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MEASURABILITY OF SETS OF PAIRS OF PARALLEL STRAIGHT LINES IN THE GALILEAN PLANE*

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The measurable sets of pairs of parallel straight lines and the corresponding invariant densities with respect to the group of the general similitudes and its subgroups are described.

1. Introduction. In the affine version, the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y–axis of the basic affine coordinate system Oxy [7], [8], [10], [11]. The affine transformations leaving invariant the special direction Oy can be written in the form

$$\begin{aligned} x' &= a_1 + a_2 x, \\ y' &= a_3 + a_4 x + a_5 y, \end{aligned} \tag{1}$$

where $a_1, ..., a_5 \in \mathbb{R}$ and $a_2 a_5 \neq 0$.

It is easy to verify that the transformations (1) map a line segment and an angle of Γ_2 into a proportional line segment and a proportional angle with the coefficients of proportionality $|a_2|$ and $|a_2^{-1} a_5|$, respectively. Thus they form the group H_5 of the general similitudes of Γ_2. The infinitesimal operators of H_5 are

$$X_1 = \frac{\partial}{\partial x},\ X_2 = x\frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = x\frac{\partial}{\partial y},\ X_5 = y\frac{\partial}{\partial y}.$$

In [1], [2] we proved the following results:

I. The four–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_4^1 = (X_1, X_2, X_3, X_4),\ H_4^2 = (X_1, X_2, X_3, X_5),\ H_4^3 = (X_2, X_3, X_4, X_5),$$

$$H_4^4 = (X_1, X_3, X_4, \alpha X_2 + X_5).$$

II. The three–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_3^1 = (X_1, X_2, X_3),\ H_3^2 = (X_1, X_2, X_5),\ H_3^3 = (X_1, X_3, X_4),\ H_3^4 = (X_2, X_3, X_4),$$

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$$H_3^5=(X_2,X_3,X_5),\ H_3^6=(X_2,X_4,X_5),\ H_3^7=(X_1,X_3,\alpha X_2+\beta X_4+X_5),$$
$$H_3^8=(X_3,X_4,\alpha X_1+X_5),\ H_3^9=(X_3,X_4,\alpha X_2+X_5|\alpha\neq 0),$$
$$H_3^{10}=(X_3,X_2+2X_5,\alpha X_1+X_4|\alpha\neq 0).$$

III. The two–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_2^1=(X_1,X_2),\ H_2^2=(X_2,X_3),\ H_2^3=(X_2,X_4),\ H_2^4=(X_2,X_5),$$
$$H_2^5=(X_1,\alpha X_2+X_3),\ H_2^6=(X_1,\alpha X_2+X_5),\ H_2^7=(X_3,\alpha X_1+X_4|\alpha\neq 0),$$
$$H_2^8=(X_3,\alpha X_1+X_5),\ H_2^9=(X_3,\alpha X_2+\beta X_4+X_5|\alpha\neq 0),\ H_2^{10}=(X_4,\alpha X_2+X_3),$$
$$H_2^{11}=(X_4,\alpha X_2+X_5),\ H_2^{12}=(X_2+2X_5,\alpha X_1+X_4|\alpha\neq 0).$$

IV. The one–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_1^1=(X_1),\ H_1^2=(X_2),\ H_1^3=(X_3),\ H_1^4=(X_4),\ H_1^5=(X_5),$$
$$H_1^6=(\alpha X_1+X_4|\alpha\neq 0),\ H_1^7=(X_1+X_5),\ H_1^8=(\alpha X_2+X_3|\alpha\neq 0),$$
$$H_1^9=(\alpha X_2+X_5|\alpha\neq 0),\ H_1^{10}=(\alpha X_2+\beta X_4+X_5|\alpha\beta\neq 0).$$

Here and everywhere in the text α and β are real constants.

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [9], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6], we find the measurable sets of pairs of parallel straight lines in Γ_2 with respect to H_5 and its subgroups.

2. Measurability with respect H_5. Let $G_i:\ y=kx+n_i,\ i=1,2$, be two parallel straight lines in Γ_2, i.e.

$$k(n_2-n_1)\neq 0.$$

Under the action of (1) the pair $(G_1,G_2)(k,n_1,n_2)$ is transformed into the pair $(G_1',G_2')(k',n_1',n_2')$ as

$$\begin{aligned}&k'=a_2^{-1}(a_4+a_5k),\\&n_i'=a_2^{-1}(a_2a_3-a_1a_4-a_1a_5k+a_2a_5n_i),\\&a_2a_5\neq 0,\quad i=1,2.\end{aligned}\tag{2}$$

The transformations (2) form the so–called associated group $\overline{H_5}$ of H_5 [9; p.34]. The associated group $\overline{H_5}$ is isomorphic to H_5 and the invariant density with respect to H_5 of the pairs (G_1,G_2), if it exists, coincides with the invariant density with respect to $\overline{H_5}$ of the points (k,n_1,n_2) in the set of parameters [9; p.33]. The infinitestimal operators of $\overline{H_5}$ are

$$Y_1=kY_3,\ Y_2=kY_4,\ Y_3=\frac{\partial}{\partial n_1}+\frac{\partial}{\partial n_2},$$
$$Y_4=\frac{\partial}{\partial k},\ Y_5=k\frac{\partial}{\partial k}+n_1\frac{\partial}{\partial n_1}+n_2\frac{\partial}{\partial n_2}.$$

From $Y_4(k)\neq 0$ we deduce:

Theorem 1. *The sets of pairs of parallel straight lines are not measurable with respect to the group* H_5 *of the general similitudes and have not measurable subsets.*

3. Measurability with respect to the subgroups of H_5**.** The group $\overline{H_4^2} = (Y_1, Y_2, Y_3, Y_5)$, corresponding to the subgroup $H_4^2 = (X_1, X_2, X_3, X_5)$, is a transitive group and since $Y_3(k) = 0$ it is measurable. The integral invariant function [9;p.9] $f = f(k, n_1, n_2)$, satisfying the system of R.Deltheil [3;p.28], [9;p.11]

$$Y_1(f) = 0,\ Y_2(f) + f = 0,\ Y_3(f) = 0,\ Y_5(f) + 3f = 0$$

has the form

$$f = \frac{c}{k(n_2 - n_1)^2},$$

where $c = const \neq 0$. Thus we establish:

Theorem 2. *The pairs* (G_1, G_2) *of parallel straight lines* $G_i : y = kx + n_i,\ i = 1, 2,$ *have the invariant with respect to* H_4^2 *density*

$$d(G_1, G_2) = \frac{1}{|k|(n_2 - n_1)^2} dG_1 \wedge dn_2,$$

where $dG_1 = dk \wedge dn_1$ *denotes the metric density for the straight lines in* Γ_2*.*

Remark 1. Note that the distance between G_1 and G_2 is defined by the quantity

$$\Delta n = |n_2 - n_1|$$

and then (3) can be written in the form

$$d(G_1, G_2) = \frac{1}{|k|(\Delta n)^2} dG_1 \wedge dn_2.$$

By arguments similar to the ones used above we examine the measurability of the set of pairs of parallel straight lines with respect to all the rest subgroups of H_5. We collect the results in the following table:

subgroup	measurable set/subset	expression of the density
1	2	3
H_4^1	it is not measurable and has not measurable subsets	
H_4^2	$k \neq 0$	$\lvert k\rvert^{-1}(n_2 - n_1)^{-2} dG_1 \wedge dn_2$
H_4^3	it is not measurable and has not measurable subsets	
H_4^4		$\lvert n_2 - n_1\rvert^{\alpha-3} dG_1 \wedge dn_2$
H_3^1	$n_2 = n_1 + \lambda,\ \lambda k \neq 0$	$\lvert k\rvert^{-1} dG_1$
H_3^2	$k \neq 0$	$\lvert k\rvert^{-1}(n_2 - n_1)^{-2} dG_1 \wedge dn_2$

1	2	3
H_3^3	$n_2 = n_1 + \lambda,\ \lambda \neq 0$	dG_1
H_3^4	it is not measurable and has not measurable subsets	
H_3^5	$k \neq 0$	$\|k\|^{-1}(n_2 - n_1)^{-2} dG_1 \wedge dn_2$
H_3^6	it is not measurable and has not measurable subsets	
H_3^7 $\alpha \neq 1$	$k = \frac{1}{1-\alpha}\left[\lambda(n_2 - n_1)^{1-\alpha} - \beta\right]$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$
H_3^7 $\alpha = 1,$ $\beta = 0$	$k = \lambda$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$
H_3^7 $\alpha = 1,$ $\beta \neq 0$	$k = \beta \ln \|n_2 - n_1\| + \lambda$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$
H_3^8		$(n_2 - n_1)^{-3} dG_1 \wedge dn_2$
H_3^9		$\|n_2 - n_1\|^{\alpha-3} dG_1 \wedge dn_2$
H_3^{10}		$\|n_2 - n_1)^{-\frac{5}{2}} dG_1 \wedge dn_2$
H_2^1	$n_2 = n_1 + \lambda,\ \lambda k \neq 0$	$\|k\|^{-1} dG_1$
H_2^2	$n_2 = n_1 + \lambda,\ \lambda k \neq 0$	$\|k\|^{-1} dG_1$
H_2^3	it is not measurable and has not measurable subsets	
H_2^4	$n_2 = \lambda n_1,\ kn_1 \neq 0,\ \lambda \neq 1$	$\|kn\|^{-1} dG_1$
H_2^5 $\alpha \neq 0$	$n_2 = n_1 + \lambda,\ \lambda k \neq 0$	$\|k\|^{-1} dG_1$
H_2^5 $\alpha = 0$	it is not measurable and has not measurable subsets	
H_2^6	$k = \lambda(n_2 - n_1)^{1-\alpha}$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$
H_2^7	$n_2 = n_1 + \lambda,\ \lambda \neq 0$	dG_1
H_2^8	$n_2 = n_1 + \lambda k,\ \lambda k \neq 0$	$\|k\|^{-2} dG_1$
H_2^9 $\alpha \neq 1$	$k = \frac{1}{1-\alpha}\left[\lambda(n_2 - n_1)^{1-\alpha} - \beta\right],$ $(n_2 - n_1)^{-2} dn_1 \wedge dn_2$	$n_1 \neq n_2$
H_2^9 $\alpha = 1,$ $\beta \neq 0$	$k = \lambda\beta \ln \|n_2 - n_1\|$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$

1	2	3
H_2^9 $\alpha=1$, $\beta=0$	$k=\lambda$	$(n_2-n_1)^{-2}dn_1\wedge dn_2$
H_2^{10}	$n_2=n_1+\lambda,\ \lambda\neq 0,$	$e^{\alpha n_1}dG_1$
H_2^{11}	$n_2=\lambda n_1,\ n_1\neq 0,\ \lambda\neq 1$	$\lvert n_1\rvert^{\alpha-2}dG$
1	2	3
H_2^{12}	$n_2=\lambda n_1+\frac{1}{2}(\lambda-1)\alpha k^2,\ \lambda\neq 1,$ $n_1+\frac{1}{2}\alpha k^2\neq 0$	$\lvert n_1+\frac{1}{2}\alpha k^2\rvert^{-\frac{3}{2}}dG_1$
H_1^1	$k=\lambda_1,\ n_2=n_1+\lambda_2,\ \lambda_2\neq 0$	dn_1
H_1^2	$n_1=\lambda_1,\ n_2=\lambda_2,\ k(\lambda_1-\lambda_2)\neq 0$	$\lvert k\rvert^{-1}dk$
H_1^3	$k=\lambda_1,\ n_2=n_1+\lambda_2,\ \lambda_2\neq 0,$	dn_1
H_1^4	$n_1=\lambda_1,\ n_2=\lambda_2,\ \lambda_1\neq\lambda_2$	dk
H_1^5	$n_1=\lambda_1 k,\ n_2=\lambda_2 k,\ (\lambda_1-\lambda_2)k\neq 0$	$\lvert k\rvert^{-1}dk$
H_1^6	$n_1=\frac{1}{2}\alpha k^2+\lambda_1,\ n_2=\frac{1}{2}\alpha k^2+\lambda_2,$ $\lambda_1\neq\lambda_2$	dk
H_1^7	$n_1=k(\lambda_1-\ln\lvert k\rvert),$ $n_2=k(\lambda_2-\ln\lvert k\rvert),\ (\lambda_1-\lambda_2)k\neq 0$	$\lvert k\rvert^{-1}dk$
H_1^8	$n_1=-\frac{1}{\alpha}\ln\lvert k\rvert+\lambda_1,$ $n_2=-\frac{1}{\alpha}\ln\lvert k\rvert+\lambda_2,\ (\lambda_1-\lambda_2)k\neq 0$	$\lvert k\rvert^{-1}dk$
H_1^9 $\alpha\neq 1$	$n_1=\lambda_1 k^{\frac{1}{1-\alpha}},\ n_2=\lambda_2 k^{\frac{1}{1-\alpha}}$ $(\lambda_1-\lambda_2)k\neq 0$	$\lvert k\rvert^{-1}dk$
H_1^9 $\alpha=1$	$k=\lambda_1,\ n_1\neq 0,\ n_2=\lambda_2 n_1,\ \lambda_2\neq 1$	$\lvert n_1\rvert^{-1}dn_1$
H_1^{10} $\alpha\neq 1$	$n_1=\lambda_1\left[(1-\alpha)k+\beta\right]^{\frac{1}{1-\alpha}}$ $n_2=\lambda_2\left[(1-\alpha)k+\beta\right]^{\frac{1}{1-\alpha}},$ $\lambda_1\neq\lambda_2,\ (1-\alpha)k+\beta\neq 0$	$\lvert(1-\alpha)k+\beta\rvert^{-1}dk$
H_1^{10} $\alpha=1$	$k=\beta\ln\lvert n_1\rvert+\lambda_1,\ n_1\neq 0,$ $n_2=\lambda_2 n_1,\ \lambda_2\neq 1$	$\lvert n_1\rvert^{-1}dn_1$

Remark 2. In the table $\lambda,\lambda_1,\lambda_2,\in R$.

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ ПАРАЛЕЛНИ ПРАВИ В ГАЛИЛЕЕВАТА РАВНИНА

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Описани са измеримите множества от двойки паралелни прави и са намерени съответните им инвариантни гъстоти относно групата на общите подобности.