete cases, such as the first open ones) of the Delsarte-Goethals-Seidel bound [14]. In other application, LP techniques are used to estimate other parameters of codes such as their distance distribution. This results [3] in obtaining a new bound on the exponent of error probability of decoding for the best possible codes in the Gaussian channel.

2. Universal LP bounds.

2.1. Gegenbauer polynomials. The theory of spherical codes and designs is naturally connected with the family of Gegenbauer polynomials $\{P_i^{(n)}(t)\}_{i=0}^{\infty}$ [1, Chapter 22] (n is nothing but the relevant dimension). They can be defined by

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t),\ i \geq 1,$$

where $P_0^{(n)}(t) = 1$ and $P_1^{(n)}(t) = t$. It is well known that any real polynomial $f(t)$, $\deg(f) = m$, can be uniquely expanded in terms of Gegenbauer polynomials as

$$f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t).$$

The following definition joins two properties of very different nature. It is convenient for explanation of the LP approach for bounding sizes of codes and designs.

Definition 2. *Denote*

$$\begin{aligned} A_{n,s} &= \{f(t) : f(t) \leq 0 \textit{ for } t \in [-1,s], f_i \geq 0 \textit{ for } i=1,...,m=\deg f(t), f_0 > 0\}, \\ B_{n,\tau} &= \{f(t) : f(t) \geq 0 \textit{ for } t \in [-1,1], f_i \leq 0 \textit{ for } i=\tau+1,...,m=\deg f(t)\}. \end{aligned}$$

In applications, one needs polynomials in the good sets $A_{n,s}$ or $B_{n,\tau}$ for specified values of n and s, or n and τ, respectively.

2.2. The LP approach. The LP approach works well for a class of metric spaces called polynomial metric space [24, 26]. The Euclidean spheres appear to be among the most interesting examples. We describe in brief the idea behind these bounds as it is exposed in [24, 26]. Another description can be found in [18].

The spaces $\mathbf{S}^{n-1}$ are under consideration with the usual Euclidean metric. It is well known that the Hilbert space $\mathcal{L}_2(\mathbf{S}^{n-1},\mu)$ of complex-valued functions defined on $\mathbf{S}^{n-1}$ with inner product

$$\langle u,v\rangle = \int_{\mathbf{S}^{n-1}} u(x)\overline{v(x)}\,d\mu(x)$$

can be decomposed into a countable direct sum of mutually orthogonal subspaces V_i, $i=0,1,\ldots$, where $\dim(V_i)=r_i$. The subspace V_i is in fact the space of all functions, which are represented on $\mathbf{S}^{n-1}$ by harmonic homogeneous polynomials on n variables of total degree i. If $\{v_{ij}(x) : j=1,2,\ldots,r_i\}$ is an orthonormal basis of V_i, one considers the so-called zonal spherical functions

$$P_i^{(n)}(\langle x,y\rangle) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)}. \tag{2}$$

In this way one obtains nothing but the Gegenbauer polynomials exactly as defined in 2.1.

The next statement [18, 25, 26] gives an equivalent definition of spherical designs.

Lemma 1. *A spherical code C is a τ-design iff $\sum_{x\in C} v(x) = 0$ for every function $v(x)\in \cup_{i=1}^{\tau} V_i$.*

It is easy to see that, for any finite set $C\subset \mathbf{S}^{n-1}$ and any polynomial $f(t)$ of degree m, the following identity holds [24, 26] (see also [18, Corollary 3.8])

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |C|^2 f_0 + \sum_{i=1}^{m}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\left|\sum_{x\in C} v_{ij}(x)\right|^2. \tag{3}$$

Using (3) and Lemma 1 one obtains the LP bounds for codes and designs [18, 23, 24, 25, 26].

Theorem 1. (i) *For any polynomial $f(t)\in A_{n,s}$ and any (n,M,s) code we have $M\le f(1)/f_0$.*
(ii) *For any polynomial $f(t)\in B_{n,\tau}$ and any τ-design $C\subset\mathbf{S}^{n-1}$ we have $|C|\ge f(1)/f_0$.*

2.3. Levenshtein bound for codes and Delsarte-Goethals-Seidel bound for designs. A general choice for good polynomials was proposed by Levenshtein [24, 25, 26] for Theorem 1(i) and by Delsarte, Goethals and Seidel [18] for Theorem 2(ii). Both bounds are extremal in some sense: they can not be improved by using polynomials of the same or lower degree. The bounds themselves can be described in terms of adjacent orthogonal polynomials and their zeroes as follows.

The adjacent polynomials are simply Jacobi polynomials $P_i^{a+\frac{n-3}{2},b+\frac{n-3}{2}}(t)$ [1, Chapter 22], where $a,b\in\{0,1\}$. We shall write them shortly as $P_i^{a,b}(t)$. Set $T_k^{1,\varepsilon}(t,s) = \sum_{i=0}^{k} r_i^{1,\varepsilon}P_i^{1,\varepsilon}(t)P_i^{1,\varepsilon}(s)$, where $r_i^{1,\varepsilon} = \left(\frac{n+2i-1+\varepsilon}{n+\varepsilon-1}\right)^{2-\varepsilon}\binom{n+i-2+\varepsilon}{i}$ and $\varepsilon\in\{0,1\}$ (i.e. $a=1$ and $b=\varepsilon$). Let $t_k^{1,\varepsilon}$ be the greatest zero of the polynomial $P_k^{1,\varepsilon}(t)$.

Then the polynomials $f_m^{(s)}(t) = (t+1)^\varepsilon (t-s)[T_{k-1}^{1,\varepsilon}(t,s)]^2$, where $m = 2k-1+\varepsilon$ and $t_{k-1+\varepsilon}^{1,1-\varepsilon} \leq s < t_k^{1,\varepsilon}$, were used by Levenshtein [24, 25, 26].

Theorem 2 ([24, 25, 26])**.** *If C is an (n,M,s) code, then*

$$M \leq \begin{cases} L_{2k-1}(n,s) = \binom{k+n-3}{k-1}\left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] \\ \qquad \text{for } t_{k-1}^{1,1} \leq s < t_k^{1,0}, \\ L_{2k}(n,s) = \binom{k+n-2}{k}\left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s) - P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s) + P_{k+1}^{(n)}(s))}\right] \\ \qquad \text{for } t_k^{1,0} \leq s < t_k^{1,1}. \end{cases} \tag{4}$$

Moreover, if $M = L_m(n,s)$, then C is a spherical m-design and all inner products $\langle x,y\rangle$, $x,y \in C$, $x \neq y$, are zeros of $f_m(t)$.

The Delsarte-Goethals-Seidel bound was obtained in [18, Theorems 5.11, 5.12]. It can be obtained by using the polynomial $f^{(\tau)}(t) = (t+1)^\varepsilon (P_e^{1,\varepsilon}(t))^2$, where $\varepsilon \in \{0,1\}$, $\tau = 2e+\varepsilon$.

Theorem 3 [18]**.** *If $C \subset \mathbf{S}^{n-1}$ is a τ-design, $\tau = 2e+\varepsilon$, $\varepsilon \in \{0,1\}$, then*

$$|C| \geq R(n,\tau) = \binom{n+e-1}{n-1} + \binom{n+e-2+\varepsilon}{n-1}. \tag{5}$$

Moreover, if $|C| = R(n,\tau)$, then C is an $(n, L_\tau(n,s), s)$ code and all inner products $\langle x,y\rangle$, $x,y \in C$, $x \neq y$, are zeros of $f^{(\tau)}(t)$.

2.4. On maximal codes and minimal designs. A spherical design is called tight if it attains the bound (5). There exist tight τ-designs for $\tau = 1,2,3$ in all dimensions. Bannai and Damerell [5, 6] proved that for $n \geq 3$ tight spherical τ-designs on $\mathbf{S}^{n-1}$ do not exist if $\tau = 2e$ and $e \geq 3$ or $\tau = 2e+1$ and $e \geq 4$ except for $\tau = 11$, $n = 24$. Exactly eight tight τ-designs with $\tau \geq 4$ are known (see [5, 6, 8]).

All spherical codes which meet even Levenshtein bounds $L_{2k}(n,s)$ were characterized in [13].

Theorem 4. *Let C be an $(n, L_{2k}(n,s), s)$ code ($k \geq 2$, $n \geq 3$). Then one of the following holds:*

a) *$k=2$, $n = m^2-3$, $m \geq 3$ is odd, $s = 1/(m+1)$, and W is a tight spherical 4-design;*

b) *$k=2$, $n=3$, $s = 1/\sqrt{5}$, and W is the icosahedron (which is a tight spherical 5-design);*

c) *$k=2$, $n = m^2-2$, $m \geq 3$ is odd, $s = 1/m$, and W is a tight spherical 5-design;*

d) *$k=3$, $n = 3m^2-4$, $m \geq 2$ is integer, $s = 1/m$, and W is a tight spherical 7-design;*

e) *$k=5$, $n=24$, $s=1/2$, and W is the unique tight spherical 11-design formed by the vectors of minimal norm in the Leech lattice.*

This classification and other results (on the odd Levenshtein bounds) from [13, 16] are based on the simple idea to find the distance distribution of maximal codes and to check whether it is integral. This approach was used for designs in [8] to establish nonexistence results for tight 4- and 5-designs which are used in Theorem 4 a)–c).

3. Necessary and sufficient conditions for improving universal bounds by LP. In [12], test functions $Q_j(n,s)$ have been defined to check the global optimality of the Levenshtein bound. Let the numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1+\varepsilon} = s$ $(-1 \le \alpha_0,\ \varepsilon \in \{0,1\})$ be all different zeros of the Levenshtein polynomial $f_{2k-1+\varepsilon}^{(s)}(t)$. In [26, Theorems 4.1 and 4.3], positive weights ρ_i, $i = 0,1,\ldots,k+\varepsilon$, have been defined with the property that, for any real polynomial $f(t)$ of degree at most $2k-1+\varepsilon$ the equality

$$f_0 = \rho_{k+\varepsilon} f(1) + \sum_{i=0}^{k-1+\varepsilon} \rho_i f(\alpha_i) \tag{6}$$

holds. Obviously, we have $L_{2k-1+\varepsilon}(n,s) = 1/\rho_{k+\varepsilon}$. The test functions are defined by

$$Q_j(n,s) = \rho_{k+\varepsilon} + \sum_{i=0}^{k-1+\varepsilon} \rho_i P_j^{(n)}(\alpha_i)$$

for $t_{k-1+\varepsilon}^{1,1-\varepsilon} \le s < t_k^{1,\varepsilon}$.

The main result in [12] (obtained by an attempt to adopt (6) for higher degrees; see also [26, Theorem 5.47]) is the following.

Theorem 5 [12, Theorem 3.1]**.** *The bound $L_m(n,s)$ can be improved by a polynomial from $A_{n,s}$ of degree at least $m+1$ if and only if $Q_j(n,s) < 0$ for some $j \ge m+1$. Moreover, if $Q_j(n,s) < 0$ for some $j \ge m+1$, then $L_m(n,s)$ can be improved by a polynomial from $A_{n,s}$ of degree j.*

The test functions can be effectively calculated both by computer and by analytical methods [12]. The sign of $Q_{2k+3}(n,s)$ was investigated in [12] implying many results. We formulate only the most general statement.

Theorem 6 [12]**.** *For any fixed $n \ge 3$ there exists $m_0 \ge 4$ such that every Levenshtein bound $L_m(n,s)$, $m \ge m_0$, can be improved by LP for all s in its range.*

This theorem shows that the Levenshtein bounds can be improved by LP in wide ranges of parameters. However, in another sense, the Levenshtein bounds are strong. The following conjectures were suggested by great amount of numerical data.

Conjecture 1. *For any fixed $s \in (0,1)$ there exists $n_0 \ge 3$ such that no Levenshtein bound $L_m(n,s)$, $n \ge n_0$, can be improved by LP.*

Conjecture 2. *If $Q_{m+3}(n,s) \ge 0$ and $Q_{m+4}(n,s) \ge 0$, then $Q_j(n,s) \ge 0$ for every $j \ge m+1$.*

4. Obtaining new bounds. Investigations in [12] show that one always has $Q_{m+1}(n,s) \ge 0$ and $Q_{m+2}(n,s) \ge 0$. Hence, the first degree which could work is $m+3$. It turns out that results by degrees $m+3$ and $m+4$ are already strong enough (see Conjecture 2 above). This fits very well with a computer method which was proposed earlier [7]. We have developed program SCOD which can be used in a wide range (for degrees between 6 and 19). In particular, SCOD was used for calculation of a column (of upper bounds on the minimum distance of codes of prescribed size) in all tables of best known spherical codes in a forthcoming book on spherical codes by Ericson and Zinoviev [19].

Test functions for designs were defined in [27] (see also [28]). They appear to be just special values of the test functions for codes. Earlier, LP improvements of the bound (5) for some $\tau \geq 6$ were obtained in [17, 34]. Analytic forms of the new bounds were proposed in [27, 28, 29].

5. Beyond the pure LP.

5.1. Investigating the structure of small designs. In [14], it was shown that the LP approach can be modified for obtaining new nonexistence and characterization results as follows. Another characterization of spherical τ-designs (see, for example, [20, Equation 1.10]) says that for any τ-design $C \subset \mathbf{S}^{n-1}$ and for any point $y \in C$ the equality

$$\sum_{x \in C \setminus \{y\}} f(\langle x, y \rangle) = |C| f_0 - f(1) \tag{7}$$

holds for every real polynomial $f(t)$ of degree at most τ.

This definition suggests the following idea [14, 9]. Using suitable polynomials in (7) we derive restrictions on the distributions of the inner products of a τ-design $C \subset \mathbf{S}^{n-1}$. This implies conditions of the existence of designs in terms of τ, n, and the cardinality $|C| = R(n, \tau) + k$. These conditions imply nonexistence results for designs with odd strengths and odd cardinalities (i.e. for odd k) in many cases. Set

$$B_{\mathrm{odd}}(n, \tau) = \min\{|C| : C \subset \mathbf{S}^{n-1} \text{ is a } \tau\text{-design, } |C| \text{ is odd.}\}$$

For $\tau = 3$, we prove the nonexistence of spherical 3-designs on $\mathbf{S}^{n-1}$ with $R(n,3)+k = 2n + k$ points for all odd $k < (2^{1/3} - 1)n + p$, where $p = 2(14 - 5.2^{1/3} - 4.2^{2/3})/9 \approx 0.30018$. For $\tau = 5$, we prove the nonexistence of spherical 5-designs on $\mathbf{S}^{n-1}$ with $R(n, 5) + k = n^2 + n + k$ points for all odd $k < n^2(2^{1/5} - 1)/2 + p_1 n + p_2$, where $p_1 = (7.2^{1/5} - 2.2^{3/5} - 5)/10 \approx 0.00095$ and $p_2 \approx 0.0428$.

In general, [14, Theorem 2.8] shows that for $\tau = 2e + 1$ and for every positive $p < (2^{1/\tau} - 1)/e!$ there exists a constant $n_0 = n_0(p)$ such that for $n \geq n_0$ there do not exist τ-designs on $\mathbf{S}^{n-1}$ with cardinality $R(n, \tau) + k$ for all odd positive $k \leq pn^e$. Therefore,

$$B_{\mathrm{odd}}(n, 2e + 1) \geq \frac{1 + 2^{1/\tau}}{e!} n^e \text{ as } n \to \infty,$$

while (5) gives $B_{\mathrm{odd}}(n, 2e + 1) \geq 2n^e/e!$ as $n \to \infty$.

In three dimensions, the above rules out the first open cases by showing the nonexistence of 3-designs with 7 points and 5-designs with 13 points. On the other hand, Bajnok [4] has constructed 3-designs on $\mathbf{S}^2$ with m points for $m = 8$ and all $m \geq 10$. Hardin-Sloane [22] and [31] have constructed 5-designs on $\mathbf{S}^2$ with m points for $m = 12$, 16, 18, 20, and all $m \geq 22$ and conjectured that the remaining cardinalities are impossible.

When the nonexistence argument does not work, we obtain bounds on the maximal inner product $s(C) = \max\{\langle x, y \rangle : x, y \in C, x \neq y\}$ of $(2e + 1)$-designs on $\mathbf{S}^{n-1}$ of odd sizes in terms of e, n, and $|C|$. Fazekas-Levenshtein [20, Theorem 4] note that a combination of (4) and (5) implies a lower bound on $s(C)$. The asymptotic form of this bound is

$$s(C) \geq \sqrt{\frac{2}{n}}h_e + O(n^{-3/2}) \text{ as } n \to \infty, \tag{8}$$

where $\tau = 2e + 1$ and h_e is the greatest zero of the Hermite polynomial $H_e(t)$. For odd $k = \gamma n^e$, we show that

$$s(C) \geq \frac{1 - 2\gamma e! - \gamma^2 (e!)^2}{(1 + \gamma e!)^2} \text{ as } n \to \infty,$$

which is positive for $(2^{1/\tau} - 1)/e! < \gamma < (\sqrt{2} - 1)/e!$, and therefore is better for large enough n than (8). Interestingly, our bounds are better in many small cases as well.

By the argument from the last paragraph, further improvements of the bounds for $B_{\text{odd}}(n, 2e + 1)$ both in small cases and asymptotically were obtained recently [10].

5.2. Some other applications. It was known from the early years of the information theory [32, 33] that spherical codes play important role in the theory of transmission of information over noisy channels. Computing the so-called reliability function (the best attainable error exponent) of the so-called the Gaussian channel (which is one of the most important ones in the practice) was dominating through the end of 1960s [21].

Let C be an (n, M, s) code. For $s_1, s_2 \in [-1, s]$, denote $a(s_1, s_2) = |\{(x, y) \in C \times C : s_1 \leq \langle x, y \rangle \leq s_2\}|/|C|$. For suitable s_1 and s_2, the numbers $a(s_1, s_2)$ are important for estimations on error probabilities in the Gaussian channel. Ashikhmin, Barg and Litsyn [3] observed that LP method can be used in the following form.

Theorem 7 [3]. *Let C be an (n, M, s) code and m be an integer. Let $-1 \leq u_0 < s$ and suppose that $u_0 < u_1 < \cdots < u_{m-1} < u_m = s$ define a partition of the interval $[u_0, s]$ into m equal segments $U_i = [u_i, u_{i+1}]$. Suppose that $f(t) \in A_{n,u_0}$ and $f(t) \geq 0$ for $t \in [u_0, 1]$. Then there exists a number i, $0 \leq i \leq m - 1$, and a point $s \in U_i$ such that $a(u_i, u_{i+1}) \geq [f_0|C| - f(1)]/mf(s)$.*

Using suitable polynomials in Theorem 7, Ashikhmin, Barg and Litsyn [3] obtain new bounds on the reliability function of the Gaussian channel thus advancing in a longstanding problem.

6. Some new results and open problems. Apart from nonexistence results, investigations of concrete objects can rely on LP. LP methods for finding indexes (degrees i, for which Lemma 1 holds, i.e. for all $v(x) \in V_i$) were explored in [15]. For example, it is proved in [15] that the 600-cell, which is only minimal 11-design [2], has (non-trivial) indexes 14, 16, 18, 22, 26, 28, 34, 38, 46 and 58. This fact inspired us to prove that there exist a unique 120-point 11-design in four dimensions and a unique $(4, 120, \cos(\pi/5))$ code and both these are nothing but the 600-cell [11].

It is still an open problem to find closed analytic form of the improvements of the Levenshtein bounds (4) (i.e. for bounds which can be obtained by higher degree polynomials). In particular, it would be interesting to know formulae for the bounds which can be obtained by degrees $m + 3$ and $m + 4$ improving $L_m(n, s)$. The analogous problem for designs was solved by Nikov and Nikova [28, 29].

Another situation in which the authors would like to see advances is the application of LP methods for investigation of the structure of putative optimal spherical codes and, in particular, for the so-called kissing numbers problem. Section 5.1 shows the power of this approach for designs.

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ГРАНИЦИ НА ЛИНЕЙНОТО ПРОГРАМИРАНЕ ЗА СФЕРИЧНИ КОДОВЕ И ДИЗАЙНИ

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Разгледани са техники на линейното програмиране, които се използват за получаване на горни/долни граници за обема на сферични кодове/дизайни. Представен е обзор на универсалните граници заедно с описание на необходими и достатъчни условия за тяхната оптималност. Когато са възможни подобрения, са описани методи за намиране на нови граници. И в двата случая се оказва възможно получаването на нови граници за голям брой параметри. Разгледани са и възможностите за достигане на универсалните граници. Описани са и някои изследвания, чиито характер е вече зад чистото линейно програмиране.